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From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

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- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

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At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.

VLADIMIR DMITRIEVICH STEPANOV

(to the 70th birthday)

Vladimir Dmitrievich Stepanov was born on December 13, 1949 in a small town Belovo, Kemerovo region. In 1966 he finished the Lavrentiev school of physics and mathematics at Novosibirsk academic town-ship and the same year he entered the Faculty of Mathematics of the Novosibirsk State University (NSU) from which he has graduated in 1971 and started to teach mathematics at the Khabarovsk Technical University till 1981 with interruption for postgraduate studies (1973-1976) in the NSU.

In 1977 he has defended the PhD dissertation and in 1985 his doctoral thesis "Integral convolution operators in Lebesgue spaces" in the S.L. Sobolev Institute of Mathematics. Scientific degree "Professor of Mathematics" was awarded to him in 1989. In 2000 V.D. Stepanov was elected a corresponding member of the Russian Academy of Sciences (RAS).

Since 1985 till 2005 V.D. Stepanov was the Head of Laboratory of Functional Analysis at the Computing Center of the Far Easten Branch of the Russian Academy of Science.

In 2005 V.D. Stepanov moved from Khabarovsk to Moscow with appointment at the Peoples Friendship University of Russia as the Head of the Department of Mathematical Analysis (retired in 2018). Also, he was hired at the V.A. Steklov Mathematical Institute of RAS at the Function Theory Department.

Research interests of V.D. Stepanov are: the theory of integral and differential operators, harmonic analysis in Euclidean spaces, weighted inequalities, duality in function spaces, approximation theory, asymptotic estimates of singular, approximation and entropy numbers of integral transformations, and estimates of the Schatten-Neumann type. Main achievements: the theory of integral convolution operators is constructed, the criteria for the boundedness and compactness of integral operators in function spaces are obtained, weighted inequalities and the behaviour of approximation numbers of the Volterra, Riemann-Liouville, Hardy integral operators are studied, etc.

Under his scientific supervision 15 candidate theses in Russia and 5 PhD theses in Sweden were successfully defended. Professor V.D. Stepanov has over 100 scientific publications including 3 monographs. Participation in scientific and organizational activities of V.D. Stepanov is well known. He is a member of the American Mathematical Society (since 1987) and a member of the London Mathematical Society (since 1996), Deputy Editor of the Analysis Mathematica, member of the Editorial Board of the Eurasian Mathematical Journal, invited speaker at many international conferences and visiting professor of universities in USA, Canada, UK, Spain, Sweden, South Korea, Kazakhstan, etc.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Vladimir Dmitrievich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF ANALYSIS, DIFFERENTIAL EQUATIONS AND ALGEBRA" (EMJ-2019), DEDICATED TO THE 10TH ANNIVERSARY OF THE EURASIAN MATHEMATICAL JOURNAL

From October 16 to October 19, 2019 at the L.N. Gumilyov Eurasian National University (ENU) the International Conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2019) was held. The conference was dedicated to the 10th anniversary of the Eurasian Mathematical Journal (EMJ).

The purposes of the conference were to discuss the current state of development of mathematical scientific directions, expand the number of potential authors of the Eurasian Mathematical Journal and further strengthen the scientific cooperation between the Faculty of Mechanics and Mathematics of the ENU and scientists from other cities of Kazakhstan and abroad.

The partner universities for the organization of the conference were the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia (the RUDN University, Moscow) and the University of Padua (Italy).

The conference was attended by more than 80 mathematicians from the cities of Almaty, Aktobe, Karaganda, Nur-Sultan, Shymkent, Taraz, Turkestan, as well as from several foreign countries: from Azerbaijan, Germany, Greece, Italy, Japan, Kyrgyzstan, Russia, Tajikistan and Uzbekistan.

The chairman of the International Programme Committee of the conference was Ye.B. Sydykov, rector of the ENU, co-chairmen were Chief editors of the EMJ: V.I. Burenkov, professor of the RUDN University, M. Otelbaev, academician of the National Academy of Sciences of the Republic of Kazakhstan (NAS RK), V.A. Sadovnichy, academician of the Russian Academy of Sciences (RAS), rector of the M.V. Lomonosov Moscow State University (MSU).

There were three sections at the conference: "Function Theory and Functional Analysis", "Differential Equations and Equations of Mathematical Physics" and "Algebra and Model Theory". 16 plenary presentations of 30 minutes each and more than 60 sectional presentations of 20 minutes each, devoted to contemporary areas of mathematics, were given.

It was decided to recommend selected reports of the participants for publication in the Eurasian Mathematical Journal and the Bulletin of the Karaganda State University (series "Mathematics").

Before the conference, a collection of abstracts of the participants' talks was published.

PROGRAMME OF THE INTERNATIONAL CONFERENCE EMJ-2019

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Conference Schedule:

16.10.2019
09.00 – 10.00 Registration
10.00 – 10.30 Opening of the conference
10.30 – 12.50 Plenary talks
12.50 – 14.00 Lunch
14.00 – 18.00 Session talks

17.10.2019
09.30 – 12.20 Plenary talks
12.20 – 14.00 Lunch
14.00 – 18.00 Session talks
18.00 – Dinner for participants of the conference

18.10.2019
09.30 – 13.00 Plenary talks
12.20 – 14.00 Lunch
14.00 – 17.00 Excursion around the city

19.10.2019
09.30 – 12.30 Plenary talks
12.30 – 13.00 Closing of the conference

At the opening ceremony welcome speeches were given by Ye.B. Sydykov, rector of the ENU, chairman of the Program Committee of the conference; V.I. Burenkov, professor of the RUDN University, editor-in-chief of the EMJ; L. Mukasheva, official representative of the international company Clarivate Analytics in the Central Asian region; A. Ospanova, official representative of Scopus.

Plenary talks were given by

T.Sh. Kalmenov (Kazakhstan), M. Otelbaev and B.D. Koshanov (Kazakhstan), P.D. Lamberti and V. Vespri (Italy) – on 16.10.2019;

V.I. Burenkov (Russia), T. Ozawa (Japan), H. Begehr (Germany), M.A. Sadybekov and A.A. Dukenbaeva (Kazakhstan), D. Suragan (Kazakhstan) – on 17.10.2019;

M.L. Goldman (Russia), A. Bountis (Greece), A.K. Kerimbekov (Kyrgyzstan), S.N. Kharin (Kazakhstan), M.I. Dyachenko (Russia) – on 18.10.2019;

E.D. Nursultanov (Kazakhstan), M.A. Ragusa (Italy), P.D. Lamberti and V. Vespri (Italy), M.G. Gadoev (Russia) and F.S. Iskhokov (Tajikistan) – on 19.10.2019.

At the closing ceremony all participants unanimously congratulated the staff of the L.N. Gumilyov Eurasian National University and the Editorial Board of the Eurasian Mathematical Journal with the 10th anniversary of the journal and wished further creative successes.

They expressed hope that the journal will continue to play an important role in the development of mathematical science and education in Kazakhstan in the future.

V.I. Burenkov, K.N. Ospanov, A.M. Temirkhanova

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STABILIZATION OF SOLUTIONS OF TWO-DIMENSIONAL PARABOLIC EQUATIONS AND RELATED SPECTRAL PROBLEMS

M. Jenaliyev, K. Imanberdiyev, A. Kassymbekova, K. Sharipov

Communicated by V.I. Korzyuk

Key words: boundary stabilization, heat equation, spectrum, loaded Laplace operator.

