ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2020, Volume 11, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 562
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

MIKHAIL L'VOVICH GOLDMAN

(to the 75th birthday)

Mikhail L'vovich Goldman was born on April 13, 1945 in Moscow. In 1963 he graduated from school in Moscow and entered the Physical Faculty of the M.V. Lomonosov Moscow State University (MSU) from which he graduated in 1969 and became a PhD student (1969–1972) at the Mathematical Department of this Faculty. In 1972 he has defended the PhD thesis, and in 1988 his DSc thesis "The study of spaces of differentiable functions of many variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor in Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he is a Professor of the S.M. Nikol'skii Mathematical Institute at the Peoples Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman are: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators. Main achievements: optimal embeddings of spaces with generalized smoothness, sharp conditions of the convergence of spectral expansions, descriptions of integral and differential properties of the generalized Bessel and Riesz-type potentials, sharp estimates for operators on cones and optimal envelopes for the cones of functions with properties of monotonicity. Professor M.L. Goldman has over 140 scientific publications in leading mathematical journals.

Under his scientific supervision, 8 candidate theses in Russia and 1 thesis in Kazakhstan were successfully defended. Some of his former students are now professors in Ethiopia, Columbia, Mongolia.

Participation in scientific and organizational activities of M.L. Goldman is well known. He is a member of the DSc Councils at RUDN and MSU, of the PhD Council in the Lulea Technical University (Sweden), a member of the Editorial Board of the Eurasian Mathematical Journal, an invited lector and visiting professor at universities of Russia, Germany, Sweden, UK etc., an invited speaker at many international conferences.

The mathematical community, friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Mikhail L'vovich Goldman on the occasions of his 75th birthday and wish him good health, happiness, and new achievements in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 11, Number 2 (2020), 09 – 18

DERIVATIONS IN SEMIGROUP ALGEBRAS

A.V. Alekseev, A.A. Arutyunov

Communicated by J.A. Tussupov

Key words: group theory, category theory, derivations.

AMS Mathematics Subject Classification: 34A55, 34B05, 58C40.

Abstract. The main goal of this paper is to describe derivations in semigroup algebras, where the semigroup satisfies the Maltsev conditions. We prove that the derivation algebra of such semigroup algebra is embedded into the derivation algebra of some group. In addition, we describe the so-called quasi-inner derivations as an ideal in a derivation algebra, which is useful for the further study of differentiations in semigroup algebras.

DOI: https://doi.org/10.32523/2077-9879-2020-11-2-09-18

1 Introduction

Derivations of semigroup algebras have been studied in several works. M. Lashkarizadeh Bami [11] considered first cohomology group of Clifford topological semigroups, D.E. Bagha [5] and N. D. Gilbert [8] considered module derivations of inverse semigroups. In the present paper we will consider semigroups which can be embedded in some groups as subsemigroups and we will show how to apply the geometrical approach to study of their derivation algebra.

We apply the approach we studied in [3] and show that the derivations can be associated with a category of adjoint action and the vector space of the so-called characters, i.e. mappings from the set of morphisms of the category to complex numbers preserving the composition. Within the framework of this paper, we will also propose a natural modification of the construction of internal and external derivations, which significantly encapsulates the study of the space of derivations, taking into account the geometric constructions that we introduce.

The main goal of this paper is to describe derivations in semigroup algebras, where the semigroup satisfies the Maltsev conditions. We prove that the derivation algebra of such semigroup algebra is embedded into the derivation algebra of some group. In addition, we describe the so-called quasi-inner derivations as an ideal in the derivation algebra, which is useful for the further study of differentiations in semigroup algebras.

We will use methods and ideas from papers [1], [2], [3], [4], [5].

2 Derivations of semigroup algebras

Consider a semigroup S with the identity element that can be embedded in some group G as a subsemigroup. Necessary and sufficient conditions for the embeddability of a semigroup into a group were found by A.I. Maltsev [14]. These conditions are represented as an infinite system of conditional identities (quasi-identities). There are some examples of the quasi-identities:

$$ap = aq \to p = q,$$
$$pa = qa \to p = q,$$
$$ap = bq, ar = bs, cp = dq \to cr = ds$$

where a, b, c, d, p, q, r, s are elements of the semigroup. The class of semigroups embeddable in groups cannot be characterized by a finite number of conditional identities ([15]).

