ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2020, Volume 11, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

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- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

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- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.

VLADIMIR DMITRIEVICH STEPANOV

(to the 70th birthday)

Vladimir Dmitrievich Stepanov was born on December 13, 1949 in a small town Belovo, Kemerovo region. In 1966 he finished the Lavrentiev school of physics and mathematics at Novosibirsk academic town-ship and the same year he entered the Faculty of Mathematics of the Novosibirsk State University (NSU) from which he has graduated in 1971 and started to teach mathematics at the Khabarovsk Technical University till 1981 with interruption for postgraduate studies (1973-1976) in the NSU.

In 1977 he has defended the PhD dissertation and in 1985 his doctoral thesis "Integral convolution operators in Lebesgue spaces" in the S.L. Sobolev Institute of Mathematics. Scientific degree "Professor of Mathematics" was awarded to him in 1989. In 2000 V.D. Stepanov was elected a corresponding member of the Russian Academy of Sciences (RAS).

Since 1985 till 2005 V.D. Stepanov was the Head of Laboratory of Functional Analysis at the Computing Center of the Far Easten Branch of the Russian Academy of Science.

In 2005 V.D. Stepanov moved from Khabarovsk to Moscow with appointment at the Peoples Friendship University of Russia as the Head of the Department of Mathematical Analysis (retired in 2018). Also, he was hired at the V.A. Steklov Mathematical Institute of RAS at the Function Theory Department.

Research interests of V.D. Stepanov are: the theory of integral and differential operators, harmonic analysis in Euclidean spaces, weighted inequalities, duality in function spaces, approximation theory, asymptotic estimates of singular, approximation and entropy numbers of integral transformations, and estimates of the Schatten-Neumann type. Main achievements: the theory of integral convolution operators is constructed, the criteria for the boundedness and compactness of integral operators in function spaces are obtained, weighted inequalities and the behaviour of approximation numbers of the Volterra, Riemann-Liouville, Hardy integral operators are studied, etc.

Under his scientific supervision 15 candidate theses in Russia and 5 PhD theses in Sweden were successfully defended. Professor V.D. Stepanov has over 100 scientific publications including 3 monographs. Participation in scientific and organizational activities of V.D. Stepanov is well known. He is a member of the American Mathematical Society (since 1987) and a member of the London Mathematical Society (since 1996), Deputy Editor of the Analysis Mathematica, member of the Editorial Board of the Eurasian Mathematical Journal, invited speaker at many international conferences and visiting professor of universities in USA, Canada, UK, Spain, Sweden, South Korea, Kazakhstan, etc.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Vladimir Dmitrievich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF ANALYSIS, DIFFERENTIAL EQUATIONS AND ALGEBRA" (EMJ-2019), DEDICATED TO THE 10TH ANNIVERSARY OF THE EURASIAN MATHEMATICAL JOURNAL

From October 16 to October 19, 2019 at the L.N. Gumilyov Eurasian National University (ENU) the International Conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2019) was held. The conference was dedicated to the 10th anniversary of the Eurasian Mathematical Journal (EMJ).

The purposes of the conference were to discuss the current state of development of mathematical scientific directions, expand the number of potential authors of the Eurasian Mathematical Journal and further strengthen the scientific cooperation between the Faculty of Mechanics and Mathematics of the ENU and scientists from other cities of Kazakhstan and abroad.

The partner universities for the organization of the conference were the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia (the RUDN University, Moscow) and the University of Padua (Italy).

The conference was attended by more than 80 mathematicians from the cities of Almaty, Aktobe, Karaganda, Nur-Sultan, Shymkent, Taraz, Turkestan, as well as from several foreign countries: from Azerbaijan, Germany, Greece, Italy, Japan, Kyrgyzstan, Russia, Tajikistan and Uzbekistan.

The chairman of the International Programme Committee of the conference was Ye.B. Sydykov, rector of the ENU, co-chairmen were Chief editors of the EMJ: V.I. Burenkov, professor of the RUDN University, M. Otelbaev, academician of the National Academy of Sciences of the Republic of Kazakhstan (NAS RK), V.A. Sadovnichy, academician of the Russian Academy of Sciences (RAS), rector of the M.V. Lomonosov Moscow State University (MSU).

