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From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

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- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

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- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.

VLADIMIR DMITRIEVICH STEPANOV

(to the 70th birthday)

Vladimir Dmitrievich Stepanov was born on December 13, 1949 in a small town Belovo, Kemerovo region. In 1966 he finished the Lavrentiev school of physics and mathematics at Novosibirsk academic town-ship and the same year he entered the Faculty of Mathematics of the Novosibirsk State University (NSU) from which he has graduated in 1971 and started to teach mathematics at the Khabarovsk Technical University till 1981 with interruption for postgraduate studies (1973-1976) in the NSU.

In 1977 he has defended the PhD dissertation and in 1985 his doctoral thesis "Integral convolution operators in Lebesgue spaces" in the S.L. Sobolev Institute of Mathematics. Scientific degree "Professor of Mathematics" was awarded to him in 1989. In 2000 V.D. Stepanov was elected a corresponding member of the Russian Academy of Sciences (RAS).

Since 1985 till 2005 V.D. Stepanov was the Head of Laboratory of Functional Analysis at the Computing Center of the Far Easten Branch of the Russian Academy of Science.

In 2005 V.D. Stepanov moved from Khabarovsk to Moscow with appointment at the Peoples Friendship University of Russia as the Head of the Department of Mathematical Analysis (retired in 2018). Also, he was hired at the V.A. Steklov Mathematical Institute of RAS at the Function Theory Department.

Research interests of V.D. Stepanov are: the theory of integral and differential operators, harmonic analysis in Euclidean spaces, weighted inequalities, duality in function spaces, approximation theory, asymptotic estimates of singular, approximation and entropy numbers of integral transformations, and estimates of the Schatten-Neumann type. Main achievements: the theory of integral convolution operators is constructed, the criteria for the boundedness and compactness of integral operators in function spaces are obtained, weighted inequalities and the behaviour of approximation numbers of the Volterra, Riemann-Liouville, Hardy integral operators are studied, etc.

Under his scientific supervision 15 candidate theses in Russia and 5 PhD theses in Sweden were successfully defended. Professor V.D. Stepanov has over 100 scientific publications including 3 monographs. Participation in scientific and organizational activities of V.D. Stepanov is well known. He is a member of the American Mathematical Society (since 1987) and a member of the London Mathematical Society (since 1996), Deputy Editor of the Analysis Mathematica, member of the Editorial Board of the Eurasian Mathematical Journal, invited speaker at many international conferences and visiting professor of universities in USA, Canada, UK, Spain, Sweden, South Korea, Kazakhstan, etc.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Vladimir Dmitrievich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF ANALYSIS, DIFFERENTIAL EQUATIONS AND ALGEBRA" (EMJ-2019), DEDICATED TO THE 10TH ANNIVERSARY OF THE EURASIAN MATHEMATICAL JOURNAL

From October 16 to October 19, 2019 at the L.N. Gumilyov Eurasian National University (ENU) the International Conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2019) was held. The conference was dedicated to the 10th anniversary of the Eurasian Mathematical Journal (EMJ).

The purposes of the conference were to discuss the current state of development of mathematical scientific directions, expand the number of potential authors of the Eurasian Mathematical Journal and further strengthen the scientific cooperation between the Faculty of Mechanics and Mathematics of the ENU and scientists from other cities of Kazakhstan and abroad.

The partner universities for the organization of the conference were the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia (the RUDN University, Moscow) and the University of Padua (Italy).

The conference was attended by more than 80 mathematicians from the cities of Almaty, Aktobe, Karaganda, Nur-Sultan, Shymkent, Taraz, Turkestan, as well as from several foreign countries: from Azerbaijan, Germany, Greece, Italy, Japan, Kyrgyzstan, Russia, Tajikistan and Uzbekistan.

The chairman of the International Programme Committee of the conference was Ye.B. Sydykov, rector of the ENU, co-chairmen were Chief editors of the EMJ: V.I. Burenkov, professor of the RUDN University, M. Otelbaev, academician of the National Academy of Sciences of the Republic of Kazakhstan (NAS RK), V.A. Sadovnichy, academician of the Russian Academy of Sciences (RAS), rector of the M.V. Lomonosov Moscow State University (MSU).

There were three sections at the conference: "Function Theory and Functional Analysis", "Differential Equations and Equations of Mathematical Physics" and "Algebra and Model Theory". 16 plenary presentations of 30 minutes each and more than 60 sectional presentations of 20 minutes each, devoted to contemporary areas of mathematics, were given.

It was decided to recommend selected reports of the participants for publication in the Eurasian Mathematical Journal and the Bulletin of the Karaganda State University (series "Mathematics").