Let us consider a group G, a semigroup $S \in G$ and its semigroup algebra $\mathbb{C}[S]$. A linear map

$$D : \mathbb{C}[S] \longrightarrow \mathbb{C}[S],$$

satisfying the Leibniz rule

$$D(ab) = D(a)b + aD(b), \quad a, b \in \mathbb{C}[S],$$

is called a derivation of the semigroup algebra.

Let $\mathrm{Der}(S)$ be the algebra of all derivations. A derivation of the following form

$$\mathrm{d}_x(a) \overset{def}{=} xa - ax, \quad x, a \in \mathbb{C}[S],$$

is called an inner derivation. Let $\mathrm{Der}_{Inn}(S)$ be the algebra of all inner derivations. It is easy to see that $\mathrm{Der}(S) \rhd \mathrm{Der}_{Inn}(S)$.

2.1 General definitions

Consider a semigroup $S \in G$ with the identity element. We construct the category C associated with the semigroup S.

1. A set of objects $Obj(C) = G$.

2. For each pair of objects a and $b \in Obj(C)$ a set of maps $\mathbf{Hom}\ (a, b) = \{(u, v) \in G \times S | v^{-1}u = a, uv^{-1} = b\}$. Let $\mathbf{Hom}(C)$ be the set of all maps.

3. A composition of maps $\varphi = (u_1, v_1) \in \mathbf{Hom}\ (a, b)$, $\psi = (u_2, v_2) \in \mathbf{Hom}\ (b, c)$ is a map $\varphi \circ \psi \in \mathbf{Hom}\ (a, c)$, such that

$$\varphi \circ \psi = (u_2 v_1, v_2 v_1).$$

The composition of morphisms is associative, since the product of elements in a semigroup is associative. The role of the identity map in $\mathbf{Hom}(a, a)$ is played by the map (a, e), where e is the semigroup identity element. It is also worth to mention that two objects $a, b \in Obj(C)$ are connected if and only if there exists a semigroup element $s \in S$, such that $b = sas^{-1}$. That means that the group elements a and b belong to the same conjugated class $[u]$, which leads us to the following statement:

Proposition 2.1. *The category C is a disjoint union of the categories $C_{[u]}$, where $[u]$ is a congugated class of the element u:*

$$C = \bigsqcup_{[u] \in [S]} C_{[u]}.$$

Proof. The proof is the same as for Theorem 2 in [3]. □

If the considered semigroup S is a group then the corresponding category becomes a groupoid which is studied in [3].

2.2 Relation between semigroups and categories

Let G be a group generated by a semigroup S, then the category C, which is assosiated with S is a subcategory of the groupoid Γ, which is associated with the group G.

In the following paragraph we are going to show that for every category C can be found a semigroup S and a categoty C_S, which is associated with S, so that the given category will be a subcategory of the C_S.

Consider a category C and its set of maps $\mathbf{Hom}(C)$. We associate each map with a pair of numbers, i.e. $\forall \varphi_i \in \mathbf{Hom}(C)\ \varphi_i \to (a_i, b_i)$. Consider a free semigroup $SF < \ldots a_i, \ldots, b_i, \ldots. >$. Compositions of maps in the category C define the relation in the semigroup, i.e. $\varphi_i \circ \varphi_j = \varphi_k$ or $(a_i, b_i) \circ (a_j, b_j) = (a_k, b_k)$. The composition rule gives us a relation in the semigroup $a_k = a_j b_i$, $b_k = b_j b_i$. Let R be the set of all relations. Consider the semigroup

$$S = SF < \ldots, a_i, \ldots, b_i, \ldots. > /R.$$

Proposition 2.2. *The category C is a subcategory of C_S which is associated with the semigroup S. The algebra of characters of C is a subalgebra of characters of C_S.*

Proof. Every map $\varphi \in \mathbf{Hom}(C)$ also exists in $\mathbf{Hom}(C_S)$ and each pair of maps $\varphi, \psi \in \mathbf{Hom}(C)$ is composable if and only if it is composable in $\mathbf{Hom}(C_S)$ due to the construction. □

2.3 Characters and derivations

The central definition of this paper is a definition of the character on the category C which has been studied before in [3].

