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From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

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The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

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- compliance of the title of the paper to its content;

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- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

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At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.

VLADIMIR DMITRIEVICH STEPANOV

(to the 70th birthday)

Vladimir Dmitrievich Stepanov was born on December 13, 1949 in a small town Belovo, Kemerovo region. In 1966 he finished the Lavrentiev school of physics and mathematics at Novosibirsk academic town-ship and the same year he entered the Faculty of Mathematics of the Novosibirsk State University (NSU) from which he has graduated in 1971 and started to teach mathematics at the Khabarovsk Technical University till 1981 with interruption for postgraduate studies (1973-1976) in the NSU.

In 1977 he has defended the PhD dissertation and in 1985 his doctoral thesis "Integral convolution operators in Lebesgue spaces" in the S.L. Sobolev Institute of Mathematics. Scientific degree "Professor of Mathematics" was awarded to him in 1989. In 2000 V.D. Stepanov was elected a corresponding member of the Russian Academy of Sciences (RAS).

Since 1985 till 2005 V.D. Stepanov was the Head of Laboratory of Functional Analysis at the Computing Center of the Far Easten Branch of the Russian Academy of Science.

In 2005 V.D. Stepanov moved from Khabarovsk to Moscow with appointment at the Peoples Friendship University of Russia as the Head of the Department of Mathematical Analysis (retired in 2018). Also, he was hired at the V.A. Steklov Mathematical Institute of RAS at the Function Theory Department.

Research interests of V.D. Stepanov are: the theory of integral and differential operators, harmonic analysis in Euclidean spaces, weighted inequalities, duality in function spaces, approximation theory, asymptotic estimates of singular, approximation and entropy numbers of integral transformations, and estimates of the Schatten-Neumann type. Main achievements: the theory of integral convolution operators is constructed, the criteria for the boundedness and compactness of integral operators in function spaces are obtained, weighted inequalities and the behaviour of approximation numbers of the Volterra, Riemann-Liouville, Hardy integral operators are studied, etc.

Under his scientific supervision 15 candidate theses in Russia and 5 PhD theses in Sweden were successfully defended. Professor V.D. Stepanov has over 100 scientific publications including 3 monographs. Participation in scientific and organizational activities of V.D. Stepanov is well known. He is a member of the American Mathematical Society (since 1987) and a member of the London Mathematical Society (since 1996), Deputy Editor of the Analysis Mathematica, member of the Editorial Board of the Eurasian Mathematical Journal, invited speaker at many international conferences and visiting professor of universities in USA, Canada, UK, Spain, Sweden, South Korea, Kazakhstan, etc.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Vladimir Dmitrievich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF ANALYSIS, DIFFERENTIAL EQUATIONS AND ALGEBRA" (EMJ-2019), DEDICATED TO THE 10TH ANNIVERSARY OF THE EURASIAN MATHEMATICAL JOURNAL

From October 16 to October 19, 2019 at the L.N. Gumilyov Eurasian National University (ENU) the International Conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2019) was held. The conference was dedicated to the 10th anniversary of the Eurasian Mathematical Journal (EMJ).

The purposes of the conference were to discuss the current state of development of mathematical scientific directions, expand the number of potential authors of the Eurasian Mathematical Journal and further strengthen the scientific cooperation between the Faculty of Mechanics and Mathematics of the ENU and scientists from other cities of Kazakhstan and abroad.

The partner universities for the organization of the conference were the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia (the RUDN University, Moscow) and the University of Padua (Italy).

The conference was attended by more than 80 mathematicians from the cities of Almaty, Aktobe, Karaganda, Nur-Sultan, Shymkent, Taraz, Turkestan, as well as from several foreign countries: from Azerbaijan, Germany, Greece, Italy, Japan, Kyrgyzstan, Russia, Tajikistan and Uzbekistan.

The chairman of the International Programme Committee of the conference was Ye.B. Sydykov, rector of the ENU, co-chairmen were Chief editors of the EMJ: V.I. Burenkov, professor of the RUDN University, M. Otelbaev, academician of the National Academy of Sciences of the Republic of Kazakhstan (NAS RK), V.A. Sadovnichy, academician of the Russian Academy of Sciences (RAS), rector of the M.V. Lomonosov Moscow State University (MSU).

There were three sections at the conference: "Function Theory and Functional Analysis", "Differential Equations and Equations of Mathematical Physics" and "Algebra and Model Theory". 16 plenary presentations of 30 minutes each and more than 60 sectional presentations of 20 minutes each, devoted to contemporary areas of mathematics, were given.

It was decided to recommend selected reports of the participants for publication in the Eurasian Mathematical Journal and the Bulletin of the Karaganda State University (series "Mathematics").

Before the conference, a collection of abstracts of the participants' talks was published.

PROGRAMME OF THE INTERNATIONAL CONFERENCE EMJ-2019

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Chairman: Ye.B. Sydykov, rector of the ENU;

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Conference Schedule:

16.10.2019

09.00 – 10.00 Registration
10.00 – 10.30 Opening of the conference
10.30 – 12.50 Plenary talks
12.50 – 14.00 Lunch
14.00 – 18.00 Session talks

17.10.2019

09.30 – 12.20 Plenary talks
12.20 – 14.00 Lunch
14.00 – 18.00 Session talks
18.00 – Dinner for participants of the conference

18.10.2019

09.30 – 13.00 Plenary talks
12.20 – 14.00 Lunch
14.00 – 17.00 Excursion around the city

19.10.2019

09.30 – 12.30 Plenary talks
12.30 – 13.00 Closing of the conference

At the opening ceremony welcome speeches were given by Ye.B. Sydykov, rector of the ENU, chairman of the Program Committee of the conference; V.I. Burenkov, professor of the RUDN University, editor-in-chief of the EMJ; L. Mukasheva, official representative of the international company Clarivate Analytics in the Central Asian region; A. Ospanova, official representative of Scopus.

Plenary talks were given by

T.Sh. Kalmenov (Kazakhstan), M. Otelbaev and B.D. Koshanov (Kazakhstan), P.D. Lamberti and V. Vespri (Italy) – on 16.10.2019;

V.I. Burenkov (Russia), T. Ozawa (Japan), H. Begehr (Germany), M.A. Sadybekov and A.A. Dukenbaeva (Kazakhstan), D. Suragan (Kazakhstan) – on 17.10.2019;

M.L. Goldman (Russia), A. Bountis (Greece), A.K. Kerimbekov (Kyrgyzstan), S.N. Kharin (Kazakhstan), M.I. Dyachenko (Russia) – on 18.10.2019;

E.D. Nursultanov (Kazakhstan), M.A. Ragusa (Italy), P.D. Lamberti and V. Vespri (Italy), M.G. Gadoev (Russia) and F.S. Iskhokov (Tajikistan) – on 19.10.2019.

At the closing ceremony all participants unanimously congratulated the staff of the L.N. Gumilyov Eurasian National University and the Editorial Board of the Eurasian Mathematical Journal with the 10th anniversary of the journal and wished further creative successes.

They expressed hope that the journal will continue to play an important role in the development of mathematical science and education in Kazakhstan in the future.

V.I. Burenkov, K.N. Ospanov, A.M. Temirkhanova

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LCT BASED INTEGRAL TRANSFORMS AND HAUSDORFF OPERATORS

P. Jain, S. Jain, V.D. Stepanov

Communicated by R. Oinarov

Key words: LCT, linear canonical cosine transform, linear canonical sine transform, Hausdorff operators.