There were three sections at the conference: "Function Theory and Functional Analysis", "Differential Equations and Equations of Mathematical Physics" and "Algebra and Model Theory". 16 plenary presentations of 30 minutes each and more than 60 sectional presentations of 20 minutes each, devoted to contemporary areas of mathematics, were given.

It was decided to recommend selected reports of the participants for publication in the Eurasian Mathematical Journal and the Bulletin of the Karaganda State University (series "Mathematics").

Before the conference, a collection of abstracts of the participants' talks was published.

PROGRAMME OF THE INTERNATIONAL CONFERENCE EMJ-2019

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Conference Schedule:

16.10.2019
09.00 – 10.00 Registration
10.00 – 10.30 Opening of the conference
10.30 – 12.50 Plenary talks
12.50 – 14.00 Lunch
14.00 – 18.00 Session talks

17.10.2019
09.30 – 12.20 Plenary talks
12.20 – 14.00 Lunch
14.00 – 18.00 Session talks
18.00 – Dinner for participants of the conference

18.10.2019
09.30 – 13.00 Plenary talks
12.20 – 14.00 Lunch
14.00 – 17.00 Excursion around the city

19.10.2019
09.30 – 12.30 Plenary talks
12.30 – 13.00 Closing of the conference

At the opening ceremony welcome speeches were given by Ye.B. Sydykov, rector of the ENU, chairman of the Program Committee of the conference; V.I. Burenkov, professor of the RUDN University, editor-in-chief of the EMJ; L. Mukasheva, official representative of the international company Clarivate Analytics in the Central Asian region; A. Ospanova, official representative of Scopus.

Plenary talks were given by

T.Sh. Kalmenov (Kazakhstan), M. Otelbaev and B.D. Koshanov (Kazakhstan), P.D. Lamberti and V. Vespri (Italy) – on 16.10.2019;

V.I. Burenkov (Russia), T. Ozawa (Japan), H. Begehr (Germany), M.A. Sadybekov and A.A. Dukenbaeva (Kazakhstan), D. Suragan (Kazakhstan) – on 17.10.2019;

M.L. Goldman (Russia), A. Bountis (Greece), A.K. Kerimbekov (Kyrgyzstan), S.N. Kharin (Kazakhstan), M.I. Dyachenko (Russia) – on 18.10.2019;

E.D. Nursultanov (Kazakhstan), M.A. Ragusa (Italy), P.D. Lamberti and V. Vespri (Italy), M.G. Gadoev (Russia) and F.S. Iskhokov (Tajikistan) – on 19.10.2019.

At the closing ceremony all participants unanimously congratulated the staff of the L.N. Gumilyov Eurasian National University and the Editorial Board of the Eurasian Mathematical Journal with the 10th anniversary of the journal and wished further creative successes.

They expressed hope that the journal will continue to play an important role in the development of mathematical science and education in Kazakhstan in the future.

V.I. Burenkov, K.N. Ospanov, A.M. Temirkhanova

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 11, Number 1 (2020), 86 – 94

AN ESTIMATE OF APPROXIMATION OF A MATRIX-VALUED FUNCTION BY AN INTERPOLATION POLYNOMIAL

V.G. Kurbatov, I.V. Kurbatova

Communicated by T.V. Tararykova

Key words: matrix function, polynomial interpolation, estimate.

AMS Mathematics Subject Classification: 65F60, 97N50.

Abstract. Let A be a square complex matrix; $z_1, \dots, z_n \in \mathbb{C}$ be (possibly repetitive) points of interpolation; f be a function analytic in a neighborhood of the convex hull of the union of the spectrum of A and the points $z_1, \dots, z_n$; and p be the interpolation polynomial of f constructed by the points $z_1, \dots, z_n$. It is proved that under these assumptions

$$\|f(A) - p(A)\| \leq \frac{1}{n!} \max_{\substack{t\in[0,1] \\ \mu\in\operatorname{co}\{z_1,z_2,\dots,z_n\}}} \big\|\Omega(A) f^{(n)}\big((1-t)\mu\mathbf{1} + tA\big)\big\|,$$

where $\Omega(z) = \prod_{k=1}^{n}(z - z_k)$ and the symbol co means the convex hull.