Definition 1. A map $\chi : \mathbf{Hom}(C) \to \mathbb{C}$ such that for every maps φ and ψ

$$\chi(\varphi \circ \psi) = \chi(\varphi) + \chi(\psi)$$

is called a 1-character on the 1-category.

Let $X(C)$ be the space of all characters on C. It is easy to see that $X(C)$ is an infinite-dimensional vector space and, as we will show in Section 3, a Lie algebra.

Let G be the group generated by S. Then there is an embedding of the algebras

$$\lambda : \mathbb{C}[S] \to \mathbb{C}[G].$$

To simplify the construction we will consider the derivations $d : \mathbb{C}[S] \to \mathbb{C}[G]$ with the property $\mathrm{Im}(d) \subset \mathbb{C}[S]$. Consider an element $s \in S$. Then for some $d : \mathbb{C}[S] \to \mathbb{C}[G]$ an element $d(s)$ can be represented as (see [1], p. 17)

$$d(s) = \sum_{h \in G} d_s^h h, \tag{2.1}$$

where $d_s^h \in \mathbb{C}$ are coefficients that depend only on the derivation d.

The linear operator d satisfies the Leibniz rule $d(xy) = d(x)y + xd(y)$. We represent d as a matrix and find the relationship between the coefficients:

$$d_{s_1 s_2}^h = \sum_i d_{s_1}^{h_i} + \sum_j d_{s_2}^{h_j}, \tag{2.2}$$

$$s_2 h_i = h,$$
$$h_j s_1 = h.$$

Suppose that the coefficients h_i and h_j can be taken from the group G so that the system of equations always has a solution, but for the sake of accuracy, we set $d_{s_1}^{h_i} = 0$ if $h_i \notin S$ and $d_{s_2}^{h_j} = 0$ if $h_j \notin S$.

Define the map $\chi_d : \mathbf{Hom}(C) \to \mathbb{C}$ as follows:

$$\chi_d(h, g) = d_g^h. \tag{2.3}$$

Proposition 2.3. *The map χ_d is a character.*

Proof. The maps (h_i, s_1) and (h_j, s_2) such that $s_2 h_i = h_j s_1 = h$ are composable in category C and $(h_j, s_2) \circ (h_i, s_1) = (h, s_1 s_2)$. In accordance with formulas (2.2) and (2.3) one obtains that $\chi(h_i, s_1) + \chi(h_j, s_2) = \chi(h, s_1 s_2)$ □

We say that the character χ defines a derivation, if there exists a derivation d, such that $\chi(h, g) = d_g^h$. Next we formulate a criterion for the character χ to define a derivation.

Theorem 2.1. *The character χ forms a derivation if and only if for any semigroup element $v \in S$ the character $\chi(x, v) = 0$ for all $x \in G$, and also $\chi(u, v) = 0$, if $u \notin S$.*

Proof. Proof is the same as for group algebras in [3]. □

2.4 Character algebra

The linear space $\mathrm{Der}(S)$ is a Lie algebra with the commutator ([10], p. 206)

$$[d_1, d_2] = d_1 d_2 - d_2 d_1.$$

Proposition 2.4. *The values of the 1-character $\chi_{[d_1,d_2]} = \{\chi_{d_1}, \chi_{d_2}\}$ are defined by χ_{d_1} and χ_{d_2} as follows*

$$\{\chi_{d_1}, \chi_{d_2}\}(a, g) = \sum_{h \in G} \chi_{d_1}(a, h)\chi_{d_2}(h, g) - \chi_{d_2}(a, h)\chi_{d_1}(h, g). \tag{2.4}$$

Proof. Let $g \in G$, then

$$\begin{aligned} d_1(g) &= \sum_{h \in G} \chi_{d_1}(h, g)h, \\ d_2(g) &= \sum_{h \in G} \chi_{d_2}(h, g)h, \\ [d_1, d_2](g) &= \sum_{h \in G} \{\chi_{d_1}, \chi_{d_2}\}(a, g)a. \end{aligned}$$