AMS Mathematics Subject Classification: 44A35, 26D20.

Abstract. In this paper, it is shown that certain Hausdorff operator and its adjoint are connected by linear canonical sine as well as linear canonical cosine transforms. The results have been proved in one as well as in two dimensions.

DOI: https://doi.org/10.32523/2077-9879-2020-11-1-57-71

1 Introduction

The Fourier cosine transform of a suitable function f is defined by

$$F_c f(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos yt \, dt.$$

According to a well known result of Titchmarsh ([16], Theorem 69), if $f \in L^2(\mathbb{R}_+)$ and $g = F_c f$, then the Fourier cosine transform of the Hardy averaging operator

$$Hf(x) = \frac{1}{x} \int_0^x f(y) \, dy$$

is the adjoint operator of the Fourier cosine transform of g

$$H^* g(y) = \int_y^\infty \frac{g(x)}{x} \, dx,$$

i.e., $F_c H f = H^* F_c f$. By the same way one can prove that $H F_c f = F_c H^* f$.

Goldberg [4] extended this result by considering more general operators replacing H and H^* by, respectively,

$$G_\psi g(x) = \frac{1}{x} \int_0^\infty \psi\left(\frac{y}{x}\right) g(y) \, dy$$

and

$$G_\psi^* f(y) = \int_0^\infty \psi\left(\frac{y}{x}\right) \frac{f(x)}{x} \, dx,$$

with some restrictions on the function ψ. It is obvious that for $\psi = \chi_{(0,1)}$, the operators G_ψ and G_ψ^* become H and H^*, respectively. The above results of Titchmarsh and Goldberg are also true for the Fourier sine transform

$$F_s f(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin yt \, dt.$$

Let us mention that the operator G_ψ is related with the Hausdorff operator. It was proved in [4] that G_ψ is a bounded operator from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R}_+)$ and

$$\|G_\psi\|_2 \le K =: \int_0^\infty \frac{|\psi(y)|}{\sqrt{y}}dy.$$

In this paper, we prove that the operator G_ψ is bounded from $L^p(\mathbb{R}_+)$ to $L^p(\mathbb{R}_+)$, $1 < p < \infty$. Moreover, we provide the precise value of $\|G_\psi\|$ with a new proof of the lower bound. This is done in Section 2.

Next, during the recent past, people have worked with more general transforms such as fractional Fourier transform and linear canonical transform (LCT). The LCT was first studied in [3], [11] and is connected with the 2×2 matrix M given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{with } ad - bc = 1.$$

The LCT is defined by

$$L_M f(x) = \int_{\mathbb{R}} K_M(x, y) f(y)\, dy,$$

where the kernel K_M is defined by

$$K_M(x, y) = \begin{cases} \frac{1}{\sqrt{2\pi bi}} \exp \frac{i}{2}\left(\frac{a}{b}y^2 - \frac{2}{b}xy + \frac{d}{b}x^2\right), & \text{if } b \ne 0 \\ \frac{1}{\sqrt{a}} e^{i\left(\frac{c}{2a}\right)y^2} \delta\left(x - \frac{y}{a}\right), & \text{if } b = 0. \end{cases}$$

In our case, we shall be dealing with the situation when $b \ne 0$.

In the modern time, "Fourier Analysis" is usually termed as "Time Frequency Analysis". In this context, the Fourier transform rotates the signals from the time axis to the frequency axis by 90 degrees. It has been observed that certain optical systems rotated the signals by an arbitrary angle which requires the notion of fractional Fourier transforms, which is a one parameter family of transforms. The linear canonical transforms (LCT) form a class of three parameter family of transforms which include many known transforms. For notational convenience, if we write the matrix M as $(a, b; c, d)$, then the matrixes $(0, 1; -1, 0)$ and $(\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$ corresponds, respectively, to the Fourier and fractional Fourier transforms. More special matrices lead to some other known integral transforms, e.g., Fresnel transform, chirp functions etc. Various applications of LCT have been realized in the field of electromagnatic, acoustic and other wave propagation problems. As mentioned in [9], LCT is known by other terminolgy as well such as quadractic phase integral [1], generalized Huygens integral [14], generalized Fresnel transform [7], [12] etc. Recently, in [13], the authors have studied certain mapping properties of LCT and the associated pseudo-differential operators in a variant of Schwartz space.

The next aim of the paper is to extend Goldberg's result in the framework of linear canonical cosine and sine transforms with respect to a generalized Hausdorff operator

$$(T_\psi g)(x) = \frac{1}{x}\int_0^\infty \psi\left(\frac{y}{x}\right) g(y) e^{\frac{ia}{2b}\left(x^2 - y^2\right)} dy.$$

In fact, we shall prove that under certain condition on ψ, if g is the linear cosine (sine) transform of f, then $T_\psi g$ is the linear cosine (sine) transform of $T_\psi^* g$, where T_ψ^* is the adjoint of T_ψ. This is done in Section 3.

We study and discuss the two-dimensional Hausdorff operator in Section 4 giving a sharp value of the L^p-L^p norm of it with a new proof of the lower bound. Finally, in Section 5, the two-dimensional Hausdorff operator is discussed in respect of the two-dimensional linear canonical cosine (sine) transform.

2 The Hausdorff operator

In this section, we consider the following operator:

$$G_\psi g(x) = \frac{1}{x}\int_0^\infty \psi\left(\frac{y}{x}\right) g(y)dy.$$

By the replacement $\psi(s) = \frac{1}{s}\phi\left(\frac{1}{s}\right)$, $s > 0$ it can be seen that the operator G_ψ becomes the Hausdorff operator

$$H_\phi g(x) = \int_0^\infty \frac{\phi(y)}{y} f\left(\frac{x}{y}\right) dy.$$

The Hausdorff operator H_ϕ and its multidimensional extensions are well known in the literature, see for instance recent survey [10] and references given therein.

Remark 1. The operator H_ϕ (and consequently G_ψ) includes several well known integral operators as follows:

(i) For $\phi(t) = \frac{1}{t}\chi_{(1,\infty)}(t)$, the operator H_ϕ reduces to the standard Hardy averaging operator

$$Hg(x) = \frac{1}{x}\int_0^x g(y)dy$$

while for $\phi(t) = \chi_{[0,1]}(t)$, it reduces to the adjoint of Hardy averaging operator

$$H^*g(x) = \int_x^\infty \frac{g(y)}{y} dy.$$

(ii) For $\phi(t) = \frac{1}{\max(1,t)}$, the operator H_ϕ gives rise to the Calderon operator ([2], Definition 5.1)

$$H_\phi g(x) = H + H^*.$$

(iii) For $\phi(t) = \gamma(1-t)^{\gamma-1}\chi_{(0,1)}(t)$, $\gamma > 0$, the operator H_ϕ becomes the Cesáro operator

$$H_\phi g(x) = \gamma \int_x^\infty \frac{(y-x)^{\gamma-1}}{y^\gamma} g(y)dy.$$

(iv) For $\phi(t) = \frac{1}{t}\left(1-\frac{1}{t}\right)^{\beta-1}\chi_{(1,\infty)}(t)$, $\beta > 0$, the operator H_ϕ becomes the fractional Riemann Liouville operator

$$H_\phi g(x) = x^{-\beta}\int_0^x (x-y)^{\beta-1} g(y)dy.$$

In ([4], Theorem 1), Goldberg proved the following

Theorem A. Let $\psi \geq 0$ on $\mathbb{R}_+$ be such that $\displaystyle\int_0^\infty \frac{\psi(y)}{\sqrt{y}} dy =: K < \infty$. Then the operator G_ψ is a bounded operator on $L^2(\mathbb{R}_+)$ and $\|G_\psi\| \leq K$.