DOI: https://doi.org/10.32523/2077-9879-2020-11-1-86-94

1 Introduction

An approximate calculation of analytic functions of matrices [8, 16] arises in many applications. One of the often used methods for the approximate calculation of a function f of a large matrix A is the replacement of f by its polynomial approximation p. For the approximation by the Taylor polynomial, a good estimate of accuracy is known [23], see Corollary 2.1; this estimate is used by many authors, see, e.g., [2, 4, 16, 22, 28]. In this paper, we propose an estimate of the norm $\|f(A) - p(A)\|$, which is a generalization to the case when p is an interpolation polynomial of f, of the estimate established in [23]. This estimate may help to choose an interpolation polynomial for an approximate calculation of a matrix function in an optimal way. Our estimate can be considered as a matrix analogue of the estimate [5, Theorem 3.1.1]

$$|f(z) - p(z)| \leq |\Omega(z)| \max_{\lambda\in\operatorname{co}\{z_1,z_2,\dots,z_n,\, z\}} \left|\frac{f^{(n)}(\lambda)}{n!}\right|,$$

where

$$\Omega(z) = \prod_{k=1}^{n}(z - z_k),$$

for the difference between an analytic function f and its interpolation polynomial p with respect to the points of interpolation $z_1, \dots, z_n$ provided that f is analytic in a neighborhood of the convex hull of the points $z_1, \dots, z_n$ and z.

Many estimates of $\|f(A)\|$ are known, see, e.g., [3, 11, 12, 13, 15, 21, 25, 26, 27]. All of them can be equivalently written as estimates of $\|f(A)-p(A)\|$, see, e.g., [14, Theorem 11.2.2]. The difference between these estimates and the proposed one (Theorem 2.1) is that the latter is adapted for the approximation by an interpolation polynomial.

We also note that the consideratrion of $f(A)-p(A)$, where p is an interpolation polynomial, is important [6, 7, 18, 19, 24] for so-called restarted Krylov subspace methods.

The paper is organized as follows. In Section 2, we prove our estimate (Theorem 2.1) and describe some of its variants for special cases. In Section 3, we give a numerical application.

2 The estimate

Theorem 2.1. *Let A be a square complex matrix, $z_1, \dots, z_n \in \mathbb{C}$ be arbitrary (possibly repetitive) points of interpolation, f be an analytic function defined in a neighborhood of the convex hull of the union of the spectrum of A and the points $z_1, \dots, z_n$, and p be the interpolation polynomial of f constructed by the points $z_1, \dots, z_n$ (taking into account their multiplicities). Then (for any norm on the space of matrices)*

$$\|f(A)-p(A)\| \le \frac{1}{n!} \max_{\substack{t\in[0,1]\\ \mu\in\operatorname{co}\{z_1,z_2,\dots,z_n\}}} \big\|\Omega(A) f^{(n)}\big((1-t)\mu\mathbf{1}+tA\big)\big\|,$$

where $\mathbf{1}$ is the identity matrix, the symbol co *means the convex hull, and*

$$\Omega(z)=\prod_{k=1}^{n}(z-z_k).$$

Proof. It is well-known (see, e.g., [5, Theorem 3.4.1] or [10, formula (52)]) that

$$f(z)-p(z)=\Omega(z) f[z_1,z_2,\dots,z_n,z],$$

where $f[z_1,z_2,\dots,z_n,z]$ is the divided difference [5, 10, 20]. On the other hand, by [10, formula (47)], we have

$$f[z_1,z_2,\dots,z_n,z]=\int_0^1\int_0^{t_1}\dots\int_0^{t_{n-1}} f^{(n)}\big(z_1+(z_2-z_1)t_1+\dots \\ +(z_n-z_{n-1})t_{n-1}+(z-z_n)t_n\big)\,dt_n dt_{n-1}\dots dt_1, \quad (2.1)$$

or

$$f[z_1,z_2,\dots,z_n,z]=\int_0^1\int_0^{t_1}\dots\int_0^{t_{n-1}} f^{(n)}\big((1-t_1)z_1+(t_1-t_2)z_2+\dots \\ +(t_{n-1}-t_n)z_n+t_n z\big)\,dt_n dt_{n-1}\dots dt_1.$$