Before the conference, a collection of abstracts of the participants' talks was published.

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Conference Schedule:

16.10.2019
09.00 – 10.00 Registration
10.00 – 10.30 Opening of the conference
10.30 – 12.50 Plenary talks
12.50 – 14.00 Lunch
14.00 – 18.00 Session talks

17.10.2019
09.30 – 12.20 Plenary talks
12.20 – 14.00 Lunch
14.00 – 18.00 Session talks
18.00 – Dinner for participants of the conference

18.10.2019
09.30 – 13.00 Plenary talks
12.20 – 14.00 Lunch
14.00 – 17.00 Excursion around the city

19.10.2019
09.30 – 12.30 Plenary talks
12.30 – 13.00 Closing of the conference

At the opening ceremony welcome speeches were given by Ye.B. Sydykov, rector of the ENU, chairman of the Program Committee of the conference; V.I. Burenkov, professor of the RUDN University, editor-in-chief of the EMJ; L. Mukasheva, official representative of the international company Clarivate Analytics in the Central Asian region; A. Ospanova, official representative of Scopus.

Plenary talks were given by

T.Sh. Kalmenov (Kazakhstan), M. Otelbaev and B.D. Koshanov (Kazakhstan), P.D. Lamberti and V. Vespri (Italy) – on 16.10.2019;

V.I. Burenkov (Russia), T. Ozawa (Japan), H. Begehr (Germany), M.A. Sadybekov and A.A. Dukenbaeva (Kazakhstan), D. Suragan (Kazakhstan) – on 17.10.2019;

M.L. Goldman (Russia), A. Bountis (Greece), A.K. Kerimbekov (Kyrgyzstan), S.N. Kharin (Kazakhstan), M.I. Dyachenko (Russia) – on 18.10.2019;

E.D. Nursultanov (Kazakhstan), M.A. Ragusa (Italy), P.D. Lamberti and V. Vespri (Italy), M.G. Gadoev (Russia) and F.S. Iskhokov (Tajikistan) – on 19.10.2019.

At the closing ceremony all participants unanimously congratulated the staff of the L.N. Gumilyov Eurasian National University and the Editorial Board of the Eurasian Mathematical Journal with the 10th anniversary of the journal and wished further creative successes.

They expressed hope that the journal will continue to play an important role in the development of mathematical science and education in Kazakhstan in the future.

V.I. Burenkov, K.N. Ospanov, A.M. Temirkhanova

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BESOV-TYPE SPACES AND DIFFERENCES

M. Hovemann, W. Sickel

Communicated by M.L. Gol'dman

Key words: Nikol'skii-Besov spaces, Morrey spaces, Besov-type spaces, characterizations by differences.

AMS Mathematics Subject Classification: 46E35.

Abstract. We study characterizations by differences of the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Our focus is on necessary and sufficient conditions on s for the validity of those characterizations.

DOI: https://doi.org/10.32523/2077-9879-2020-11-1-25-56

1 Introduction and main results

In the last few years smoothness spaces built upon Morrey spaces have attracted some attention. One variant has already been considered in the monograph of Besov, Il'in and Nikol'skii [4, Section 27], see also Netrusov [29]. However, here we are interested in a modification, so-called Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$, originally introduced by El Baraka in 2002, see [13]. We refer to the next section for the definition and more detailed comments on the literature. A rough interpretation of the role of the parameters is as follows. As in case of Nikol'skii-Besov spaces s is related to the smoothness, p is related to integrability properties and q is a fine-index. The new parameter, which we shall call *the Morrey parameter*, is not well understood at this moment. Partly τ influences smoothness, partly it influences integrability. In view of the coincidence $B^{s,0}_{p,q}(\mathbb{R}^d) = B^{s}_{p,q}(\mathbb{R}^d)$ the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ may be understood as generalizations of Nikol'skii-Besov spaces $B^{s}_{p,q}(\mathbb{R}^d)$. In the center of our approach always stands the definition in Fourier-analytical terms (for $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ as well as for $B^{s}_{p,q}(\mathbb{R}^d)$). Then an immediate question concerns the characterization by differences (and this is not only because of the fact that the historical roots can be found there, see Nikol'skii [30], Besov [1], [2]). More exactly, we shall deal with the following problem.

Under which restrictions on the parameters s, τ, p, q, d the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ can be described by using only $\Delta^N_h f(x)$?

In our answer given below there is absolutely no surprise if we restrict ourselves to the Banach space case $p, q \in [1,\infty]$. However, if $p < 1$, there is a surprising new phenomenon. To explain this, first we recall the characterization by differences of the Nikol'skii-Besov spaces. Definitions of the underlying function spaces will be recalled in Section 2, see Definition 1.