Theorem A has L^p-L^p version ([5], Theorem 319) and many other extensions with sharp constants (see [8], Theorem 6.4 and bibliographic notes to Chapter 2 therein).

In the following, we extend Theorem A by proving it for $L^p(\mathbb{R}_+)$ and moreover, the precise value of $\|G_\psi\|$ will be obtained.

Theorem 2.1. *Let $\psi \geq 0$ on $\mathbb{R}_+$ be such that $\int_0^\infty \frac{\psi(y)}{y^{1/p}} dy =: K_p < \infty$. Then the operator G_ψ is a bounded operator on $L^p(\mathbb{R}_+)$, $1 < p < \infty$. Moreover, $\|G_\psi\| = K_p$.*

Proof. Let $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$, where $p' = \frac{p}{p-1}$. We have by change of variables, Fubini's Theorem and Hölder's inequality

$$\begin{aligned}
\int_0^\infty |G_\psi f(x)||g(x)|dx &\leq \int_0^\infty \left(\frac{1}{x} \int_0^\infty \psi\left(\frac{y}{x}\right) |f(y)|dy \right) |g(x)|dx \\
&= \int_0^\infty \left(\int_0^\infty \psi(y)|f(xy)|dy \right) |g(x)|dx \\
&= \int_0^\infty \psi(y) \left(\int_0^\infty |g(x)||f(xy)|dx \right) dy \\
&\leq \int_0^\infty \psi(y) \left(\int_0^\infty |f(xy)|^p dx \right)^{1/p} \left(\int_0^\infty |g(x)|^{p'} dx \right)^{1/p'} dy \\
&\leq K_p \|f\|_p \|g\|_{p'}.
\end{aligned}$$

It now follows by duality principle that $G_\psi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ is bounded and $\|G_\psi\| \leq K_p$.

To prove the reverse inequality, we use an idea of [15]. Let $0 \leq f \in L^p(\mathbb{R}_+)$ and $0 \leq g \in L^{p'}(\mathbb{R}_+)$. We have

$$\begin{aligned}
J &:= \int_0^\infty \psi(y) \left(\int_0^\infty f(xy)g(x)dx \right) dy \qquad (2.1)\\
&= \frac{1}{x} \int_0^\infty \psi\left(\frac{y}{x}\right) \left(\int_0^\infty f(y)g(x)dx \right) dy \\
&= \int_0^\infty g(x) \left(\frac{1}{x} \int_0^\infty \psi\left(\frac{y}{x}\right) f(y)dy \right) dx \\
&= \int_0^\infty g(x) \left(G_\psi f\right)(x)dx.
\end{aligned}$$

Since $G_\psi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ is bounded, by applying Hölder's inequality in the last expression, we get

$$J \leq \|G_\psi\| \|f\|_p \|g\|_{p'}. \qquad (2.2)$$

Now, for $t \in (0,1)$, we define the test functions

$$f_t(x) = \frac{1}{x^{1/p}} \chi_{(t,1/t)}(x), \qquad g_t(x) = \frac{1}{x^{1/p'}} \chi_{(t,1/t)}(x).$$

Then it can be calculated that

$$\|f\|_p^p = \|g\|_{p'}^{p'} = 2\log(1/t). \qquad (2.3)$$

Also, we have

$$h_t(y) := \int_0^\infty f_t(xy)g_t(x)dx = y^{-1/p} \int_0^\infty \chi_{(t,1/t)}(xy)\chi_{(t,1/t)}(x)\frac{dx}{x} = y^{-1/p} \int_{\left(t,\frac{1}{t}\right)\cap\left(\frac{t}{y},\frac{1}{ty}\right)} \frac{dx}{x}.$$

We divide $(0,\infty)$ as

$$(0,\infty) = (0,t^2) \cup [t^2, 1] \cup \left(1, \frac{1}{t^2}\right] \cup \left(\frac{1}{t^2}, \infty\right).$$

If $y \in (0, t^2) \cup \left(\frac{1}{t^2}, \infty\right)$, then $\left(t, \frac{1}{t}\right) \cap \left(\frac{t}{y}, \frac{1}{ty}\right) = \varnothing$. So that in this case

$$h_t(y) = 0. \tag{2.4}$$

If $y \in [t^2, 1]$, then $\left(t, \frac{1}{t}\right) \cap \left(\frac{t}{y}, \frac{1}{ty}\right) = \left(\frac{t}{y}, \frac{1}{t}\right)$ and

$$h_t(y) = y^{-1/p}\left(2\log\frac{1}{t} + \log y\right). \tag{2.5}$$

If $y \in \left(1, \frac{1}{t^2}\right]$, then $\left(t, \frac{1}{t}\right) \cap \left(\frac{t}{y}, \frac{1}{ty}\right) = \left(t, \frac{1}{ty}\right)$ and we have

$$h_t(y) = y^{-1/p}\left(2\log\frac{1}{t} - \log y\right). \tag{2.6}$$

Thus, taking f and g as f_t and g_t in (2.1) and using (2.4), (2.5), (2.6), we obtain

$$\begin{aligned} J &= \left(\int_0^{t^2} + \int_{t^2}^{1} + \int_1^{1/t^2} + \int_{1/t^2}^{\infty}\right)\psi(y)h_t(y)dy \\ &= 2\log\frac{1}{t}\int_{t^2}^{1/t^2}\frac{\psi(y)}{y^{1/p}}\left(1 - \frac{\xi(y)}{2\log\frac{1}{t}}\right)dy, \end{aligned} \tag{2.7}$$

where

$$\xi(y) = \begin{cases} \log 1/y, & \text{if } 0 < y < 1 \\ \log y, & \text{if } y > 1. \end{cases}$$

Now, using (2.7) and (2.3), (2.2) gives for $f = f_t$ and $g = g_t$

$$\int_{t^2}^{1/t^2}\frac{\psi(y)}{y^{1/p}}\left(1 - \frac{\xi(y)}{2\log\frac{1}{t}}\right)dy \le \|G_\psi\|.$$

By the Monotone Convergence Theorem the LHS $\uparrow K_p$ as $t \to 0$ and we conclude that $K_p \le \|G_\psi\|$. $\square$

Example 1. *In view of Theorem 2.1, the precise norm of the integral operators mentioned in Remark 1 can be calculated given as follows:*

(i) $\|H\| = \displaystyle\int_0^1 \frac{dt}{t^{1/p}} = \frac{p}{p-1}$

(ii) $\|H^*\| = \displaystyle\int_1^\infty \frac{dt}{t^{1+1/p}} = p$

(iii) $\|H + H^*\| = \dfrac{p^2}{p-1}$

(iv) $\|G_\psi\| = \gamma\displaystyle\int_1^\infty \frac{(t-1)^{\gamma-1}}{t^{\gamma+1/p}}dt$

(v) $\|G_\psi\| = \displaystyle\int_0^1 \frac{(1-t)^{\beta-1}}{t^{1/p}}dt$

3 LCT

Given a function f, define f_e by

$$f_e(x) = \begin{cases} f(x), & \text{if } x \geq 0 \\ f(-x), & \text{if } x < 0. \end{cases}$$