Represent the expression for the commutator by definition:

$$\begin{aligned} [d_1, d_2] &= d_1 d_2 - d_2 d_1, \\ d_1 d_2(g) &= \sum_{h \in G} \chi_{d_2}(h, g)(\sum_{a \in G} \chi_{d_1}(a, h)a), \\ d_2 d_1(g) &= \sum_{h \in G} \chi_{d_1}(h, g)(\sum_{a \in G} \chi_{d_2}(a, h)a). \end{aligned}$$

Change the sum order in the last expressions:

$$[d_1, d_2](h) = \sum_{a \in G}(\sum_{h \in G} \chi_{d_2}(h, g)\chi_{d_1}(a, h) - \chi_{d_1}(h, g)\chi_{d_2}(a, h))a.$$

The formula for $\{\chi_{d_1}, \chi_{d_2}\}(a, h)$ is a coefficient of a, i.e.

$$\{\chi_{d_1}, \chi_{d_2}\}(a, g) = \sum_{h \in G} \chi_{d_1}(a, h)\chi_{d_2}(h, g) - \chi_{d_2}(a, h)\chi_{d_1}(h, g).$$

□

2.5 Quasi-inner derivations

Here is an example of a character that defines an inner derivation.

Consider an element $a \in G$. Let $\chi^a : \mathbf{Hom}(\Gamma) \to \mathbb{C}$ be a map defined as follows. If $b \neq a$ is an element of the group G, then for every map $\phi \in \mathbf{Hom}(a, b)$ with the source a and the target b, let $\chi^a(\phi) = 1$, for every map $\psi \in \mathbf{Hom}(b, a)$ let $\chi^a(\psi) = -1$. For all remaining maps let χ^a equals to zero.

Proposition 2.5. *The map χ^a is the character defined by the inner derivation*

$$d_a : x \to [x, a],\ a \in S. \tag{2.5}$$

Proof. Make sure that χ^a is a character. Consider two maps $\varphi \in \mathbf{Hom}(a, b)$, $\psi \in \mathbf{Hom}(b, c)$, then the composition $\varphi \circ \psi \in \mathbf{Hom}(a, c)$. Due to the definition of the χ^a one obtains that $\chi(\varphi) + \chi(\psi) = 1 + 0 = \chi(\varphi \circ \psi)$. All other cases can be considered in the same way.

To make sure that χ^a is defined by the given derivation one can obtain $d_a(s) = sa - as$, $\lambda^s a_s = 1$, $\lambda^a s_s = -1$ in accordance to the formula (2.5). The map $(as, s) \in \mathbf{Hom}(s^{-1}as, a)$ and $(sa, s) \in \mathbf{Hom}(a, s^{-1}as)$, whence we obtain the required statement. ◻

Note, that every element $u \in \mathbb{C}[S]$ can be represented as $u = \sum_{s \in S} \lambda^s s$, whence it follows that character χ^u, which defines the inner derivation d_u, can be represented as $\chi^u = \sum_{s \in S} \lambda^s \chi^s$, where χ^s defines the inner derivation d_s.

This example motivates the definition of the space of quasi-inner derivations.

Definition 2. A map $p : Obj(C) \to \mathbb{C}$ is called a potential on the category C.

Definition 3. A character χ, such that there exists a potential p and character's value of the map $\varphi : a \to b$ equals to $\chi(\varphi) = p(b) - p(a)$, is called a potential character.

Definition 4. A derivation d which defines a potential character is called a quasi-inner derivation. In the other words a space of the quasi-inner derivations defines as follows:

$$\mathrm{Der}^*_{Inn}(S) = \{d \in \mathrm{Der}(S) \,|\, \chi_d \text{ is potential } \}.$$

Every potential derivation satisfies two following statements:

Proposition 2.6.

1. *Potential derivation χ is trivial on loops, $\forall a \in Obj(C)$ and $\forall \varphi \in \boldsymbol{Hom}(a, a)$, $\chi(\varphi) = 0$.*

2. *It is constant on $\boldsymbol{Hom}(a, a)$: $\forall \varphi, \psi \in \boldsymbol{Hom}(a, a)$, $\chi(\varphi) = \chi(\psi)$.*

Proof. The proof is straightforward. ◻

It is worth to mention here the fact that two different potentials p_1 and p_2 define same derivation if and only if $p_1 - p_2 = const$.