AMS Mathematics Subject Classification: 35K05, 39B82, 47A75.

Abstract. One of the important properties that characterize the behaviour of solutions of boundary value problems for differential equations is stabilization, which has a direct relationship with the problems of controllability. In this paper, the problems of solvability are investigated for stabilization problems of two-dimensional loaded equations of parabolic type with the help of feedback control given on the boundary of the region. These equations have numerous applications in the study of inverse problems for differential equations.

The problem consists in the choice of boundary conditions (controls), so that the solution of the boundary value problem tends to a given stationary solution at a certain speed at $t \to \infty$. This requires that the control is feedback, i.e. that it responds to unintended fluctuations in the system, suppressing the results of their impact on the stabilized solution. The spectral properties of the loaded two-dimensional Laplace operator, which are used to solve the initial stabilization problem, are also studied. The paper presents an algorithm for solving the stabilization problem, which consists of constructively implemented stages.

The idea of reducing the stabilization problem for a parabolic equation by means of boundary controls to the solution of an auxiliary boundary value problem in the extended domain of independent variables belongs to A.V. Fursikov. At the same time, recently, the so-called loaded differential equations are actively used in problems of mathematical modeling and control of nonlocal dynamical systems.

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1 Introduction

The idea of reducing the stabilization problem for a parabolic equation by means of boundary controls to an auxiliary boundary value problem in the extended domain of independent variables belongs to A.V. Fursikov. It was proposed in his work [1] and developed further in the works [2, 3, 4]. At the same time, recently, the so-called loaded differential equations [5, 6, 7, 8, 9, 10, 11] are actively used in problems of mathematical modeling and control of nonlocal dynamical systems. We have previously studied stabilization problems for a loaded one- and two-dimensional heat equations [12, 13, 14]. In [15] we have studied stabilization problems for a two-dimensional heat equations loaded trace of a solution. In this work, we investigate the case of loading by the trace of the derivative of a solution, the spectral properties of the loaded two-dimensional Laplace operator, which are applied to solving of the stabilization problem. A one-dimensional analogue of problems (4.2) is studied in [14]. Note that the results of this work differ significantly from our previous results.

Note also that in recent years due to the intensive study of problems of optimal control of agroecosystem, for example, problems of long-term forecasting and regulation of the groundwater level of soil moisture, interest in the loaded equations [5], [6] has increased significantly.

2 Statement of the stabilization problem

Let $\Omega = \{x, y : -\pi/2 < x, y < \pi/2\}$ be a domain with the boundary $\partial\Omega$. In the cylinder $Q = \Omega \times \{t > 0\}$ with lateral surface $\Sigma = \partial\Omega \times \{t > 0\}$ we consider the boundary value problem for the loaded heat equation

$$u_t - \Delta u + \alpha \cdot u_x(0, y, t) + \beta \cdot u_y(x, 0, t) = 0, \ \{x, y, t\} \in Q, \tag{2.1}$$

$$u(x, y, 0) = u_0(x, y), \ \{x, y\} \in \Omega, \tag{2.2}$$

$$u(x, y, t) = p(x, y, t), \ \{x, y, t\} \in \Sigma, \tag{2.3}$$

where $\alpha, \ \beta \in \mathbb{C}$ are given complex numbers, $u_0(x, y) \in L_2(\Omega)$ is a given function.

The aim is to find a function $p(x, y, t)$ such that a solution of boundary value problem (2.1)–(2.3) satisfies the inequality

$$\|u(x, y, t)\|_{L_2(\Omega)} \le C_0 e^{-\sigma t}, \ \sigma > 0, \ t > 0. \tag{2.4}$$

Note that here σ is a given constant and $C_0 \ge \|u_0(x, y)\|_{L_2(\Omega)}$ is an arbitrary finite constant.

Remark 1. In Sections 6 and 7 it will be shown that the solution of stabilization problem (2.1)–(2.4) significantly depends on the values of the coefficients α and β, including the signs of their real parts.

Equation (2.1) is called a loaded equation [5, 7]. We note that problem (2.1)–(2.4) with a multi-points load was studied in [14].

One of the applications of the stabilization problem is explained in Figure 1. Figure 1 shows a diagram of a closed system of automatic control (CSAC). It is assumed that the control object (CO), the main characteristic of which is its thermal mode, is described by the heat equation (without of loadings) and initial condition (2.2). The output of the CO is the solution of the heat equation. This solution is measured by a measuring device (MD), which captures the derivative of thermal regime only on some manifolds.

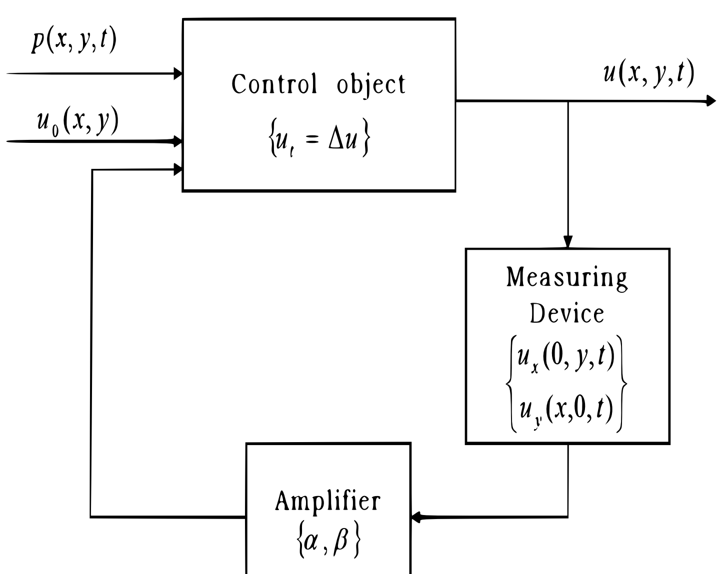

Figure 1. Closed system of automatic control

In our case, they are the manifolds, obtained for $x = 0$ and $y = 0$. Further, the signals from the MD are "amplified" with the help of the coefficients α and β and are fed to the input of the CO. In this way, we get the CSAC, which is described by the loaded heat equation (2.1). But we need the properties of the thermal regime of exponential stabilization (2.4). The latter must be provided by the boundary controls $p(x, y, t)$, which, in our stabilization problem, must be determined.

For problem (2.1)–(2.4) in Section 2, an auxiliary stabilization problem is associated with it by expanding the region of independent variables. To solve this auxiliary problem (3.1)–(3.3) in Section 3 we consider spectral properties of the loaded two-dimensional Laplace operator in Section 4. Section 5 contains the main results of the work which were formulated in the form of Theorems 5.1 and 5.2, establishing the desired spectral properties and the proofs of Theorems 5.1 and 5.2. On the basis of these results, an algorithm for solving stabilization problem (2.1)–(2.4) is proposed. In Section 6, an algorithm for solving problem (2.1)–(2.4) is given, where the solution of problem (2.1)–(2.4) is found as a trace of solution of auxiliary problem (3.1)–(3.3). The numerical implementation of the algorithm of Section 6 for a specific example is carried out in Section 7.