Clearly, if $f \in L^1(\mathbb{R})$, then $f_e \in L^1(\mathbb{R})$. The Fourier transform of f_e is given by

$$F f_e(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_e(y) e^{-\iota xy} dy = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(y) \cos xy dy.$$

This is one of the motivations to define and study the Fourier cosine transform. Motivated by this, we apply the definition of LCT and obtain the so-called linear canonical cosine transform. On the similar lines, the linear canonical sine transform can be defined. Precisely we give the following

Definition 1. For a function $f \in L^1(\mathbb{R}_+)$, the linear canonical cosine transform is defined by

$$L_c f(x) = \sqrt{\frac{2}{\pi b i}} \int_0^{\infty} f(y) e^{\frac{i}{2b}\left(ay^2 + dx^2\right)} \cos\left(\frac{xy}{b}\right) dy$$

and the linear canonical sine transform is defined by

$$L_s f(x) = \sqrt{\frac{2}{\pi b i}} \int_0^{\infty} f(y) e^{\frac{i}{2b}\left(ay^2 + dx^2\right)} \sin\left(\frac{xy}{b}\right) dy$$

where $(a, b; c, d)$ represents a real matrix. Throughout in this section, we assume that $a = d$. In that sense, the linear canonical cosine transform reduces to

$$L_c f(x) = \sqrt{\frac{2}{\pi b i}} \int_0^{\infty} f(y) e^{\frac{ia}{2b}\left(y^2 + x^2\right)} \cos\left(\frac{xy}{b}\right) dy.$$

In this section, we are concerned with the generalized Hausdorff type operator

$$(T_\psi g)(x) = \frac{1}{x} \int_0^{\infty} \psi\left(\frac{y}{x}\right) g(y) e^{\frac{ia}{2b}\left(x^2 - y^2\right)} dy \tag{3.1}$$

and its adjoint

$$(T_\psi^* f)(y) = \int_0^{\infty} \frac{f(x)}{x} \psi\left(\frac{y}{x}\right) e^{\frac{ia}{2b}\left(x^2 - y^2\right)} dx. \tag{3.2}$$

In the following theorem, we prove the boundedness of the operator T_ψ.

Theorem 3.1. *Let $\psi \in L^1_{loc}(\mathbb{R}_+)$ be such that $\int_0^{\infty} \frac{|\psi(y)|}{\sqrt{y}} dy =: K < \infty$. The operator T_ψ in (3.1) is a bounded operator on $L^2(\mathbb{R}_+)$ and $\|T_\psi\| \leq K$.*

Proof. For any $h, g \in L^2(\mathbb{R}_+)$, by applying Schwartz's inequality, we obtain

$$\begin{aligned} \int_0^{\infty} |(T_\psi g(x)) h(x)| dx &= \int_0^{\infty} \frac{|h(x)|}{x} \left| \int_0^{\infty} \psi\left(\frac{y}{x}\right) g(y) e^{\frac{ia}{2b}\left(x^2 - y^2\right)} dy \right| dx \\ &\leq \int_0^{\infty} \frac{|h(x)|}{x} \int_0^{\infty} \left|\psi\left(\frac{y}{x}\right)\right| |g(y)| dy dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty |h(x)| \int_0^\infty |\psi(y)||g(xy)|dydx \\
&= \int_0^\infty |\psi(y)| \int_0^\infty |h(x)g(xy)|dxdy \\
&\leq K\|h\|_2\|g\|_2.
\end{aligned}
$$

Thus, by the converse of Schwartz's inequality, we get

$$
\|(T_\psi g)\|_2 \leq K\|g\|_2,
$$

so that $T_\psi g \in L^2(\mathbb{R}_+)$. Hence, T_ψ is a bounded linear operator on $L^2(\mathbb{R}_+)$ and $\|T_\psi\| \leq K$. □

Remark 2. It is clear that the adjoint operator T_ψ^* in (3.2) is also a bounded linear operator with

$$
\|T_\psi^*\| = \|T_\psi\| \leq K.
$$

Now, we prove the main result of this section.

Theorem 3.2. *Let $\psi \in L^1_{loc}(\mathbb{R}_+)$ be such that $\displaystyle\int_0^\infty \frac{|\psi(y)|}{\sqrt{y}}dy =: K < \infty$. If g is the linear canonical cosine transform of f, then $T_\psi g$ is the linear canonical cosine transform of $T_\psi^* f$.*

Proof. Let $f \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Using $g(x) = (L_c f)(x)$, we get

$$
\begin{aligned}
(T_\psi g)(x) &= \sqrt{\frac{2}{\pi bi}}\frac{1}{x}\int_0^\infty \psi\left(\frac{y}{x}\right) e^{\frac{ia}{2b}(x^2-y^2)} \left(\int_0^\infty f(t)e^{\frac{ia}{2b}(t^2+y^2)}\cos\left(\frac{ty}{b}\right)dt\right)dy \\
&= \sqrt{\frac{2}{\pi bi}}\frac{1}{x}\int_0^\infty \psi\left(\frac{y}{x}\right) \left(\int_0^\infty f(t)e^{\frac{ia}{2b}(x^2+t^2)}\cos\left(\frac{ty}{b}\right)dt\right)dy.
\end{aligned}
$$

Since the above integral is absolutely convergent, we change the order of integration and obtain

$$
\begin{aligned}
(T_\psi g)(x) &= \sqrt{\frac{2}{\pi bi}}\frac{1}{x}\int_0^\infty f(t) \left(\int_0^\infty \psi\left(\frac{y}{x}\right) e^{\frac{ia}{2b}(x^2+t^2)}\cos\left(\frac{ty}{b}\right)dy\right)dt \\
&= \sqrt{\frac{2}{\pi bi}}\int_0^\infty \frac{f(t)}{t} \left(\int_0^\infty \psi\left(\frac{y}{t}\right) e^{\frac{ia}{2b}(x^2+t^2)}\cos\left(\frac{xy}{b}\right)dy\right)dt \\
&= \sqrt{\frac{2}{\pi bi}}\int_0^\infty e^{\frac{ia}{2b}(x^2+y^2)}\cos\left(\frac{xy}{b}\right) \left(\int_0^\infty \frac{f(t)}{t}\psi\left(\frac{y}{t}\right) e^{\frac{ia}{2b}(t^2-y^2)}dt\right)dy \\
&= \sqrt{\frac{2}{\pi bi}}\int_0^\infty e^{\frac{ia}{2b}(x^2+y^2)}\cos\left(\frac{xy}{b}\right) T_\psi^* f(y)dy \\
&= (L_c(T_\psi^* f))(x)
\end{aligned}
$$

and we are done. □

Theorem 3.2 is also true if the linear canonical cosine transform is replaced by the linear canonical sine transform. Since the proof is similar, we only state it below.

Theorem 3.3. *Let $\psi \in L^1_{loc}(\mathbb{R}_+)$ be such that $\displaystyle\int_0^\infty \frac{|\psi(y)|}{\sqrt{y}}dy =: K < \infty$ If g is the linear canonical sine transform of f, then $T_\psi g$ is the linear canonical sine transform of $T_\psi^* f$.*

Remark 3. (i) If we take the matrix $(a, b; c, a)$ as $(\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$, then the linear canonical cosine transform reduces to the fractional cosine transform which is still a generalization of Goldberg's result [4] which can be obtained by taking $\alpha = \pi/2$.