Theorem 2.2. *The space of all quasi-inner derivations $\mathrm{Der}^*_{Inn}(S)$ forms an ideal in the derivation algebra $\mathrm{Der}(S)$:*

$$d_0 \in \mathrm{Der}^*_{Inn}(S), d \in \mathrm{Der}(S) \Rightarrow [d_0, d], [d, d_0] \in D^*_{Inn}.$$

Proof. Let us prove that $\mathrm{Der}^*_{Inn}(S) \subset \mathrm{Der}(S)$ is a subalgebra, i.e.

$$d_1, d_2 \in \mathrm{Der}^*_{Inn}(S) \Rightarrow [d_1, d_2] \in \mathrm{Der}^*_{Inn}(S).$$

A 1-character χ_{d_1} can be represented as:

$$\chi_{d_1} = \sum_{a \in G} \lambda^a \chi^a,\ \lambda^a \in \mathbb{C},$$

since χ_{d_1} is trivial on loops, where χ^a is defined by formula (14).

Similarly with a 1-character χ_{d_2}:

$$\chi_{d_2} = \sum_{b \in G} \mu^b \chi^b,\ \mu^b \in \mathbb{C}.$$

By the bilinearity of the commutator,

$$\{\chi_{d_1}, \chi_{d_2}\} = \sum_{a \in G} \sum_{b \in G} \lambda^a \mu^b \{\chi^a, \chi^b\}.$$

The commutator of the d^a Pë d^b can be represented as follows:

$$[d^a, d^b] = d^{ab} - d^{ba}.$$

Then we define the 1-character $\{\chi^a, \chi^b\}$ by the following formula

$$\{\chi^a, \chi^b\} = \chi^{ab} - \chi^{ba}$$

and obtain the final expression for $\{\chi_{d_1}, \chi_{d_2}\}$:

$$\{\chi_{d_1}, \chi_{d_2}\} = \sum_{a \in G} \sum_{b \in G} \lambda^a \mu^b \chi^{ab} - \sum_{a \in G} \sum_{b \in G} \lambda^a \mu^b \chi^{ba}.$$

Note that$\{\chi_{d_1}, \chi_{d_2}\} \in Der^*_{Inn}$. To prove this consider the value of the character on loops $(uz, z), z \in Z_G(u)$:

$$\{\chi_{d_1}, \chi_{d_2}\}(uz, z) = \sum_{ab = zuz^{-1}} \lambda^a \mu^b - \sum_{ab = u} \lambda^a \mu^b - \sum_{ba = u} \lambda^a \mu^b + \sum_{ba = zuz^{-1}} \lambda^a \mu^b = 0.$$

Now we pass directly to the proof of the theorem. Represent χ_{d_0} as

$$\chi_{d_0} = \sum_{a \in G} \lambda^a \chi^a.$$

Let us prove the statement of the theorem for χ^a and extend the result to the χ_{d_0}, using the bilinearity of the commutator. Consider the 1-character $\{\chi_d, \chi^a\}$. Let us prove that it is trivial on loops, i.e. $\forall b \in G$ and for $\forall z \in Z_G(b)$ we have $\{\chi_d, \chi^a\}(bz, z) = 0$. By formula (2.4):

$$\{\chi_d, \chi^a\}(bz, z) = \sum_{h \in G} \chi_d(bz, h)\chi^a(h, z) - \chi^a(bz, h)\chi_d(h, z).$$

Note that if $bz \notin S$, then by definition $\{\chi_d, \chi^a\}(bz, z) = 0$. If $bza^{-1} \notin S$, then the map $(bz, bza^{-1}) \notin \mathbf{Hom}(C)$, and $\chi_d(bza^{-1}, z) = 0$ by Theorem 2.1. If $a^{-1}bz \notin S$, then the map $(bz, a^{-1}bz) \notin \mathbf{Hom}(C)$, and $\chi_d(a^{-1}bz, z) = 0$. That means that all of the following statements remain true. Therefore, without loss of generality, we state that all of the above elements lie in the semigroup.