3 The auxiliary boundary value problem (BVP)

Let $\Omega_1 = \{x, y : -\pi < x, y < \pi\}$ and $Q_1 = \Omega_1 \times \{t > 0\}$. We introduce the auxiliary boundary value problem:

$$z_t - \Delta z + \alpha \cdot z_x(0, y, t) + \beta \cdot z_y(x, 0, t) = 0, \ \{x, y, t\} \in Q_1, \tag{3.1}$$

$$z(x, y, 0) = z_0(x, y), \ \{x, y\} \in \Omega_1, \tag{3.2}$$

$$\frac{\partial^j z(-\pi, y, t)}{\partial x^j} = \frac{\partial^j z(\pi, y, t)}{\partial x^j}, \ \{y, t\} \in (-\pi, \pi) \times \{t > 0\},$$

$$\frac{\partial^j z(x, -\pi, t)}{\partial y^j} = \frac{\partial^j z(x, \pi, t)}{\partial y^j}, \ \{x, t\} \in (-\pi, \pi) \times \{t > 0\}, \ j = 0, 1. \tag{3.3}$$

The problem is to find an initial function $z_0(x, y)$ such that a solution of BVP (3.1)–(3.3) satisfies the inequality

$$\|z(x, y, t)\|_{L_2(\Omega_1)} \le C_0 e^{-\sigma t}, \ \sigma > 0, \ t > 0, \tag{3.4}$$

where the constants C_0 and σ are the same as in original problem (2.1)–(2.4).

4 Statement of the spectral problems for the loaded two-dimensional Laplace operator

Let us look for a solution of problem (3.1)–(3.3) in the form

$$z(x, y, t) = \sum_{k,l \in \mathbb{Z}} Z_{kl}(t) \psi_{kl}(x, y), \ Z_{kl}(t) = (z(x, y, t), \varphi_{kl}(x, y)), \tag{4.1}$$

where $(\cdot, \cdot)$ denotes the scalar product, and $\{\varphi_{kl}(x, y), \ k, l \in \mathbb{Z}\}$, $\{\psi_{kl}(x, y), \ k, l \in \mathbb{Z}\}$ is a biorthogonal basis of the space $L_2(\Omega_1)$ and $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$. The following two spectral problems are considered for construction of a biorthogonal basis $\{\varphi_{kl}(x, y), \ k, l \in \mathbb{Z}\}$, $\{\psi_{kl}(x, y), \ k, l \in \mathbb{Z}\}$ in the domain $\Omega_1 = \{x, y : -\pi < x < \pi, -\pi < y < \pi\}$:

$$\begin{cases} -\Delta \varphi(x, y) + \alpha \cdot \varphi_x(0, y) + \beta \cdot \varphi_y(x, 0) = \lambda \varphi(x, y), \\[2ex] \dfrac{\partial^j \varphi(-\pi, y)}{\partial x^j} = \dfrac{\partial^j \varphi(\pi, y)}{\partial x^j}, \ \dfrac{\partial^j \varphi(x, -\pi)}{\partial y^j} = \dfrac{\partial^j \varphi(x, \pi)}{\partial y^j}, \end{cases} \tag{4.2}$$

where $j = 0, 1$, Δ is the Laplace operator, α, $\beta \in \mathbb{C}$ are given complex numbers, $\lambda \in \mathbb{C}$ is a spectral parameter.

5 Main results

Let $\mathbb{Z}=\{0,\pm 1,\pm 2,...\}$ and $\mathbb{Z}'=\mathbb{Z}\setminus\{0\}$. The following propositions are valid.

Theorem 5.1. *The system of eigenfunctions and eigenvalues for problem (4.2) has the following form:*

$$\{\varphi_{kl}(x,y),\ \lambda_{kl},\ k,\, l\in\mathbb{Z}\}=\Big\{1,\ \lambda_{00}=0;\ \ e^{ilx}-i\frac{\alpha}{l},\ \lambda_{0l}=l^2,\ l\in\mathbb{Z}';$$

$$e^{iky}-i\frac{\beta}{k},\ \lambda_{k0}=k^2,\ k\in\mathbb{Z}';$$

$$\left(e^{ilx}-i\frac{\alpha}{l}\right)\left(e^{iky}-i\frac{\beta}{k}\right),\ \lambda_{kl}=k^2+l^2,\ k,\, l\in\mathbb{Z}'\Big\}. \tag{5.1}$$

Proof. By using the method of separation of variables we set

$$\varphi_{kl}(x,y)=X_l(x)Y_k(y),\ k,\, l\in\mathbb{Z}, \tag{5.2}$$

$$\frac{-X_l''(x)+\alpha\cdot X_l'(0)}{X_l(x)}+\frac{-Y_k''(y)+\beta\cdot Y_k'(0)}{Y_k(y)}=\lambda_{kl}\equiv\lambda_l^{(1)}+\lambda_k^{(2)},\ k,\, l\in\mathbb{Z}, \tag{5.3}$$

and for the solution of (4.2) we obtain the following spectral problems

$$\begin{cases} -X_l''(x)+\alpha\cdot X_l'(0)=\lambda_l^{(1)}X_l(x),\\ X_l^j(-\pi)=X_l^j(\pi),\ \ j=0,1,\, l\in\mathbb{Z},\end{cases} \tag{5.4}$$

$$\begin{cases} -Y_k''(y)+\beta\cdot Y_k'(0)=\lambda_k^{(2)}Y_k(y),\\ Y_k^j(-\pi)=Y_k^j(\pi),\ \ j=0,1,\, k\in\mathbb{Z}.\end{cases} \tag{5.5}$$

We note that the general solution of loaded differential equation (5.4) is represented as a linear combination of the following complete system of periodic functions:

$$\left\{\Phi_l(x)=e^{i\,l\,x},\ l\in\mathbb{Z}\right\},$$

as:

$$X_l(x)=A_le^{i\,l\,x}+C_l,\ l\in\mathbb{Z}, \tag{5.6}$$

where the undetermined coefficients $A_l,\ C_l$ are to be determined from loaded differential equation (5.4). We have:

$$C_l=-iA_l\frac{\alpha}{l},\ l\in\mathbb{Z}',\ C_0=A_0.$$

Carrying out the normalization of these coefficients, from this and from (5.6) we finally obtain the system of eigenfunctions and eigenvalues for spectral problem (5.4) in the form of

$$\{X_l(x),\ \lambda_l^{(1)},\ l\in\mathbb{Z}\}=\left\{e^{i\,l\,x}-i\frac{\alpha}{l},\ \lambda_l^{(1)}=l^2,\ l\in\mathbb{Z}';\ \ 1,\ \lambda_0^{(1)}=0\right\}. \tag{5.7}$$

Similarly to (5.7) we obtain the system of eigenfunctions and eigenvalues for spectral problem (5.5) in the form of

$$\{Y_k(y),\ \lambda_k^{(2)},\ k\in\mathbb{Z}\}=\left\{e^{i\,k\,y}-i\frac{\beta}{k},\ \lambda_k^{(2)}=k^2,\ k\in\mathbb{Z}';\ \ 1,\ \lambda_0^{(2)}=0\right\}. \tag{5.8}$$