(ii) According to Theorem 3.2, $T_\psi L_c = L_c T_\psi^*$.

(iii) Working on the similar lines as in ([4], § 5)) the condition $\psi \in L^1_{loc}(\mathbb{R}_+)$ in Theorem 3.2 can be dropped.

4 Two-dimensional Hausdorff operator

The two-dimensional Hausdorff operator is given by

$$G_{\psi_2} f(x_1, x_2) = \frac{1}{x_1 x_2} \int_0^\infty \int_0^\infty \psi_2\left(\frac{y_1}{x_1}, \frac{y_2}{x_2}\right) f(y_1, y_2) dy_1 dy_2$$

and by taking

$$\psi_2(s_1, s_2) = \frac{1}{s_1 s_2} \phi_2\left(\frac{1}{s_1}, \frac{1}{s_2}\right), \quad s_1, s_2 > 0$$

it can be written in the equivalent form as

$$H_{\phi_2} f(x_1, x_2) = \int_0^\infty \int_0^\infty \frac{\phi_2(y_1, y_2)}{y_1 y_2} f\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}\right) dy_1 dy_2.$$

Remark 4. As mentioned in Remark 1 for one-dimensional case, in two-dimensional also, the operator H_{ϕ_2} generalizes many of the known integral operators such as the following:

(i) The two-dimensional Hardy averaging operator

$$H_2 f(x_1, x_2) = \frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} f(y_1, y_2) dy_1 dy_2$$

and its dual

$$H_2^* f(x_1, x_2) = \int_{x_1}^\infty \int_{x_2}^\infty \frac{f(y_1, y_2)}{y_1 y_2} dy_1 dy_2,$$

when ϕ_2 is taken, respectively, as

$$\phi_2(t_1, t_2) = \frac{1}{t_1 t_2} \chi_{(1,\infty)\times(1,\infty)}(t_1, t_2)$$

and

$$\phi_2(t_1, t_2) = \chi_{[0,1]\times[0,1]}(t_1, t_2).$$

(ii) For $\phi_2(t_1, t_2) = \gamma(1-t_1)^{\gamma-1}(1-t_2)^{\gamma-1}\chi_{(0,1)\times(0,1)}(t_1, t_2)$, $\gamma > 0$, the operator H_{ϕ_2} becomes the two dimensional Cesáro operator

$$H_{\phi_2} f(x_1, x_2) = \gamma \int_{x_1}^\infty \int_{x_2}^\infty \frac{(y_1 - x_1)^{\gamma-1}(y_2 - x_2)^{\gamma-1}}{(y_1 y_2)^\gamma} f(y_1, y_2) dy_1 dy_2.$$

(iii) $\phi_2(t_1, t_2) = \frac{1}{t_1 t_2}\left(1 - \frac{1}{t_1}\right)^{\beta-1}\left(1 - \frac{1}{t_2}\right)^{\beta-1} \chi_{(1,\infty)\times(1,\infty)}(t_1, t_2)$, $\beta > 0$, the operator H_{ϕ_2} becomes the two-dimensional fractional Riemann Liouville operator

$$H_{\phi_2} f(x_1, x_2) = (x_1 x_2)^{-\beta} \int_0^{x_1} \int_0^{x_2} (x_1 - y_1)^{\beta-1}(x_2 - y_2)^{\beta-1} f(y_1, y_2) dy_1 dy_2.$$

Below we prove two-dimensional version of Theorem 2.1.

Theorem 4.1. *Let $\psi_2 \geq 0$, on $\mathbb{R}_+^2$ be such that*

$$\mathcal{K}_p := \int_0^\infty \int_0^\infty \frac{\psi_2(y_1, y_2)}{(y_1 y_2)^{1/p}} dy_1 dy_2 < \infty.$$

Then the operator G_{ψ_2} is a bounded operator on $L^p(\mathbb{R}_+^2)$, $1 < p < \infty$. Moreover

$$\|G_{\psi_2}\| = \mathcal{K}_p.$$

Proof. The fact that G_{ψ_2} is bounded and that $\|G_{\psi_2}\| \leq \mathcal{K}_p$ can be proved similarly as in Theorem 2.1 by using change of variables, Fubini's Theorem and Hölder's inequality in two dimensions. We prove $\|G_{\psi_2}\| \geq \mathcal{K}_p$.

Let $0 \leq f \in L^p(\mathbb{R}_+^2)$ and $0 \leq g \in L^{p'}(\mathbb{R}_+^2)$. We have

$$\begin{aligned}
\mathcal{J} &:= \int_0^\infty \int_0^\infty \psi_2(y_1, y_2) \left(\int_0^\infty \int_0^\infty f(x_1 y_1, x_2 y_2) g(x_1, x_2) dx_1 dx_2 \right) dy_1 dy_2 \\
&= \frac{1}{x_1 x_2} \int_0^\infty \int_0^\infty \psi_2\left(\frac{y_1}{x_1}, \frac{y_2}{x_2}\right) \left(\int_0^\infty \int_0^\infty f(y_1, y_2) g(x_1, x_2) dx_1 dx_2 \right) dy_1 dy_2 \\
&= \int_0^\infty \int_0^\infty g(x_1, x_2) \left(\frac{1}{x_1 x_2} \int_0^\infty \int_0^\infty \psi_2\left(\frac{y_1}{x_1}, \frac{y_2}{x_2}\right) f(y_1, y_2) dy_1 dy_2 \right) dx_1 dx_2 \\
&= \int_0^\infty \int_0^\infty g(x_1, x_2) \left(G_{\psi_2} f\right)(x_1, x_2) dx_1 dx_2.
\end{aligned}$$

Since $G_{\psi_2} : L^p(\mathbb{R}_+^2) \to L^p(\mathbb{R}_+^2)$ is bounded, by applying Hölder's inequality, we obtain

$$\mathcal{J} \leq \|G_{\psi_2}\| \|f\|_p \|g\|_{p'}. \tag{4.1}$$

Now, for $t \in (0, 1)$, define the test functions

$$f_t(x_1, x_2) = \frac{1}{(x_1 x_2)^{1/p}} \chi_{(t,1/t)\times(t,1/t)}(x_1, x_2)$$

and

$$g_t(x_1, x_2) = \frac{1}{(x_1 x_2)^{1/p'}} \chi_{(t,1/t)\times(t,1/t)}(x_1, x_2).$$

It is easy to calculate that

$$\|f_t\|_p^p = \|g_t\|_{p'}^{p'} = 4(\log(1/t))^2. \tag{4.2}$$

Next, we find that

$$\begin{aligned}
h_t(y_1, y_2) &:= \int_0^\infty \int_0^\infty f_t(x_1 y_1, x_2 y_2) g_t(x_1, x_2) dx_1 dx_2 \\
&= \frac{1}{(y_1 y_2)^{1/p}} \int_0^\infty \int_0^\infty \chi_{(t,1/t)\times(t,1/t)}(x_1 y_1, x_2 y_2) \chi_{(t,1/t)\times(t,1/t)}(x_1, x_2) \frac{dx_1 dx_2}{x_1 x_2} \\
&= \frac{1}{(y_1 y_2)^{1/p}} \int_{I_1} \int_{I_2} \frac{dx_1 dx_2}{x_1 x_2},
\end{aligned}$$

where

$$I_1 = \left(t, \frac{1}{t}\right) \cap \left(\frac{t}{y_1}, \frac{1}{t y_1}\right), \quad I_2 = \left(t, \frac{1}{t}\right) \cap \left(\frac{t}{y_2}, \frac{1}{t y_2}\right)$$

It is observed that if $y_1 \in (0,t^2) \cup \left(\frac{1}{t^2},\infty\right)$, then $I_1 = \varnothing$ and therefore, in this case $h_t(y_1,y_2) = 0$. The same is the situation if $y_2 \in (0,t^2) \cup \left(\frac{1}{t^2},\infty\right)$, since then $I_2 = \varnothing$. We deal with the remaining cases as follows.