$\chi^a(h, z) \neq 0$ only in 2 cases: when either $h = za$ or $h = az$. $\chi^a(bz, h) \neq 0$ only in 2 cases: when either $h = bza^{-1}$ or $h = a^{-1}bz$. It means that

$$\{\chi_d, \chi^a\}(bz, z) = \chi_d(bz, za) - \chi_d(bz, az) + \chi_d(a^{-1}bz, z) - \chi_d(bza^{-1}, z).$$

At the same time

$$(a^{-1}bz, z) \circ (bz, za) = (bz, az) \circ (bza^{-1}, z).$$

That means

$$\chi_d(bz, za) + \chi_d(a^{-1}bz, z) = \chi_d(bz, az) + \chi_d(bza^{-1}, z)$$
$$\{\chi_d, \chi^a\}(bz, z) = 0.$$

□

The last statement leads us to the following definition.

Definition 5. An algebra $\mathrm{Der}^*_{Out}(S) = \mathrm{Der}(S)/\mathrm{Der}^*_{Inn}(S)$ is called an algebra of an quasi-outer derivations of the semigroup algebra $\mathbb{C}[S]$.

2.6 Distinction between group and semigroup derivations

Let the semigroup S satisfy the Maltsev conditions and let a group G be generated by the semigroup S. Let us show that the derivation algebra $\mathrm{Der}(S)$ of the semigroup algebra $\mathbb{C}[S]$ can be embedded in the derivation algebra $\mathrm{Der}(G)$ of the group algebra $\mathbb{C}[G]$.

Consider a group element $g \in G$. Since the group G is generated by the semigroup S, for each element $g \in G$ there is exists a finite set of elements $\{s_i\} \subset S$ such that $g = s_1^{i_1} \dots s_n^{i_n}$, where $i_k = \pm 1$.

Proposition 2.7. *A map $\lambda : \mathrm{Der}(S) \to \mathrm{Der}(G)$, such that*

$$\lambda(D)(s) = D(s),\ \lambda(D)(s^{-1}) = s^{-1}D(s)s^{-1},\ \forall s \in S,$$
$$\lambda(D)(g) = \sum_{k=1}^{n} s_1^{i_1} \dots \lambda(D)(s_k^{i_k}) \dots s_n^{i_n},\ \forall g \in G,$$

is an epimorphism.

Proof. The equation for $\lambda(D)(g)$ is obtained by applying the Leibniz rule several times. Due to the fact that $\lambda(D)(s) = D(s)$, we obtain that if $D_1 \neq D_2$ then $\lambda(D_1) \neq \lambda(D_2)$. □

The last proposition lead us to the theorem:

Theorem 2.3. *Consider the map λ from the last proposition. Then*

$$\lambda(\mathrm{Der}(S)) \subset \mathrm{Der}(G).$$

We can rewrite the last statement in the terms of characters. Consider a category C, associated with semigroup S, a groupoig Γ, associated with the group G, and the corresponding character algebras $X(C)$ and $X(\Gamma)$. It is easy to show, that the category C is a subcategory in Γ.

Proposition 2.8. *A map $\lambda^* : X(C) \to X(\Gamma)$, such that*

$$\lambda^*(\chi)(g, s) = \chi(g, s)\,\forall g \in G, \forall s \in S,$$

$$\lambda^*(\chi)(g, s^{-1}) = -\chi(sgs, s)\,\forall g \in G, \forall s \in S,$$

and for such morphisms (g_i, s_i) that

$$(g_1, s_1^{i_1}) \circ \dots \circ (g_n, s_n^{i_n}) = (g, s_1^{i_1} \dots s_n^{i_n}),\ i_k = \pm 1, \tag{2.6}$$

the following condition is satisfied

$$\lambda^*(\chi)(g, s_1^{i_1} \dots s_n^{i_n}) = \sum_{k=1}^{n} \lambda^*(\chi)(g_k, s_k^{i_k}), \tag{2.7}$$

then λ^* is an epimorphism.