Relations (5.2), (5.7) and (5.8) imply statement (5.1) of Theorem 5.1. □

Theorem 5.2. *The following biorthogonal sequence*

$$\{\psi_{kl}(x,y),\ k,\,l\in\mathbb{Z}\}=\left\{\frac{1}{4\pi^2}\left(1-i\alpha\sum_{n\in\mathbb{Z}'}\frac{1}{n}e^{inx}\right)\left(1-i\beta\sum_{m\in\mathbb{Z}'}\frac{1}{m}e^{imy}\right),\ l=k=0;\right.$$

$$\frac{1}{4\pi^2}\left[e^{ilx}-i\beta\sum_{m\in\mathbb{Z}'}\frac{1}{m}e^{i(my+lx)}\right],\ k=0,\ l\in\mathbb{Z}';$$

$$\left.\frac{1}{4\pi^2}\left[e^{iky}-i\alpha\sum_{n\in\mathbb{Z}'}\frac{1}{n}e^{i(nx+ky)}\right],\ l=0,\ k\in\mathbb{Z}';\ \frac{1}{4\pi^2}e^{i(ky+lx)},\ k,\,l\in\mathbb{Z}'\right\}\tag{5.9}$$

is a biorthogonal basis in $L_2(\Omega_1)$*.*

Proof. Let us find a biorthogonal sequence for (5.1). We search for it in the form:

$$\{\psi_{kl}(x,y),\ k,l\in\mathbb{Z}\}=$$

$$=\left\{f_0(x)g_0(y),\ \frac{1}{2\pi}g_0(y)e^{ilx},\ \frac{1}{2\pi}f_0(x)e^{iky},\ \frac{1}{4\pi^2}e^{i(lx+ky)},\ k,\,l\in\mathbb{Z}'\right\},\tag{5.10}$$

where only functions $f_0(x)$ and $g_0(x)$ are unknown. Using basis (5.1), we look for the unknown function $f_0(x)$ in the form:

$$f_0(x)=C_0+\sum_{n\in\mathbb{Z}'}C_n\left(e^{inx}-i\frac{\alpha}{l}\right)$$

where $\{C_n,\ n\in\mathbb{Z}'\}$ are the unknown coefficients, which must be determined by the biorthogonality conditions:

$$(1,\ f_0(x))=1;\ \left(e^{inx}-i\frac{\alpha}{n},\ f_0(x)\right)=0,\ \ \forall\ n\in\mathbb{Z}'.$$

The last conditions imply that

$$C_0=\frac{1}{2\pi}\left(1+i\sum_{n\in\mathbb{Z}'}C_n\cdot\frac{\alpha}{n}\right),\quad C_n=-i\frac{\alpha}{n},\ \ \forall\ n\in\mathbb{Z}'.$$

Using the values C_l, we rewrite C_0:

$$C_0=\frac{1}{2\pi}\left(1+\frac{\alpha^2\pi^2}{3}\right).$$

Further using the value C_0, we represent the desired function $f_0(x)$ in the form:

$$f_0(x)=\frac{1}{2\pi}\left(1-i\alpha\sum_{n\in\mathbb{Z}'}\frac{1}{n}e^{inx}\right).\tag{5.11}$$

Similarly we find $g_0(y)$:

$$g_0(y)=\frac{1}{2\pi}\left(1-i\beta\sum_{m\in\mathbb{Z}'}\frac{1}{m}e^{imy}\right).\tag{5.12}$$

Remark 2. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} < \infty$ the series $\sum\limits_{n\in\mathbb{Z}'} \dfrac{1}{n} e^{inx}$ and $\sum\limits_{n\in\mathbb{Z}'} \dfrac{1}{n} e^{iny}$ converge in $L_2(-\pi,\pi)$ to certain $f_0(x),\ g_0(y) \in L_2(-\pi,\pi)$ and Parceval's equalities hold:

$$\int\limits_{-\pi}^{\pi} |f_0(x)|^2 dx = \frac{1}{2\pi}\left(1 + \frac{|\alpha|^2\pi^2}{3}\right), \quad \int\limits_{-\pi}^{\pi} |g_0(y)|^2 dy = \frac{1}{2\pi}\left(1 + \frac{|\beta|^2\pi^2}{3}\right).$$

Moreover,

$$\int\limits_{\Omega_1} |f_0(x) g_0(y)|^2 d\,x\,d\,y = \frac{1}{4\pi^2}\left(1 + \frac{|\alpha|^2\pi^2}{3}\right)\left(1 + \frac{|\beta|^2\pi^2}{3}\right).$$

Thus, by formula (5.10) we obtain that $\psi_{kl}(x,y) \in L_2(\Omega_1)\ \forall \{k,l\} \in \mathbb{Z}$.

It remains to apply the Paley-Wiener theorem ([16], pp. 206–207). Then relations (5.10), (5.11) and (5.12) imply statement (5.9) of Theorem 5.2. □

6 Algorithm for solving the stabilization problem

We propose the following algorithm for solving the stabilization problem for the heat equation with a loaded two-dimensional Laplace operator. It consists of the following constructively implemented steps.

Step 1. We define the function $z_0(x,y)$ as a continuation of the given function $u_0(x,y)$. So that requirement (3.4) is satisfied for the solution $z(x,y,t)$ of problem (3.1)–(3.3). In this case condition (2.4) holds as well for its restriction $u(x,y,t)$ and a required boundary control $p(x,y,t)$, $\{x,y\} \in \Sigma$ is defined as the trace of the function $z(x,y,t)$ for $\{x,y,t\} \in \Sigma$.

Step 2. We construct a complete biorthogonal system of functions on the square Ω_1 by solving the appropriate spectral problems.

Step 3. Find the coefficients of the decomposition for the desired function $z_0(x,y)$ on the square Ω_1 along the constructed at the previous step complete biorthogonal system so that condition (3.4) holds.

We will show estimates of values C_0 and σ in inequality (3.4). For this purpose the solution of initial-boundary value problem (3.1)–(3.3) can be written in form (4.1):

$$z(x,y,t) = z_{000}\psi_{00}(x,y) + \sum_{k\in\mathbb{Z}'} z_{0k0} e^{-k^2 t}\psi_{k0}(x,y) +$$
$$+ \sum_{l\in\mathbb{Z}'} z_{00l} e^{-l^2 t}\psi_{0l}(x,y) + \sum_{k,l\in\mathbb{Z}'} z_{0kl} e^{-(k^2+l^2)t}\psi_{kl}(x,y), \tag{6.1}$$

where $z_{0kl} = \int\limits_{\Omega_1} \overline{\varphi_{kl}(x,y)} z_0(x,y) dx\,dy$, $k,l \in \mathbb{Z}$, are the Fourier coefficients of $z_0(x,y)$, where $\{\varphi_{kl}(x,y),\ k,l \in \mathbb{Z}\}$ and $\{\psi_{kl}(x,y),\ k,l \in \mathbb{Z}\}$ are defined respectively by formulas (5.1) and (5.9).