Clearly, the complex numbers

$$(1-t_1)z_1+(t_1-t_2)z_2+\dots+(t_{n-1}-t_n)z_n+t_n z$$

form the convex hull of the set $\{z_1,z_2,\dots,z_n,z\}$ when $t_1,\dots,t_n$ run through the set specified by the inequalities $0\le t_n\le\dots\le t_1\le 1$. Thus, considering integral (2.1), we use the fact that $f^{(n)}$ is defined on the convex hull of the points $z_1, \dots, z_n$, and z.

Substituting A for z into the previous formulas (thus, we assume that any point of the spectrum of A can be taken as z, which can be done, since f is analytic in a neighborhood of the convex hull of the union of the spectrum of A and the points $z_1, \dots, z_n$), we obtain

$$f(A)-p(A)=\Omega(A)\int_0^1\int_0^{t_1}\dots\int_0^{t_{n-1}} f^{(n)}\big((1-t_1)z_1\mathbf{1}+(t_1-t_2)z_2\mathbf{1}+\dots \\ +(t_{n-1}-t_n)z_n\mathbf{1}+t_nA\big)\,dt_n dt_{n-1}\dots dt_1.$$

Let ξ be a linear functional on the space of matrices (equipped by an arbitrary norm) such that $\|\xi\|=1$ and

$$\|f(A)-p(A)\|=\xi\big(f(A)-p(A)\big).$$

Such a functional exists by the Hahn-Banach theorem [17, Theorem 2.7.4]. Then we have the estimate

$$\begin{aligned}\|f(A)-p(A)\|&=\xi\bigg(\int_0^1\int_0^{t_1}\dots\int_0^{t_{n-1}}\Omega(A)f^{(n)}\big((1-t_1)z_1\mathbf{1}+\dots \\ &\qquad +(t_{n-1}-t_n)z_n\mathbf{1}+t_nA\big)\,dt_n dt_{n-1}\dots dt_1\bigg)\\ &\le\bigg\|\int_0^1\int_0^{t_1}\dots\int_0^{t_{n-1}}\Omega(A)f^{(n)}\big((1-t_1)z_1\mathbf{1}+\dots \\ &\qquad +(t_{n-1}-t_n)z_n\mathbf{1}+t_nA\big)\,dt_n dt_{n-1}\dots dt_1\bigg\| \\ &\le\int_0^1\int_0^{t_1}\dots\int_0^{t_{n-1}}\big\|\Omega(A)f^{(n)}\big((1-t_1)z_1\mathbf{1}+\dots \\ &\qquad +(t_{n-1}-t_n)z_n\mathbf{1}+t_nA\big)\big\|\,dt_n dt_{n-1}\dots dt_1 \\ &\le\int_0^1\int_0^{t_1}\dots\int_0^{t_{n-1}}\max_{0\le t_n\le\dots\le t_1\le 1}\big\|\Omega(A)f^{(n)}\big((1-t_1)z_1\mathbf{1}+\dots \\ &\qquad +(t_{n-1}-t_n)z_n\mathbf{1}+t_nA\big)\big\|\,dt_n dt_{n-1}\dots dt_1.\end{aligned}\tag{2.2}$$