Case 1 : $y_1, y_2 \in [t^2, 1]$. In this case, it can be worked out that

$$I_i = \left(\frac{t}{y_i}, \frac{1}{t}\right), \quad i = 1,2$$

and therefore,

$$h_t(y_1,y_2) = \frac{1}{(y_1y_2)^{1/p}}(2\log\frac{1}{t} + \log y_1)(2\log\frac{1}{t} + \log y_2).$$

Case 2 : $y_1 \in (1, \frac{1}{t^2}]$, $y_2 \in [t^2, 1]$. In this case

$$I_1 = \left(t, \frac{1}{ty_1}\right), \quad I_2 = \left(\frac{t}{y_2}, \frac{1}{t}\right)$$

so that

$$h_t(y_1,y_2) = \frac{1}{(y_1y_2)^{1/p}}(2\log\frac{1}{t} - \log y_1)(2\log\frac{1}{t} + \log y_2).$$

Case 3 : $y_1 \in [t^2, 1]$, $y_2 \in (1, \frac{1}{t^2}]$. In this case

$$I_1 = \left(\frac{t}{y_1}, \frac{1}{t}\right), \quad I_2 = \left(t, \frac{1}{ty_2}\right)$$

so that

$$h_t(y_1,y_2) = \frac{1}{(y_1y_2)^{1/p}}(2\log\frac{1}{t} + \log y_1)(2\log\frac{1}{t} - \log y_2).$$

Case 4 : $y_1, y_2 \in (1, \frac{1}{t^2}]$. In this case

$$I_i = \left(t, \frac{1}{ty_i}\right), \quad i = 1,2$$

so that

$$h_t(y_1,y_2) = \frac{1}{(y_1y_2)^{1/p}}(2\log\frac{1}{t} - \log y_1)(2\log\frac{1}{t} - \log y_2).$$

Combining the above information, we obtain that

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \psi_2(y_1,y_2)h_t(y_1,y_2)dy_1dy_2 \\
&\quad = 4\left(\log\frac{1}{t}\right)^2 \int_{t^2}^{1/t^2}\int_{t^2}^{1/t^2} \frac{\psi_2(y_1,y_2)}{(y_1y_2)^{1/p}}\left(1 - \frac{\xi(y_1)}{2\log\frac{1}{t}}\right)\left(1 - \frac{\xi(y_2)}{2\log\frac{1}{t}}\right)dy_1dy_2, \qquad (4.3)
\end{aligned}$$

where for $i = 1,2$

$$\xi(y_i) = \begin{cases} \log 1/y_i, & \text{if } 0 < y_i < 1 \\ \log y_i, & \text{if } y_i > 1. \end{cases}$$

Now, by using the test functions f_t, g_t in (4.1) and using (4.2), (4.3), we get

$$\int_{t^2}^{1/t^2}\int_{t^2}^{1/t^2} \frac{\psi_2(y_1,y_2)}{(y_1y_2)^{1/p}}\left(1 - \frac{\xi(y_1)}{2\log\frac{1}{t}}\right)\left(1 - \frac{\xi(y_2)}{2\log\frac{1}{t}}\right)dy_1dy_2 \le \|G_{\psi_2}\|$$

which on taking $t \to 0$ gives that

$$\mathcal{K}_p \le \|G_{\psi_2}\|$$

and we are done. □

5 Two-dimensional linear canonical transform

In this section, we consider two-dimensional LCT. The setting is the following. We shall consider the matrix

$$\mathscr{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $|\mathscr{M}| \neq 0$, where A, B, C, D are real diagonal matrices

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix}, \quad D = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}$$

and $|B| \neq 0$. The two-dimensional LCT of a function $f \in L^1(\mathbb{R}^2)$ is defined by

$$\mathscr{L}f(x_1, x_2) = \frac{1}{2\pi\sqrt{i|B|}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2) \exp\left[\frac{i}{2|B|}\left\{(k_1x_1^2 + k_2x_2^2) - 2(b_{22}x_1y_1 + b_{11}x_2y_2) + (p_1y_1^2 + p_2y_2^2)\right\}\right] dy_1dy_2,$$

where $k_1 = d_{11}b_{22}, k_2 = b_{11}d_{22}, p_1 = a_{11}b_{22}$ and $p_2 = a_{22}b_{11}$. For construction, properties and further results on two-dimensional LCT, one may refer to [17]. We use the same motivation as for defining the linear canonical cosine (sine) transforms in one dimension and define these notations in two dimensions as follows.

Definition 2. The two-dimensional linear canonical cosine transform of a function $f \in L^1(\mathbb{R}_+^2)$ is defined by

$$\mathscr{L}_cf(x_1, x_2) = \frac{2}{\pi\sqrt{i|B|}} \int_0^{\infty}\int_0^{\infty} f(y_1, y_2) \exp\left[\frac{i}{2|B|}\left\{(k_1x_1^2 + k_2x_2^2) + (p_1y_1^2 + p_2y_2^2)\right\}\right] \times \cos\left(\frac{b_{22}x_1y_1 + b_{11}x_2y_2}{|B|}\right) dy_1dy_2 \tag{5.1}$$

and the two-dimensional linear canonical sine transform of a function $f \in L^1(\mathbb{R}_+^2)$ is defined by

$$\mathscr{L}_sf(x_1, x_2) = \frac{2}{\pi\sqrt{i|B|}} \int_0^{\infty}\int_0^{\infty} f(y_1, y_2) \exp\left[\frac{i}{2|B|}\left\{(k_1x_1^2 + k_2x_2^2) + (p_1y_1^2 + p_2y_2^2)\right\}\right] \times \sin\left(\frac{b_{22}x_1y_1 + b_{11}x_2y_2}{|B|}\right) dy_1dy_2.$$

We assume that the matrices A and D are same., i.e. $a_{11} = d_{11}$ and $a_{22} = d_{22}$ so that $k_1 = p_1$ and $k_2 = p_2$. Therefore, (5.1) reduces to

$$\mathscr{L}_cf(x_1, x_2) = \frac{2}{\pi\sqrt{i|B|}} \int_0^{\infty}\int_0^{\infty} f(y_1, y_2) \exp\left[\frac{i}{2|B|}\left\{k_1(x_1^2 + y_1^2) + k_2(x_2^2 + y_2^2)\right\}\right] \times \cos\left(\frac{b_{22}x_1y_1 + b_{11}x_2y_2}{|B|}\right) dy_1dy_2.$$

On the same line of the proof of Theorem 3.2, we can prove the following theorem.