We introduce the following sets of indices

$$\mathbb{W} = \mathbb{Z}\times\mathbb{Z} = \mathbb{W}_k \cup \mathbb{W}_l \cup \mathbb{W}' \cup \{0,0\},\ \mathbb{W}' = \mathbb{Z}'\times\mathbb{Z}',\ \mathbb{W}_0 = \bigcup_{j=1}^{3} \overline{\overline{\mathbb{W}}}_{0j} \cup \mathbb{W}_{04}, \tag{6.2}$$

$$\overline{\overline{\mathbb{W}}}_{01} = \mathbb{W}_k \setminus \mathbb{W}_{01},\ \overline{\overline{\mathbb{W}}}_{02} = \mathbb{W}_l \setminus \mathbb{W}_{02},\ \overline{\overline{\mathbb{W}}}_{03} = \mathbb{W}' \setminus \mathbb{W}_{03},\ \mathbb{W}_0' = \bigcup_{j=1}^{3} \overline{\overline{\mathbb{W}}}_{0j}. \tag{6.3}$$

where $\mathbb{W}_k = \mathbb{Z}'\times\{0\}$, $\mathbb{W}_l = \{0\}\times\mathbb{Z}'$, $\mathbb{Z}' = \mathbb{Z}\setminus\{0\}$,

$$\mathbb{W}_{01} = \big\{\{k,0\} : k^2 \ge \sigma\big\} \subset \mathbb{W}_k,\ \mathbb{W}_{02} = \big\{\{0,l\} : l^2 \ge \sigma\big\} \subset \mathbb{W}_l,$$
$$\mathbb{W}_{03} = \big\{\{k,l\} : k^2 + l^2 \ge \sigma\big\} \subset \mathbb{W}',\ \mathbb{W}_{04} = \{0,0\}.$$

Remark 3. The sets $\overline{\mathbb{W}}_{01}$, $\overline{\mathbb{W}}_{02}$, $\overline{\mathbb{W}}_{03}$ (6.3) and $\mathbb{W}'_0$ (6.3) are finite.

Thus, let the conditions of Theorem 5.1 hold. Then the following assertion is true.

Theorem 6.1. *Let the conditions*

$$z_{0kl} = 0 \ at \ \{k,l\} \in \mathbb{W}_0, \tag{6.4}$$

be satisfied for solution (6.1), then the stabilized solution $z_{stab}(x,y,t)$ of problem (3.1)–(3.3) takes the form

$$z_{stab}(x,y,t) = \sum_{\{k,0\}\in\mathbb{W}_{01}} z_{0k0}e^{-k^2t}\psi_{k0}(x,y) + \sum_{\{0,l\}\in\mathbb{W}_{02}} z_{00l}e^{-l^2t}\psi_{0l}(x,y) + \\ + \sum_{\{k,l\}\in\mathbb{W}_{03}} z_{0kl}e^{-(k^2+l^2)t}\psi_{kl}(x,y), \tag{6.5}$$

and satisfies inequality (3.4).

The proof of Theorem 6.1 directly follows from our further reasoning. Thus, each of the sets $\mathbb{W}_{01}, \mathbb{W}_{02}, \mathbb{W}_{03}$ contains a set of indices $\{k,l\}$ that do not satisfy conditions (6.4). From (6.5) we obtain that for the constant C_1 the following equality is true:

$$C_1^2 = \int_{\Omega_1} |z_0(x,y)|^2 \, dx\, dy = \int_{\Omega_1} |\sum_{\substack{\{k,0\}\in\mathbb{W}_{01}\\ \{0,l\}\in\mathbb{W}_{02}\\ \{k,l\}\in\mathbb{W}_{03}}} z_{0kl}\psi_{kl}(x,y)|^2 \, dx\, dy < \infty, \tag{6.6}$$

where

$$z_0(x,y) = \begin{cases} u_0(x,y), & \text{at } \{x,y\} \in \Omega, \\ z_1(x,y), & \text{at } \{x,y\} \in \Omega_1 \setminus \Omega, \end{cases} \tag{6.7}$$

and here the function $z_1(x,y)$ and its Fourier coefficients $\{z_{0kl}, \ \{k,l\} \in \mathbb{W}'_0\}$ are unknown, and there is a need to find them. And for this we will use equalities (6.4), from which we obtain:

$$\int_{\Omega_1\setminus\Omega} \overline{\varphi_{kl}(x,y)} z_1(x,y) \, d\,x\, d\,y = -\hat{u}_0(k,l), \ \{k,l\} \in \mathbb{W}'_0, \tag{6.8}$$

where

$$\hat{u}_0(k,l) = \int_{\Omega} \overline{\varphi_{kl}(x,y)} u_0(x,y) \, d\,x\, d\,y. \tag{6.9}$$

Now we will look for the unknown function $z_1(x,y)$ in the form of the following linear combination:

$$z_1(x,y) = \sum_{\{m,n\}\in\mathbb{W}_0} \hat{z}_1(m,n)\varphi_{mn}(x,y). \tag{6.10}$$

As a result, substituting $z_1(x,y)$ (6.10) into relation (6.8), we obtain a system of algebraic equations with respect to the unknown constant matrix $\{\hat{z}_1(m,n), \ \{m,n\} \in \mathbb{W}'_0\}$:

$$\sum_{\{m,n\}\in\mathbb{W}'_0} a_{klmn}\hat{z}_1(m,n) = -\hat{u}_0(k,l), \ \{k,l\} \in \mathbb{W}'_0, \tag{6.11}$$

where

$$a_{klmn} = \int_{\Omega_1\setminus\Omega} \overline{\varphi_{kl}(x,y)} \varphi_{mn}(x,y) \, d\,x\, d\,y, \ \{k,l\}, \{m,n\} \in \mathbb{W}'_0. \tag{6.12}$$

We fix the indices k_0 and m_0. Then we will represent the system of equations (6.11) with known matrix (6.12) as the next family of independent systems of linear equations

$$\sum_{\{m_0,n\}\in\mathbb{W}'_0} a_{k_0lm_0n}\hat{z}_1(m_0,n) = -\hat{u}_0(k_0,l),\ \{k_0,l\}\in\mathbb{W}'_0, \tag{6.13}$$

with the unknown vector $\hat{z}_1(m_0,n)$, and with the known vector in the right-hand side $-\hat{u}_0(k_0,l)$, and with the well-known matrix

$$a_{k_0lm_0n} = \int\limits_{\Omega_1\setminus\Omega} \overline{\varphi_{k_0l}(x,y)}\varphi_{m_0n}(x,y)\,dx\,dy,\ \{k_0,l\},\{m_0,n\}\in\mathbb{W}'_0. \tag{6.14}$$

Since matrices (6.14) are built by using the elements $\{\varphi_{k,l}(x,y),\ \{k,l\}\in\mathbb{W}'_0\}$, which form finite subsystems of the basis $\{\varphi_{kl}(x,y),\ k,l\in\mathbb{Z}\}$ (5.1), then for each fixed pair of indices k_0 and m_0 they are the Gram matrices. As is known, the determinants of the Gram matrices are different from zero ([17], p. 219). Therefore, we have the unique solvability for equation (6.13), and as corollary for equation (6.11) too.

Next, by finding the unknown matrix $\{\hat{z}_1(m,n),\ \{m,n\}\in\mathbb{W}'_0\}$ according to formula (6.10) we find the function $\{z_1(x,y),\ \{x,y\}\in\Omega_1\setminus\Omega\}$, and together with it, according to (6.7) we find the function $\{z_0(x,y),\ \{x,y\}\in\Omega_1\}$ as a continuation of the function $\{u_0(x,y),\{x,y\}\in\Omega\}$.