Proof. The fact that $\lambda^*(\chi) \in X(\Gamma)$ takes place because of formula (2.7). Now we prove that λ^* is an epimorphism. Note that the equation $\lambda^*(\chi)(g,s) = \chi(g,s)$ is satisfied for all maps in $\mathbf{Hom}(C)$. That means that if $\chi_1 \neq \chi_2$, namely if there exists $(g,s) \in \mathbf{Hom}(C)$ such that $\chi_1(g,s) \neq \chi_2(g,s)$, then $\lambda^*(\chi_1) \neq \lambda^*(\chi_2)$. □

As a conclusion we have the following theorem:

Theorem 2.4. *The map $\lambda^* = X(C) \to X(\Gamma)$ makes the following diagram commutative:*

$$\begin{array}{ccc} X(C) & \xrightarrow{\lambda^*} & X(\Gamma) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathrm{Der}(S) & \xrightarrow{\lambda} & \mathrm{Der}(G) \end{array}$$

Proof. The statement can be obtained by applying formulas (2.1) and (2.3) to Proposition 2.8. □

Let us prove some properties of the given embedding using the construction of λ^*.

Corollary 2.1. *The map λ maps an inner derivation to an inner derivation and an outer derivation to an outer one. In other words*

1. $\lambda^*(Der^*_{Inn}(S)) \subset Der^*_{Inn}(G)$,
2. $\lambda^*(Der^*_{Out}(S)) \subset Der^*_{Out}(G)$.

Proof. 1. Consider $\chi \in Der^*_{Inn}(S)$, so χ is potential. Let a map $(uz, z) \in \Gamma$ be a loop. Using the construction of the λ^* one obtains

$$\lambda^*(\chi)(g_1, g_2) = \sum_{k=1}^{n} (-1)^{i_k} \chi(u_k, s_k) = \sum_{k=1}^{n} (-1)^{i_k} p_k = 0,$$

where each p_k is the corresponding potential. Since the sum is calculated on a closed path in Γ, it equals to zero. In other words the character $\lambda^*(\chi)$ is trivial on loops and $\lambda^*(\chi) \in Der^*_{Inn}(G)$.

2. Since $\chi \in Der^*_{Out}(S)$, there exist two maps $\varphi, \psi \in \mathbf{Hom}(a,b)$, such that $\chi(\varphi) \neq \chi(\varphi)$. That means there is a loop $\alpha \in \mathbf{Hom}(a,a)$ in Γ, such that $\alpha \circ \psi = \varphi$. Using the construction of the λ^* one obtains that $\lambda^*(\alpha) = \chi(\varphi) - \chi(\psi) \neq 0$. That means $\lambda^*(\chi) \notin Der^*_{Inn}(G)$.

□

Example

Let a semigroup S generate a nilpotent group of rank 2 (example: the Heisenberg discrete group). The derivation algebra of such a group has been studied in [4]. It has been shown that every derivation in $Der(G)$ is quasi-inner or central. We begin with some important definitions.

Definition 6. The subgroup $Z = \{z \in G \mid zg = gz \, \forall g \in G\}$ is called the center of the group G.

It is easy to show that Z is a normal subgroup.

Definition 7. A derivation $d \in Der(G)$ is called central if there exists a homomorphism $\tau : G \to \mathbb{C}$ and an element $z \in Z$, such that

$$d(g) = \tau(g) g z.$$

Proposition 2.9. *A central derivation $d \in Der(G)$ is not quasi-inner.*

Proof. It can be shown that a corresponding character χ_d is not trivial on loops. Consider a central group element $z \in G$. Its conjugated class $[z]$ consists of only one element. Thus a character χ_d can be represented as follows ([4]):

$$\chi_d(gz, g) = \tau(g),\ \forall g \in G.$$

Since (gz, g) is a loop and $\tau(g) \neq 0$, one obtains that χ_d is not quasi-inner. □

If a semigroup S generates the group G we can describe a derivation algebra $Der(S)$, using the embedding given in Corollary 2.1.

Theorem 2.5. *If a semigroup S generates a nilpotent group of rank 2, then every derivation $d \in \mathrm{Der}(S)$ is either quasi-inner or central.*

Proof. The proof immediately follows by Corollary 2.1. □

As an example, we consider the Heisenberg discrete group G and a semigroup G^+ of matrices with positive elements. Such group in nilpotent of rank 2 and every derivation is either quasi-inner or central. We introduce a general form of the central derivation in the semigroup G^+. It can be written as follows:

$$d_j^{\mu,\nu} : \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto (\mu a + \nu c) \begin{pmatrix} 1 & a & b+j \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

$$a, b, c, j \geq 0$$

where the central element $z \in Z$ is equal to

$$\begin{pmatrix} 1 & 0 & j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the group homomorpism $\tau : G \to \mathbb{C}$ is

$$\tau_{\mu,\nu} : \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto \mu a + \nu c$$

All required calculations can be found in [4].