Theorem 5.1. *Let* $\int_0^{\infty}\int_0^{\infty} \frac{|\psi_2(y_1, y_2)|}{\sqrt{y_1y_2}} dy_1dy_2 = \mathcal{K} < \infty$. *Then a linear operator* T_{ψ_2} *on* $L^2(\mathbb{R}_+^2)$ *defined as*

$$(T_{\psi_2}g)(x_1, x_2) =$$

$$= \frac{1}{x_1x_2}\int_0^\infty\int_0^\infty \psi_2\left(\frac{y_1}{x_1},\frac{y_2}{x_2}\right) g(y_1,y_2)\exp\left[\frac{i}{2|B|}\left\{k_1(x_1^2-y_1^2)+k_2(x_2^2-y_2^2)\right\}\right]dy_1dy_2,$$

and its adjoint

$$(T^*_{\psi_2}f)(y_1,y_2) =$$
$$= \int_0^\infty\int_0^\infty \psi_2\left(\frac{y_1}{x_1},\frac{y_2}{x_2}\right)\frac{f(x_1,x_2)}{x_1x_2}\exp\left[\frac{i}{2|B|}\left\{k_1(x_1^2-y_1^2)+k_2(x_2^2-y_2^2)\right\}\right]dx_1dx_2,$$

are bounded operator and $\|T_{\psi_2}\| = \|T^*_{\psi_2}\| \le \mathcal{K}$*.*

The main result of this section is the following.

Theorem 5.2. *Let* $\psi_2 \in L^1(\mathbb{R}^2_+)$ *and* $\int_0^\infty\int_0^\infty \frac{|\psi_2(y_1,y_2)|}{\sqrt{y_1y_2}}dy_1dy_2 = \mathcal{K} < \infty$*. Let* g *be the two-dimensional linear canonical cosine transform of* f*. Then* $T_{\psi_2}g$ *is the two-dimensional linear canonical cosine transform of* $T^*_{\psi_2}f$*.*

Proof. Let $f \in L^1(\mathbb{R}^2_+)\cap L^2(\mathbb{R}^2_+)$ be any function and $g(x) = (\mathscr{L}_c f)(x)$. Now, by changing the order of variables and replacing y_i by $\frac{x_iy_i}{t_i}$, $i = 1,2$, we have

$$(T_{\psi_2}g)(x_1,x_2) =$$
$$= \frac{1}{x_1x_2}\int_0^\infty\int_0^\infty \psi_2\left(\frac{y_1}{x_1},\frac{y_2}{x_2}\right)\exp\left[\frac{i}{2|B|}\left\{k_1(x_1^2-y_1^2)+k_2(x_2^2-y_2^2)\right\}\right]g(y_1,y_2)dy_1dy_2$$
$$= \frac{2}{\pi\sqrt{i|B|}}\frac{1}{x_1x_2}\int_0^\infty\int_0^\infty \psi_2\left(\frac{y_1}{x_1},\frac{y_2}{x_2}\right)\exp\left[\frac{i}{2|B|}\left\{k_1(x_1^2-y_1^2)+k_2(x_2^2-y_2^2)\right\}\right]\int_0^\infty\int_0^\infty f(t_1,t_2)$$
$$\times\exp\left[\frac{i}{2|B|}\left\{k_1(y_1^2+t_1^2)+k_2(y_2^2+t_2^2)\right\}\right]\cos\left(\frac{b_{22}t_1y_1+b_{11}t_2y_2}{|B|}\right)dt_1dt_2dy_1dy_2$$
$$= \frac{2}{\pi\sqrt{i|B|}}\frac{1}{x_1x_2}\int_0^\infty\int_0^\infty f(t_1,t_2)\int_0^\infty\int_0^\infty \psi_2\left(\frac{y_1}{x_1},\frac{y_2}{x_2}\right)\exp\left[\frac{i}{2|B|}\left\{k_1(x_1^2+t_1^2)+k_2(x_2^2+t_2^2)\right\}\right]$$
$$\times\cos\left(\frac{b_{22}t_1y_1+b_{11}t_2y_2}{|B|}\right)dy_1dy_2dt_1dt_2$$
$$= \frac{2}{\pi\sqrt{i|B|}}\int_0^\infty\int_0^\infty \frac{f(t_1,t_2)}{t_1t_2}\int_0^\infty\int_0^\infty \psi_2\left(\frac{y_1}{t_1},\frac{y_2}{t_2}\right)\exp\left[\frac{i}{2|B|}\left\{k_1(x_1^2+t_1^2)+k_2(x_2^2+t_2^2)\right\}\right]$$
$$\times\cos\left(\frac{b_{22}x_1y_1+b_{11}x_2y_2}{|B|}\right)dy_1dy_2dt_1dt_2$$
$$= \frac{2}{\pi\sqrt{i|B|}}\int_0^\infty\int_0^\infty \exp\left[\frac{i}{2|B|}\left\{k_1(x_1^2+y_1^2)+k_2(x_2^2+y_2^2)\right\}\right]\cos\left(\frac{b_{22}x_1y_1+b_{11}x_2y_2}{|B|}\right)$$
$$\times\int_0^\infty\int_0^\infty \frac{f(t_1,t_2)}{t_1t_2}\psi_2\left(\frac{y_1}{t_1},\frac{y_2}{t_2}\right)\exp\left[\frac{i}{2|B|}\left\{k_1(t_1^2-y_1^2)+k_2(t_2^2-y_2^2)\right\}\right]dt_1dt_2dy_1dy_2$$
$$= \frac{2}{\pi\sqrt{i|B|}}\int_0^\infty\int_0^\infty \exp\left[\frac{i}{2|B|}\left\{k_1(x_1^2+y_1^2)+k_2(x_2^2+y_2^2)\right\}\right]\cos\left(\frac{b_{22}x_1y_1+b_{11}x_2y_2}{|B|}\right)$$
$$\times T^*_{\psi_2}f(y_1,y_2)dy_1dy_2$$
$$= \frac{2}{\pi\sqrt{i|B|}}\int_0^\infty\int_0^\infty T^*_{\psi_2}f(y_1,y_2)\exp\left[\frac{i}{2|B|}\left\{k_1(x_1^2+y_1^2)+k_2(x_2^2+y_2^2)\right\}\right]$$
$$\times\cos\left(\frac{b_{22}x_1y_1+b_{11}x_2y_2}{|B|}\right)dy_1dy_2$$

$$= \mathscr{L}_c(T^*_{\psi_2} f)(x_1, x_2).$$

□

Remark 5. (i) The result in Theorem 5.2 is also true for two-dimensional linear sine transform. The proof follows on the same lines.

(ii) On taking the matrix parameters $(A, B; C, A)$ as $A = (\cos\alpha_1, 0; 0, \cos\alpha_2)$, $B = (\sin\alpha_1, 0; 0, \sin\alpha_2)$ and $C = (-\sin\alpha_1, 0; 0, -\sin\alpha_2)$, the two-dimensional linear canonical cosine transform reduces to the two-dimensional fractional cosine transform. So that the result proved in Theorem 5.2 comes true in the framework of two-dimensional fractional cosine transform.

(iii) If we take $\alpha_1 = \pi/2$ and $\alpha_2 = \pi/2$, the two-dimensional fractional cosine transform reduces to two-dimensional Fourier cosine transform, so that Theorem 5.2 is multifold generalization of the result given by Goldberg [4].