Further, analyzing formula (6.5) and taking into account the definitions of sets $\mathbb{W}_{0j}, j=\overline{1,3}$ in (6.3), we obtain the following estimate for a real constant σ_r, which determines the decay order in the exponent in (3.4):

$$\sigma_r \triangleq \min\left\{\min_{\{k,0\}\in\mathbb{W}_{01}}\{k^2\};\ \min_{\{0,l\}\in\mathbb{W}_{02}}\{l^2\};\ \min_{\{k,l\}\in\mathbb{W}_{03}}\{k^2+l^2\}\right\}\geq\sigma.$$

Now, according to formulas (6.5)–(6.7), we can find the value of the finite constant C_1 in (3.4).

Step 4. By the founded solution $z(x,y,t)$ of auxiliary boundary value problem (3.1)–(3.3) as restriction of it to the cylinder Q we find the solution $u(x,y,t)$ to given boundary value problem (2.1)–(2.3), satisfying required condition (2.4). A boundary control $p(x,y,t)$, $\{x,y\}\in\Sigma$ is found as the trace of the solution $z(x,y,t)$, i.e.

$$p(x,y,t) = z(x,y,t)_{|\{x,y,t\}\in\Sigma}.$$

7 Numerical realization of Steps 1–4 of the algorithm

By Theorems 5.1 and 5.2 for the eigenfunctions, eigenvalues and biorthogonal functions we obtained formulas (5.1) and (5.9).

Example. Let $u_0(x,y) = (\pi^2/4 - x^2)\,(\pi^2/4 - y^2)\,,\ \alpha=-0.7+3i,\ \beta=-1.2+2i,\ \sigma=1.5.$

We have $\overline{\mathbb{W}}_{01}=\{\{-1,0\},\ \{1,0\}\}$, $\overline{\mathbb{W}}_{02}=\{\{0,-1\},\ \{0,1\}\}$, $\overline{\mathbb{W}}_{03}=\varnothing$. We write the eigenfunctions $\varphi_{kl}(x,y)$, eigenvalues λ_{kl}, and biorthogonal system of functions $\psi_{kl}(x,y)$ $\forall\{k,l\}\in\mathbb{W}'_0$ and give a table of computing of matrix a_{klmn} (6.12). We have

$$\varphi_{-10}(x,y)=\exp\{-iy\}+i\beta,\ \lambda_{-10}=1;\ \varphi_{10}(x,y)=\exp\{iy\}-i\beta,\ \lambda_{10}=1;$$

$$\varphi_{0-1}(x,y)=\exp\{-ix\}+i\alpha,\ \lambda_{0-1}=1;\ \varphi_{01}(x,y)=\exp\{ix\}-i\alpha,\ \lambda_{01}=1,$$

$$\psi_{-10}(x,y)=\frac{1}{4\pi^2}\left[\exp\{-iy\}-i\alpha\sum_{n\in\mathbb{Z}}\frac{1}{n}\exp\{i(nx-y)\}\right];$$

$$\psi_{10}(x,y) = \frac{1}{4\pi^2}\left[\exp\{iy\} - i\alpha\sum_{n\in\mathbb{Z}}\frac{1}{n}\exp\{i(nx+y)\}\right];$$

$$\psi_{0-1}(x,y) = \frac{1}{4\pi^2}\left[\exp\{-ix\} - i\beta\sum_{m\in\mathbb{Z}}\frac{1}{m}\exp\{i(-x+my)\}\right];$$

$$\psi_{01}(x,y) = \frac{1}{4\pi^2}\left[\exp\{ix\} - i\beta\sum_{m\in\mathbb{Z}}\frac{1}{m}\exp\{i(x+my)\}\right].$$

Table 1. Matrix $a_{klmn} = \left(\overline{\varphi_{kl}(x,y)}\varphi_{mn}(x,y)\right)$, $\{k,l\}$, $\{m,n\} \in \mathbb{W}_0'$

$\{k,l\} \in \mathbb{W}_0'$	$\{m,n\} \in \mathbb{W}_0'$			
	$\varphi_{-10}(x,y)$	$\varphi_{10}(x,y)$	$\varphi_{0-1}(x,y)$	$\varphi_{01}(x,y)$
$\varphi_{-10}(x,y)$	a_{-10-10}	a_{-1010}	a_{-100-1}	a_{-1001}
$\varphi_{10}(x,y)$	a_{10-10}	a_{1010}	a_{100-1}	a_{1001}
$\varphi_{0-1}(x,y)$	a_{0-1-10}	a_{0-110}	a_{0-10-1}	a_{0-101}
$\varphi_{01}(x,y)$	a_{01-10}	a_{0110}	a_{010-1}	a_{0101}

where

$a_{-10-10} = 3\pi^2 \cdot 6.44 + 8\pi \approx 215.814;$
$a_{-1010} = -3\pi^2 \cdot 5.44 - i4.8\pi \approx -161.072 - i\ 15.080;$
$a_{-100-1} = -4 + 3\pi^2 \cdot 6.84 + 10\pi - i(3\pi^2 \cdot 2,2 + \pi) \approx 229.940 - i\ 68.281;$
$a_{-1001} = -4 - 3\pi^2 \cdot 6.84 - 2\pi + i(3\pi^2 \cdot 2.2 - 3.8\pi) \approx -212.808 + i\ 53.201;$
$a_{10-10} = -3\pi^2 \cdot 5.44 + i4.8\pi \approx -161.072 + i\ 15.080;$
$a_{1010} = 3\pi^2 \cdot 6.44 - 8\pi \approx 165.548;$
$a_{100-1} = -4 - 3\pi^2 \cdot 6.84 + 2\pi + i(3\pi^2 \cdot 2.2 + 3.8\pi) \approx -212.808 + i\ 77.077;$
$a_{1001} = -4 + 3\pi^2 \cdot 6.84 - 10\pi - i(3\pi^2 \cdot 2.2 - \pi) \approx 167.108 - i\ 61.998;$
$a_{0-1-10} = -4 + 3\pi^2 \cdot 6.84 + 10\pi + i(3\pi^2 \cdot 2.2 + \pi) \approx 229.940 + i\ 68.281;$
$a_{0-110} = -4 - 3\pi^2 \cdot 6.84 + 2\pi - i(3\pi^2 \cdot 2.2 + 3.8\pi) \approx -212.808 - i\ 77.077;$
$a_{0-10-1} = 3\pi^2 \cdot 10.49 + 12\pi \approx 348.296;$
$a_{0-101} = -3\pi^2 \cdot 9.49 - i2.8\pi \approx -280.988 - i\ 8.797;$
$a_{01-10} = -4 - 3\pi^2 \cdot 6.84 - 2\pi - i(3\pi^2 \cdot 2.2 - 3.8\pi) \approx -212.808 - i\ 53.201;$
$a_{0110} = -4 + 3\pi^2 \cdot 6.84 - 10\pi + i(3\pi^2 \cdot 2.2 - \pi) \approx 167.108 + i\ 61.998;$
$a_{010-1} = -3\pi^2 \cdot 9.49 + i2.8\pi \approx -280.988 + i\ 8.797;$
$a_{0101} = 3\pi^2 \cdot 10.49 - 12\pi \approx 272.897.$