Acknowledgments

The authors thank Professor A.S. Mischenko for his attention to these results.

The second author was supported the Grants Council of the President of the Russian Federation, project no. MK-2364.2020.1. The results in Section 2.5 were obtained by the second author under the support of the Ministry of Science and Higher Education of the Russian Federation (Goszadaniye 075-00337-20-03, project no. 0714-2020-0005).

The results in Section 2.2 were obtained by the first author with the support of the Russian Science Foundation (grant no. 20-11-20131) in V.A. Trapeznikov Institute of Control Sciences of RAS).

References

[1] A.A. Arutyunov, A.S. Mishchenko, A.I. Shtern, *Derivations of group algebras.* Fundam. Prikl. Mat. 21 (2016), no. 6, 65–78.

[2] A.A. Arutyunov, A.S. Mishchenko, *A smooth version of Johnson's problem on derivations of group algebras.* Mat. Sb. 210 (2019), no. 6, 3–29; Sb. Math., 210 (2019), no. 6, 756–782.

[3] A.A. Arutyunov, A.V. Alekseev, *Complex of n-categories and derivations in group algebras.* Topology and its Applications, Volume 275, 2020, 107002, doi.org/10.1016/j.topol.2019.107002.

[4] A.A. Arutyunov, *Derivation algebra in noncommutative group algebras.* Differential equations and dynamical systems, Collected papers, Tr. Mat. Inst. Steklova 308 (2020), 28–41 (in Russian); Engl. translation Proc. Steklov Inst. Math. 308 (2020), 22–34.

[5] D.E. Bagha, M. Amini, *Module derivations on semigroup algebras.* Semigroup Forum 94 (2017), 176-180 doi:10.1007/s00233-015-9742-0.

[6] J. Benabou, *Introduction to bicategories.* Reports of the Midwest Category Seminar, Springer, Berlin, 1967, 1–77.

[7] C. Ehresmann, *Categories et structures.* Dunod, Paris, 1965.

[8] N.D. Gilbert, *Derivations and relation modules for inverse semigroups.* Algebra Discrete Math. 12 (2011), no. 1, 1вЂ“19.

[9] A. Hirschowitz, C. Simpson, *Descente pour les n-champs.* arXiv:math/9807049, 1998.

[10] N. Jacobson, *Abstract derivations and Lie algebras.* American Mathematical Society, DOI: 10.2307/1989656, 1937.

[11] M. Lashkarizadeh Bami, B. Mohammadzadeh, H. Samea, *Derivations on certain semigroup algebra.* Journal of Sciences, Islamic Republic of Iran, University of Tehran 18 (2007), no. 4, 339-345.

[12] V. Losert, *The derivation problem for group algebras.* Ann. of Math. (2) 168 (2008), no. 1, 221-246.

[13] S. Mac Lane, *Categories for the working mathematician.* Springer-Verlag New York, ISBN 0-387-90036-5.

[14] A.I. Maltsev, *On the inclusion of associative systems in groups. II.* Sbornik: Mathematics 6 (1939), no. 48, 331-36 (in Russian).

[15] A.I. Maltsev, *On the inclusion of associative systems in groups.* Sbornik: Mathematics 8 (1940), no. 50, 251-64 (in Russian).

[16] C. Simpson, *Homotopy theory of higher categories.* ISBN 978-0-521-51695-2.

Alekseev Aleksandr Vladimirovich
V.A. Trapeznikov Institute of Control Sciences,
Russian Academy of Sciences
65 Profsoyuznaya St
Moscow, Russian Federation
E-mail: alekandr.alekseev@frtk.ru

Arutyunov Andronick Aramovich
Department of Mathematics
Moscow Institute of Physics and Technology
9 Institutskiy Pereulok, Dolgoprudny
Moscow region, Russian Federation
E-mail: andronick.arutyunov@gmail.com

Received: 10.03.2019