(iv) In view of Theorem 5.2, we have that $T_{\psi_2}\mathscr{L}_c = \mathscr{L}_c T^*_{\psi_2}$

On taking $A = (0, 0; 0, 0), B = (1, 0; 0, 1), C = (-1, 0; 0, -1)$, the two-dimensional linear canonical cosine transform reduces to

$$F_c f(x_1, x_2) = \frac{2}{\pi\sqrt{\iota}} \int_0^\infty \int_0^\infty f(y_1, y_2) \cos(x_1 y_1 + x_2 y_2) dy_1 dy_2$$

while the operators T_{ψ_2} and $T^*_{\psi_2}$ become the two-dimensional Hausdorff operators

$$G_{\psi_2} g(x_1, x_2) = \frac{1}{x_1 x_2} \int_0^\infty \int_0^\infty \psi_2\left(\frac{y_1}{x_1}, \frac{y_2}{x_2}\right) g(y_1, y_2) dy_1 dy_2$$

and its adjoint

$$G^*_{\psi_2} f(y_1, y_2) = \int_0^\infty \int_0^\infty \psi_2\left(\frac{y_1}{x_1}, \frac{y_2}{x_2}\right) \frac{f(x_1, x_2)}{x_1 x_2} dx_1 dx_2.$$

We immediately have the following corollary of Theorem 5.2 which is two-dimensional version of Goldberg's result [4].

Corollary 5.1. *Let $\psi_2 \in L^1(\mathbb{R}^2_+)$ and $\int_0^\infty \int_0^\infty \frac{|\psi_2(y_1, y_2)|}{\sqrt{y_1 y_2}} dy_1 dy_2 < \infty$. Let g be the two-dimensional Fourier cosine transform of f. Then $G_{\psi_2} g$ is the two-dimensional Fourier cosine transform of $G^*_{\psi_2} f$.*

Note that the operator G_{ψ_2} is a special case of the operator T_{ψ_2}. According to Remark 5.4(iv), $T_{\psi_2}\mathscr{L}_c = \mathscr{L}_c T^*_{\psi_2}$. However for G_{ψ_2}, extending [4], we prove the following.

Theorem 5.3. *Let ψ_2 be defined on $(0, 1] \times (0, 1]$ with*

$$\int_0^1 \int_0^1 \frac{|\psi_2(y_1, y_2)|}{\sqrt{y_1 y_2}} dy_1 dy_2 < \infty. \tag{5.2}$$

We define

$$\psi_2(y_1, y_2) = \frac{1}{y_1 y_2} \psi_2\left(\frac{1}{y_1}, \frac{1}{y_2}\right) \text{ for } y_1, y_2 > 1. \tag{5.3}$$

Then the operator G_{ψ_2} commutes with the two-dimensional Fourier cosine(sine) transform.

Proof. It is enough to prove that G_{ψ_2} is a self adjoint operator. That is, for $0 < x_1, x_2, y_1, y_2 < \infty$, we must have

$$\frac{1}{x_1x_2}\psi_2\left(\frac{y_1}{x_1}, \frac{y_2}{x_2}\right) = \frac{1}{y_1y_2}\psi_2\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}\right)$$

i.e.

$$\psi_2\left(\frac{y_1}{x_1}, \frac{y_2}{x_2}\right) = \frac{x_1x_2}{y_1y_2}\psi_2\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}\right)$$

i.e.

$$\psi_2\left(y_1, y_2\right) = \frac{1}{y_1y_2}\psi_2\left(\frac{1}{y_1}, \frac{1}{y_2}\right) \text{ for all } 0 < y_1, y_2 < \infty.$$

By assumption ψ_2 is defined for $1 < y_1, y_2 < \infty$ and is of the form

$$\psi_2\left(y_1, y_2\right) = \frac{1}{y_1y_2}\psi_2\left(\frac{1}{y_1}, \frac{1}{y_2}\right).$$

If $0 < y_1, y_2 < 1$ then $\frac{1}{y_1}, \frac{1}{y_2} > 1$ so that

$$\psi_2\left(\frac{1}{y_1}, \frac{1}{y_2}\right) = y_1y_2\psi_2\left(y_1, y_2\right)$$

or

$$\psi_2\left(y_1, y_2\right) = \frac{1}{y_1y_2}\psi_2\left(\frac{1}{y_1}, \frac{1}{y_2}\right).$$

Consequently, the form (5.3) of ψ_2 can be extended for all $y_1, y_2 \in (0, \infty)$. Next, we have by using (5.2)

$$\begin{aligned}\int_1^\infty \int_1^\infty \frac{|\psi_2(y_1, y_2)|}{\sqrt{y_1y_2}} dy_1dy_2 &= \int_1^\infty \int_1^\infty \left|\frac{1}{y_1y_2}\psi_2\left(\frac{1}{y_1}, \frac{1}{y_2}\right)\right| \frac{1}{\sqrt{y_1y_2}} dy_1dy_2 \\ &= \int_0^1 \int_0^1 \frac{|\psi_2(y_1, y_2)|}{\sqrt{y_1y_2}} dy_1dy_2 < \infty. \end{aligned} \tag{5.4}$$

Combining (5.2) and (5.4), we get

$$\int_0^\infty \int_0^\infty \frac{|\psi_2(y_1, y_2)|}{\sqrt{y_1y_2}} dy_1dy_2 < \infty.$$

The assertion now follows from Corollary 5.1. □

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References

[1] M.J. Bastiaans, *Wigner ditribution function and its application to first-order optics*, J. Opt. Soc. Amer. 69 (1979), 1710–1716.

[2] C. Bennett, R. Sharpley *Interpolation of Operators*, Pure Applied Math., 129, Boston, MA, Academic Press, 1988.

[3] S.A. Collins, *Lens-system diffraction integral written in terms of matrix optics*, J. Opt. Soc. Amer. 60 (1970), 1168–1177.

[4] R.R. Goldberg, *Certain operators and Fourier transforms on* L^2, Proc. Amer. Math. Soc. 10 (1959), 385–390.

[5] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, 1934, xii+324 pp.

[6] P. Jain, S. Jain, R. Kumar *On fractional convolutions and distributions*, Integral Transforms and Special Functions, 26 (2015), 885–899.

[7] D.F.V. James, G.S. Agarwal, *The generalized Fresnel transform and its applications to optics*, Opt. Commun. 126 (1996), 207–212.

[8] N.K. Karapetiants, S.G. Samko, *Equations with involutive operators*, Birkhäuser Boston, Inc., Boston, MA, 2001. xxiv+427 pp

[9] A. Koc, H.M. Ozaktas, C. Candan, M.A. Kutey, *Digital computation of linear canonical transforms*, IEEE Trans. Signal Proc. 56 (2008), 2383–2394.

[10] E. Liflyand, *Hausdorff operators on Hardy spaces*, Eurasian Math. J. 4 (2013), no. 4, 101–141.

[11] M. Moshinsky, C. Quesne, *Linear canonical transformations and their unitary representations*, J. Math. Phys. 12 (1971), 1772–1783.

[12] C. Palma, V. Bagini, *Extension of the Fresnel transform to ABCD systems,* J. Opt. Soc. Amer. 14 (1997), 1774–1779.

[13] A. Prasad, Z.A. Ansari, P. Jain, *The linear canonical transform and pseudo-differential operator*, submitted.

[14] A.E. Siegman, *Lasers*, Mill Valley, CA: University Science Books, 1986.

[15] V.D. Stepanov, *On operators in* $L_p(R_n)$ *spaces which commute with shifts,* Sib. Math. J. 15 (1974), no. 3, 496–501.

[16] E.C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford, 1937.

[17] L. Zhao, J.J. Healy, J.T. Sheridan, *Two-dimensional nonseparable linear canonical transform: sampling theorem and unitary discretization*, J. Opt. Soc. Am. A. 31 (12) (2014), 2631-2641.

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