And, further, we obtain

$$\sum_{\{0,n\}\in\mathbb{W}_0} a_{0l0n}\hat{z}_1(0,n) = -\hat{u}_0(0,l), \quad \{0,l\} \in \mathbb{W}_0', \tag{7.1}$$

i.e. $\begin{pmatrix} a_{0-10-1} & a_{0-101} \\ a_{010-1} & a_{0101} \end{pmatrix} \begin{pmatrix} \hat{z}_1(0,-1) \\ \hat{z}_1(0,1) \end{pmatrix} = \begin{pmatrix} -\hat{u}_0(0,-1) \\ -\hat{u}_0(0,1) \end{pmatrix},$

$$\sum_{\{m,0\}\in\mathbb{W}_0} a_{k0m0}\hat{z}_1(m,0) = -\hat{u}_0(k,0), \quad \{k,0\} \in \mathbb{W}_0', \tag{7.2}$$

i.e. $\begin{pmatrix} a_{-10-10} & a_{-1010} \\ a_{10-10} & a_{1010} \end{pmatrix} \begin{pmatrix} \hat{z}_1(-1,0) \\ \hat{z}_1(1,0) \end{pmatrix} = \begin{pmatrix} -\hat{u}_0(-1,0) \\ -\hat{u}_0(1,0) \end{pmatrix},$

Table 2

$\{k,l\} \in \mathbb{W}'_0$			
$-\hat{u}_0(-1,0)$	$-\hat{u}_0(1,0)$	$-\hat{u}_0(0,-1)$	$-\hat{u}_0(0,1)$
$32.740 - i\ 32.046$	$-74.081 + i\ 32.046$	$59.445 - i\ 18.694$	$-100.787 + i\ 18.694$

According to (6.10), (7.1), (7.2) and Tables 2 we obtain

Table 3

$\{k,l\} \in \mathbb{W}'_0$			
$\hat{z}_1(-1,0)$	$\hat{z}_1(1,0)$	$\hat{z}_1(0,-1)$	$\hat{z}_1(0,1)$
$-0.732 - i\ 0.132$	$-1.172 + i\ 0.132$	$-0.766 - i\ 0.046$	$-1.159 + i\ 0.046$

According to (6.10) and Tables 3 we obtain the function $z_1(x,y)$:

$$z_1(x,y) = \hat{z}_1(0,-1) \cdot \left(e^{-ix} + i\alpha\right) + \hat{z}_1(0,1) \cdot \left(e^{ix} - i\alpha\right) +$$
$$+\hat{z}_1(-1,0) \cdot \left(e^{-iy} + i\beta\right) + \hat{z}_1(1,0) \cdot \left(e^{iy} - i\beta\right) =$$
$$= A_1 e^{-ix} + A_2 e^{ix} + A_3 e^{-iy} + A_4 e^{iy} + A_5, \tag{7.3}$$

where

$$A_1 = \hat{z}_1(0,-1) \approx -0.732 - i\ 0.132;\ \ A_2 = \hat{z}_1(0,1) \approx -1.172 + i\ 0.132;$$
$$A_3 = \hat{z}_1(-1,0) \approx 0.766 - i\ 0.046;\ \ A_4 = \hat{z}_1(1,0) \approx -1.159 + i\ 0.046;$$
$$A_5 = [\hat{z}_1(0,-1) - \hat{z}_1(0,1)] \cdot i\alpha + [\hat{z}_1(-1,0) - \hat{z}_1(1,0)] \cdot i\beta \approx -2.441.$$

Thus we have

$$z_0(x,y) = \begin{cases} (\pi^2/4 - x^2)\,(\pi^2/4 - y^2)\,, & \text{at } \{x,y\} \in \Omega, \\ A_1 e^{-ix} + A_2 e^{ix} + A_3 e^{-iy} + A_4 e^{iy} + A_5, & \text{at } \{x,y\} \in \Omega_1 \setminus \Omega. \end{cases} \tag{7.4}$$

According to (6.1) we define the Fourier coefficients for function $z_0(x,y)$ (7.4) and write them in Table 4.

Table 4

$\{k,l\} \in \mathbb{W}'_0 = \mathbb{W}_{01} \cup \mathbb{W}_{02} \cup \mathbb{W}_{03}$		
$z_{0k0},\ \{k,0\} \in \mathbb{W}_{01}$	$z_{00l},\ \{0,l\} \in \mathbb{W}_{02}$	$z_{0kl},\ \{k,l\} \in \mathbb{W}_{03}$
D_k	E_l	F_{kl}

Here

$$D_k = D_{k1} \sin(k\pi/2) - D_{k2} \cos(k\pi/2) - D_{k3} \sin((k+1)\pi/2) -$$
$$-D_{k4} \sin((k-1)\pi/2) + D_{k5},$$
$$E_l = E_{l1} \sin(l\pi/2) - E_{l2} \cos(l\pi/2) - E_{l3} \sin((l+1)\pi/2) -$$
$$-E_{l4} \sin((l-1)\pi/2) + E_{l5},$$
$$F_{kl} = F_{kl}^{(1)} + F_{kl}^{(2)} - F_{kl}^{(3)},$$

where

$$D_{k1} = \frac{1}{k}\left[\frac{2\pi^3}{3k^2} - \frac{\pi^5}{12} - 2A_5\pi - 4A_1 - 4A_2\right],\ D_{k2} = \frac{\pi^4}{3k^2},\ D_{k3} = \frac{2A_3\pi}{k+1},$$

$$D_{k4}=\frac{2A_4\pi}{k-1},\ D_{k5}=\frac{\pi^5}{12k}-i\frac{\overline{\beta}\pi}{k}\left[\frac{\pi^5}{24}+3\pi A_5-2\sum_{j=1}^{4}A_j\right];$$

$$E_{l1}=\frac{1}{l}\left[\frac{2\pi^3}{3l^2}-\frac{\pi^5}{12}-2A_5\pi-4A_3-4A_4\right],\ E_{l2}=\frac{\pi^4}{3l^2},\ E_{l3}=\frac{2A_1\pi}{l+1},$$

$$E_{l4}=\frac{2A_2\pi}{l-1},\ E_{l5}=\frac{\pi^5}{12k}-i\frac{\overline{\alpha}\pi}{k}\left[\frac{\pi^5}{24}+3\pi A_5-2\sum_{j=1}^{4}A_j\right];$$

$$F^{(1)}_{kl}=F(k,\beta)F(l,\alpha),\ \{k,l\}\in\mathbb{W}_3;$$

$$F(j,\gamma)=\frac{4}{j^3}\sin\frac{j\pi}{2}-i\frac{4}{j^2}\cos\frac{j\pi}{2}-i\frac{\overline{\gamma}\pi^3}{6j},$$

$$F^{(2)}_{kl}=\frac{\overline{\alpha}\overline{\beta}}{kl}A_5 4\pi^2,\ k^2\geq 4,\ l^2\geq 4,$$

$$F^{(2)}_{11}=-4\pi^2\left[i\overline{\alpha}A_4+i\overline{\beta}A_2+\overline{\alpha}\overline{\beta}A_5\right],\ \ F^{(2)}_{1-1}=4\pi^2\left[i\overline{\alpha}A_4-i\overline{\beta}A_3+\overline{\alpha}\overline{\beta}A_5\right],$$

$$F^{(2)}_{-11}=4\pi^2\left[-i\overline{\alpha}A_3+i\overline{\beta}A_2+\overline{\alpha}\overline{\beta}A_5\right],\ \ F^{(2)}_{-1-1}=4\pi^2\left[i\overline{\alpha}A_3+i\overline{\beta}A_1-\overline{\alpha}\overline{\beta}A_5\right].$$