We observe that the complex numbers

$$\mu=\frac{1}{1-t_n}\big((1-t_1)z_1+(t_1-t_2)z_2+\dots+(t_{n-1}-t_n)z_n\big)$$

are contained in the convex hull of $\{z_1,z_2,\dots,z_n\}$ when $t_1,\dots,t_n$ belong to the set specified by the inequalities $0\le t_n\le\dots\le t_1\le 1$. Indeed, for the numbers

$$\xi_1=\frac{1-t_1}{1-t_n}\qquad\text{and}\qquad\xi_k=\frac{t_{k-1}-t_k}{1-t_n}\ \text{if } k=2,\dots,n,$$

we have

$$0\le\xi_k\le 1,\qquad \xi_1+...+\xi_n=1,\qquad \mu=\xi_1z_1+\dots+\xi_nz_n.$$

Thus, we can change the expression $f^{(n)}\big((1-t_1)z_1\mathbf{1}+\dots+(t_{n-1}-t_n)z_n\mathbf{1}+t_nA\big)$ in (2.2) by $f^{(n)}\big((1-t_n)\mu\mathbf{1}+t_nA\big)$. We note that $f^{(n)}\big((1-t_n)\mu\mathbf{1}+t_nA\big)$ is defined because the spectrum of the argument $(1-t_n)\mu\mathbf{1}+t_nA$ is contained in the convex hull of the union of the spectrum of A and the points $z_1, \dots, z_n$. Besides,

$$\int_0^1\int_0^{t_1}\dots\int_0^{t_{n-1}}dt_n\dots dt_1=\frac{1}{n!}.$$

Therefore from estimate (2.2) it follows that

$$\|f(A)-p(A)\| \leq \frac{1}{n!} \max_{\substack{t\in[0,1]\\ \mu\in\operatorname{co}\{z_1,z_2,\dots,z_n\}}} \bigl\|\Omega(A)f^{(n)}\bigl((1-t)\mu\mathbf{1}+tA\bigr)\bigr\|,$$

which was to be proved. □

Remark 1. For numerical calculations, it may be useful to note that the maximum can be taken over the boundary $\partial\operatorname{co}$ of the convex hull instead of the whole convex hull:

$$\max_{\mu\in\operatorname{co}\{z_1,z_2,\dots,z_n\}} \bigl\|\Omega(A)f^{(n)}\bigl((1-t)\mu\mathbf{1}+tA\bigr)\bigr\| = \max_{\mu\in\partial\operatorname{co}\{z_1,z_2,\dots,z_n\}} \bigl\|\Omega(A)f^{(n)}\bigl((1-t)\mu\mathbf{1}+tA\bigr)\bigr\|.$$

Indeed, by the Hahn-Banach theorem,

$$\bigl\|\Omega(A)f^{(n)}\bigl((1-t)\mu\mathbf{1}+tA\bigr)\bigr\| = \max_{\|\xi\|\leq 1} \xi\bigl[\Omega(A)f^{(n)}\bigl((1-t)\mu\mathbf{1}+tA\bigr)\bigr],$$

where the functional ξ runs over the unit ball of the dual space of the space of all matrices. The function

$$\xi\mapsto \xi\bigl[\Omega(A)f^{(n)}\bigl((1-t)\mu\mathbf{1}+tA\bigr)\bigr]$$

is analytic. Therefore, by the maximum modulus principle,

$$\max_{\mu\in\operatorname{co}\{z_1,z_2,\dots,z_n\}} \bigl|\xi\bigl[\Omega(A)f^{(n)}\bigl((1-t)\mu\mathbf{1}+tA\bigr)\bigr]\bigr| = \max_{\mu\in\partial\operatorname{co}\{z_1,z_2,\dots,z_n\}} \xi\bigl[\Omega(A)f^{(n)}\bigl((1-t)\mu\mathbf{1}+tA\bigr)\bigr].$$

Taking maximum over all functionals ξ of the unit norm, we arrive at the desired equality.

Our Theorem 2.1 was inspired by the following result.

Corollary 2.1 ([23, Corollary 2], [16, Theorem 4.8]). *Let the Taylor series*

$$f(\lambda)=\sum_{k=0}^{\infty} c_k(\lambda-z_1)^k,$$

where $c_k\in\mathbb{C}$, converges on an open circle of radius r with the center at z_1, and the spectrum of a square matrix A is contained in this circle. Then

$$\Bigl\|f(A)-\sum_{k=0}^{n-1} c_k(A-z_1\mathbf{1})^k\Bigr\| \leq \frac{1}{n!}\max_{t\in[0,1]}\bigl\|(A-z_1\mathbf{1})^n f^{(n)}\bigl((1-t)z_1\mathbf{1}+tA\bigr)\bigr\|.$$

In corollaries below, we simplify the estimate of Theorem 2.1 in the case of the most important function $f(z)=e^z$.

In the notation of Theorem 2.1, we set

$$\begin{aligned}
\alpha &= \max\{\operatorname{Re}\lambda:\ \lambda\in\sigma(A)\},\\
\beta &= \max_k \operatorname{Re} z_k,\\
\gamma &= \max\{\alpha,\beta\}.
\end{aligned}$$

Corollary 2.2. *Let the assumptions of Theorem 2.1 be satisfied and $f(z)=e^z$. Then*

$$\|e^A-p(A)\| \leq \frac{1}{n!}\max_{t\in[0,1]} e^{(1-t)\beta}\|\Omega(A)e^{tA}\|.$$

Proof. Clearly, $f^{(n)}(z)=e^z$. Therefore

$$f^{(n)}\big((1-t)\mu\mathbf{1}+tA\big)=e^{(1-t)\mu}e^{tA}.$$

It remains to observe that

$$\max_{\mu\in\operatorname{co}\{z_1,z_2,\dots,z_n\}}|e^{(1-t)\mu}|=e^{(1-t)\beta},$$

which completes the proof. □

The following three corollaries are more effective (but rougher) versions of the previous one. We denote by $\|\cdot\|_{2\to2}$ the matrix norm induced by the Euclidian norm on $\mathbb{C}^n$.

Corollary 2.3. *Let the assumptions of Theorem 2.1 be satisfied and $f(z)=e^z$. Then*

$$\|e^A-p(A)\|_{2\to2}\le e^{\gamma}\frac{\|\Omega(A)\|_{2\to2}}{n!}\sum_{j=0}^{n-1}\frac{(2\|A\|_{2\to2})^j}{j!}.$$

where the matrix A has the size $n\times n$.

Proof. From Corollary 2.2 it follows that

$$\|e^A-p(A)\|_{2\to2}\le\frac{\|\Omega(A)\|_{2\to2}}{n!}\max_{t\in[0,1]}e^{(1-t)\beta}\|e^{tA}\|_{2\to2}. \tag{2.3}$$

Next, we make use of the estimate [1, p. 131, Lemma 10.2.1], [9, p. 68, formula (13)]

$$\|e^{At}\|_{2\to2}\le e^{\alpha t}\sum_{j=0}^{n-1}\frac{(2t\|A\|_{2\to2})^j}{j!},\qquad t\ge0,$$

which completes the proof. □

Corollary 2.4. *Let the assumptions of Theorem 2.1 be satisfied and $f(z)=e^z$. Let the matrix A be represented in the triangular Schur form [14] $A=Q^{-1}BQ$, where B is triangular and Q is unitary. Further, let $B=D+N$, where D is diagonal and N is strictly triangular. Then*

$$\|e^A-p(A)\|_{2\to2}\le e^{\gamma}\frac{\|\Omega(A)\|_{2\to2}}{n!}\sum_{j=0}^{n-1}\frac{\|N\|_{2\to2}^j}{j!},$$

where the matrix A has the size $n\times n$.

Proof. The proof is similar to that of Corollary 2.3: it is sufficient to substitute the estimate [26]

$$\|e^{At}\|=\|e^{Bt}\|\le e^{\alpha t}\sum_{j=0}^{n-1}\frac{\|Nt\|^j}{j!},\qquad t\ge0,$$

in (2.3). □

Corollary 2.5. *Let the assumptions of Theorem 2.1 be satisfied and $f(z)=e^z$. Let the matrix A be normal. Then*

$$\|e^A-p(A)\|_{2\to2}\le e^{\gamma}\frac{\max_{\lambda\in\sigma(A)}|\Omega(\lambda)|}{n!}.$$

Proof. For a normal matrix A, we have $N = 0$. Therefore by Corollary 2.4, we have

$$\|e^A - p(A)\|_{2\to 2} \le e^\gamma \frac{\|\Omega(A)\|_{2\to 2}}{n!}.$$

It remains to observe that $\|\Omega(A)\|_{2\to 2} = \max_{\lambda\in\sigma(A)} |\Omega(\lambda)|$ since A is normal. □

Corollary 2.5 can be also obtained from the following more general result.