$$F^{(3)}_{kl}=g(k,l,A_1,A_2)\sin\frac{k\pi}{2}\cos\frac{l\pi}{2}+g(l,k,A_3,A_4)\sin\frac{l\pi}{2}\cos\frac{k\pi}{2}+$$

$$+\frac{4A_5}{kl}\sin\frac{k\pi}{2}\sin\frac{l\pi}{2}-i\frac{2\overline{\alpha}}{kl}\left[A_1+A_2+A_5\pi\right]\sin\frac{k\pi}{2}+$$

$$+i\frac{\pi\overline{\alpha}}{2}g(l,k,A_3,A_4)\cos\frac{k\pi}{2}-i\frac{\overline{\beta}\pi}{2}g(k,l,A_1,A_2)\cos\frac{l\pi}{2}-$$

$$-i\frac{2\pi\overline{\beta}}{kl}\sum_{j=3}^{5}A_j\sin\frac{l\pi}{2}-\frac{2\pi\overline{\alpha}\overline{\beta}}{kl}\left[\sum_{j=1}^{4}A_j+A_5\pi/2\right],\ k^2\geq 4,\ l^2\geq 4,$$

$$g(m,n,A,B)=\frac{4}{m}\left[-\frac{A}{n-1}+\frac{B}{n+1}\right],$$

$$F^{(3)}_{-1-1}=h^{(0)}_{-1-1}(A_2,A_4,A_5)+i\overline{\alpha}h^{(1)}_{-1-1}(A_1,A_2,A_4,A_5)+i\overline{\beta}h^{(1)}_{-1-1}(A_3,A_4,A_2,A_5)-\overline{\alpha}\overline{\beta}h^{(2)},$$

$$F^{(3)}_{1-1}=h^{(0)}_{1-1}(A_1,A_4,A_5)+i\overline{\alpha}h^{(1)}_{1-1}(A_1,A_2,A_4,A_5)-i\overline{\beta}h^{(1)}_{1-1}(A_3,A_4,A_1,A_5)+\overline{\alpha}\overline{\beta}h^{(2)},$$

$$F^{(3)}_{-11}=h^{(0)}_{-11}(A_2,A_3,A_5)-i\overline{\alpha}h^{(1)}_{-11}(A_1,A_2,A_3,A_5)+i\overline{\beta}h^{(1)}_{-11}(A_3,A_4,A_2,A_5)+\overline{\alpha}\overline{\beta}h^{(2)},$$

$$F^{(3)}_{11}=h^{(0)}_{11}(A_1,A_3,A_5)-i\overline{\alpha}h^{(1)}_{11}(A_1,A_2,A_3,A_5)-i\overline{\beta}h^{(1)}_{11}(A_3,A_4,A_1,A_5)-\overline{\alpha}\overline{\beta}h^{(2)},$$

$$h^{(0)}_{kl}(A,B,C)=2\pi(A+B)+4C,\ h^{(1)}_{kl}(A,B,C,D)=4A+4B+\pi^2C+2\pi D,$$

$$h^{(2)}=2\pi\sum_{j=1}^{4}A_j+\pi^2A_5.$$

For solution $z_{stab}(x,y,t)$ (6.5) we have

$$z_{stab}(x,y,t)=\frac{1}{4\pi^2}\left\{\sum_{k^2\geq 4}D_k\,e^{-k^2\,t}\left[e^{iky}-i\alpha\sum_{n\in\mathbb{Z}'}\frac{1}{n}e^{i(nx+ky)}\right]+\right.$$

$$+\sum_{l^2\geq 4} E_l\, e^{-l^2 t}\left[e^{ilx}-i\beta\sum_{m\in\mathbb{Z}'}\frac{1}{m}e^{i(my+lx)}\right]+$$

$$\left.+\sum_{k^2\geq 1,\; l^2\geq 1} F_{kl}\, e^{-(k^2+l^2)\,t}e^{i(lx+ky)}\right\},\quad \{x,y,t\}\in Q_1.$$

Thus, we obtain a solution of the stabilization problem:

$$C_1^2=\|z_0(x,y)\|^2_{L_2(\Omega_1)}=\|u_0(x,y)\|^2_{L_2(\Omega)}+3\pi^2\sum_{k=1}^{5}|A_k|^2-$$

$$-8\mathrm{Re}\{(\overline{A_1+A_2})(A_3+A_4)\}-4\pi\mathrm{Re}\{\overline{A}_5\sum_{k=1}^{4}A_k\}\approx 246.203>$$

$$>C_0^2=\|u_0(x,y)\|^2_{L_2(\Omega)}=\frac{\pi^{10}}{900},\ \text{i.e.,}\ C_1\approx 15.691>C_0=\frac{\pi^5}{30}\approx 10.201,$$

$$\sigma_r=2.0=\min\{4.0,\ 4.0,\ 2.0\}>\sigma=1.5,$$

$$u_{stab}(x,y,t)=z_{stab}(x,y,t)_{|\{x,y,t\}\in Q},$$

$$p_{stab}(x,y,t)=u_{stab}(x,y,t)_{|\{x,y,t\}\in\Sigma}=\begin{cases}u_{stab}(\pm\pi/2,y,t), & y\in(-\pi/2,\pi/2),\\ u_{stab}(x,\pm\pi/2,t), & x\in(-\pi/2,\pi/2),\end{cases}$$

where

$$u_{stab}(\pm\pi/2,y,t)=\frac{1}{4\pi^2}\left\{\sum_{k^2\geq 4} D_k\, e^{-k^2\,t}\left[e^{iky}-i\alpha\sum_{n\in\mathbb{Z}'}\frac{1}{n}e^{i(\pm n\pi/2+ky)}\right]+\right.$$

$$\left.+\sum_{l^2\geq 4} E_l\, e^{-l^2 t}\left[e^{ilx}-i\beta\sum_{m\in\mathbb{Z}'}\frac{1}{m}e^{i(my\pm l\pi/2)}\right]+\sum_{k^2\geq 1,\; l^2\geq 1} F_{kl}\, e^{-(k^2+l^2)\,t}e^{i(\pm l\pi/2+ky)}\right\},$$

$$u_{stab}(x,\pm\pi/2,t)=\frac{1}{4\pi^2}\left\{\sum_{k^2\geq 4} D_k\, e^{-k^2\,t}\left[e^{\pm ik\pi/2}-i\alpha\sum_{n\in\mathbb{Z}'}\frac{1}{n}e^{i(nx\pm k\pi/2)}\right]+\right.$$

$$\left.+\sum_{l^2\geq 4} E_l\, e^{-l^2 t}\left[e^{ilx}-i\beta\sum_{m\in\mathbb{Z}'}\frac{1}{m}e^{i(\pm m\pi/2+lx)}\right]+\sum_{k^2\geq 1,\; l^2\geq 1} F_{kl}\, e^{-(k^2+l^2)\,t}e^{i(lx\pm k\pi/2)}\right\}.$$

8 Conclusion

The results of the work on the spectral properties of a loaded two-dimensional Laplace operator can be useful in solving stabilization problems for a loaded parabolic equation with the help of boundary control actions that can be used in problems of mathematical modeling by controlled loaded differential equations.

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