Corollary 2.6. *Let the assumptions of Theorem 2.1 be satisfied and the matrix A be normal. Then*

$$\|f(A) - p(A)\|_{2\to 2} \le \max_{\lambda\in\sigma(A)} \big|\Omega(\lambda)\big| \max_{\mu\in\operatorname{co}(\{z_1,z_2,\dots,z_n\}\cup\sigma(A))} \Big|\frac{f^{(n)}(\mu)}{n!}\Big|.$$

Proof. The proof follows directly by Theorem 2.1 and the normality of the matrix A. □

The most important question is whether the estimate is close to the actual accuracy of the approximation.

Example 1. Let the points of interpolation $z_1, \dots, z_n$ be taken coinciding with the points of the spectrum of A (counted according to their algebraic multiplicities). Then Ω is the characteristic polynomial of a matrix A. By the Cayley–Hamilton theorem, $\Omega(A) = 0$. Thus, in this case, Theorem 2.1 implies the well-known identity $p(A) = f(A)$. Similarly, if Ω and its derivatives are small on the spectrum of A, the factor $\Omega(A)$ is also small.

Example 2. Let A be a Hermitian matrix with the spectrum lying in $[-1, 1]$. Let the points of interpolation be the zeroes of the Chebyshev polynomial of the first kind [5, § 3.3] of degree n on $[-1, 1]$. In this case, Ω is this Chebyshev polynomial; if its leading coefficient is taken to be 1, then the maximal absolute value of Ω on $[-1, 1]$ is $\frac{1}{2^{n-1}}$. Therefore, Corollary 2.5 implies that

$$\|e^A - p(A)\| \le \frac{1}{n!}\|\Omega(A)\|e^\gamma \le \frac{1}{n!}\frac{e}{2^{n-1}}.$$

If the spectrum of A is not known exactly, the sharp estimate (for this polynomial) is

$$\|e^A - p(A)\| \le \max_{\lambda\in[0,1]} |e^\lambda - p(\lambda)|.$$

We compare these two estimates for $n = 10$: we have $\frac{1}{n!}\frac{e}{2^{n-1}} = 1.46\cdot 10^{-9}$ and $\max_{\lambda\in[0,1]} |e^\lambda - p(\lambda)| = 0.60\cdot 10^{-9}$. The comparison shows that the estimate of Corollary 2.5 is rather close to the sharp one.

3 Numerical experiment

Theorem 2.1 can help to estimate whether the accuracy of the approximation of the matrix function $f(A)$ by a matrix polynomial $p(A)$ is good enough for the given points of interpolation. We describe an example of such a verification based on Corollary 2.2.

We put $N = 1024$. We take complex numbers d_i, $i = 1, \dots, N$, uniformly distributed in $[-1, 0] + [-i\pi, i\pi]$. We consider the diagonal matrix D of the size $N \times N$ with the diagonal entries d_i. We take a matrix T, whose entries are random numbers uniformly distributed in $[-1, 1]$. Then, we consider the matrix $A = TDT^{-1}$. Clearly, $\sigma(A)$ consists of the numbers d_i. We interpret A as a random matrix whose spectrum is contained in the rectangle $[-1, 0] + [-i\pi, i\pi]$. On the left of Fig. 1 we show an example of the spectrum of such a matrix.

For $t \in [0, 1]$, we take as the sharp matrix e^{tA} the matrix $Te^{tD}T^{-1}$.

We take the following 16 points as interpolation points:

$$0, \pm i\pi, \pm i\pi/2, \pm 3i\pi/4; -1, -1 \pm i\pi, -1 \pm i\pi/2, -1 \pm 3i\pi/4; -1/2 \pm i\pi.$$

They are marked on the left of Fig. 1 by the sign $\otimes$. These points are chosen heuristically. We calculate the interpolation polynomial p and the polynomial Ω, which correspond to these points, and substitute the matrix A into them.

Next we calculate $\|\Omega(A)e^{tA}\|_{2\to 2} = \|\Omega(A)Te^{tD}T^{-1}\|_{2\to 2}$ for $t = 0.01k$, where $k = 0, \dots, 100$, and take the maximum of these numbers as an approximate value of $\max_{t\in[0,1]} e^{(1-t)\beta}\|\Omega(A)e^{tA}\|$ (we put $\beta = 0$). Finally, we divide the result by 16! and, thus, obtain the estimate of Corollary 2.2; we denote it by e_1. We also calculate the true accuracy $e_0 = \|e^A - p(A)\|_{2\to 2}$ and the condition number [16, p. 1] $\varkappa(T) = \|T\|_{2\to 2} \cdot \|T^{-1}\|_{2\to 2}$.

We repeated the described experiment 100 times. After that, we excluded 3 results when the condition number $\varkappa(T)$ of the matrix T is greater than 10^6. Finally, we calculated the average values. They are as follows: the mean value of e_0 is $9.36 \cdot 10^{-6}$ with the standard deviation $1.37 \cdot 10^{-5}$, the mean value of e_1 is $2.57 \cdot 10^{-5}$ with the standard deviation $3.33 \cdot 10^{-5}$, the mean value of e_1/e_0 is 3.03 with the standard deviation 1.11, the mean value of $\varkappa(T)$ is $0.86 \cdot 10^5$ with the standard deviation $1.56 \cdot 10^5$.

The mean value 3.03 of e_1/e_0 shows that the estimate is rather close to the true value. So, we can state that for not very bad matrices A of the size $N \times N$ with the spectrum in the rectangle $[-1, 0] + [-i\pi, i\pi]$, the interpolation polynomial with the considered interpolation points usually approaches e^A with accuracy about $3 \cdot 10^{-5}$.

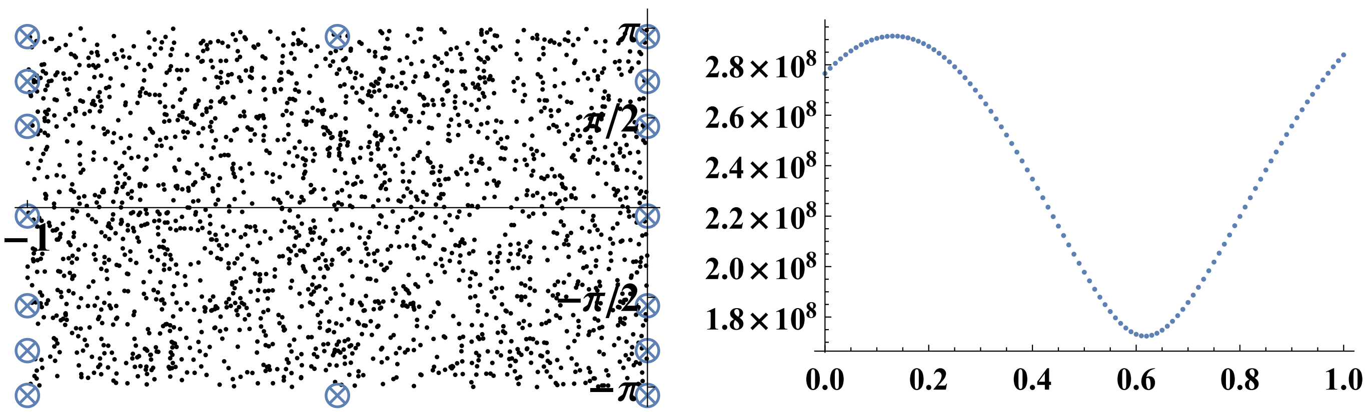

Figure 1: Left: the spectrum of A and the points of interpolation; right: the norms $\|\Omega(A)e^{tA}\|$ for $t = 0.01k$, $k = 0, \dots, 100$

Acknowledgments

The first author was supported by the Ministry of Education and Science of the Russian Federation under state order No. 3.1761.2017/4.6. The second author was supported by the Russian Foundation for Basic Research under research project No. 19-01-00732 A.

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Received: 18.03.2019