Theorem 1.1. *Let $0 < p, q \le \infty$ and let $N \in \mathbb{N}$. If*

$$d \max\Big(0, \frac{1}{p} - 1\Big) < s < N \,, \qquad (1.1)$$

then $B^s_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in L_{\max(p,1)}(\mathbb{R}^d)$ such that

$$\Big(\int_{\mathbb{R}^d} |h|^{-sq} \, \|\Delta_h^N f \, | L_p(\mathbb{R}^d)\|^q \frac{dh}{|h|^d}\Big)^{1/q} < \infty$$

(with the standard modification if $q = \infty$).

We refer to [31, 4.3.4], [5, Chapter 4] and [40, Theorem 2.5.12], [41, Theorem 3.5.3]. Let us denote by $\mathbf{B}^s_{p,q,N}(\mathbb{R}^d)$ the collection of all functions $f \in L_p(\mathbb{R}^d)$ such that

$$\| f \, |\mathbf{B}^s_{p,q,N}(\mathbb{R}^d)\| := \| f \, |L_p(\mathbb{R}^d)\| + \Big(\int_{\mathbb{R}^d} |h|^{-sq} \, \|\Delta_h^N f \, | L_p(\mathbb{R}^d)\|^q \frac{dh}{|h|^d}\Big)^{1/q} < \infty \, . \tag{1.2}$$

Then the following statement is in principal known.

Theorem 1.2. *Let $0 < p, q \le \infty$ and $N \in \mathbb{N}$. Let $s \in \mathbb{R}$ such that $s < N$.*
(i) *$B^s_{p,q}(\mathbb{R}^d)$ and $\mathbf{B}^s_{p,q,N}(\mathbb{R}^d)$ coincide if and only if (1.1) holds.*
(ii) *If (1.1) holds, then $\| \cdot \, |B^s_{p,q}(\mathbb{R}^d)\|$ and $\| \cdot \, |\mathbf{B}^s_{p,q,N}(\mathbb{R}^d)\|$ are equivalent for all $f \in L_p(\mathbb{R}^d) \cap L_1^{loc}(\mathbb{R}^d)$.*

There are easy explanations for these restrictions, which we will discuss below. But let us mention that in the case $s < \sigma_p$ Besov spaces always contain singular distributions. For example the Dirac delta distribution belongs to $B^{d/p-d}_{p,\infty}(\mathbb{R}^d)$ for $0 < p \le \infty$. Hence, a characterization of those spaces by differences does not make sense. Here our aim will be to prove a related result for the more general Besov-type spaces. The most satisfactory answer we have obtained in case $p = q$.

Theorem 1.3. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \le \tau < \frac{1}{p}$ and $N \in \mathbb{N}$.*
(i) *Then $B^{s,\tau}_{p,p}(\mathbb{R}^d)$ is the collection of all $f \in L^{loc}_{\max(p,1)}(\mathbb{R}^d)$ such that*

$$\begin{aligned} \| f \|_1 \quad &:= \quad \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \Big(\int_P |f(x)|^p dx \Big)^{\frac{1}{p}} \\ &+ \quad \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \Big(\int_0^\infty t^{-sp} \int_P \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(x)| dh \Big)^p dx \, \frac{dt}{t} \Big)^{\frac{1}{p}} \end{aligned}$$

is finite if and only if (1.1) holds. Here $\mathcal{Q}$ refers to the collection of all dyadic cubes in $\mathbb{R}^d$.
(ii) *If (1.1) holds, then $\| \cdot \|_1$ and $\| \cdot \, |B^{s,\tau}_{p,p}(\mathbb{R}^d)\|$ are equivalent on $L^{loc}_{\max(p,1)}(\mathbb{R}^d)$.*

It is really surprising that the Morrey parameter τ is not playing a role here. Let us have a closer look at the case $0 < p < 1$. It is known that

$$B^{s,\tau}_{p,q}(\mathbb{R}^d) \subset L_1^{loc}(\mathbb{R}^d) \qquad \text{if} \qquad s > d \left(\frac{1}{p} - 1 \right) - d\tau(1-p) \, ,$$

whereas in the case of Nikol'skii-Besov spaces

$$B^{s}_{p,q}(\mathbb{R}^d) \subset L_1^{loc}(\mathbb{R}^d) \qquad \text{if} \qquad s > d \left(\frac{1}{p} - 1 \right) .$$

For $d(\frac{1}{p} - 1) - d\tau(1-p) < s \le d(\frac{1}{p} - 1)$ there is a large area where the spaces $B^{s,\tau}_{p,p}(\mathbb{R}^d)$ do not contain any singular distribution but also can not be described with the quasinorm $\| \cdot \|_1$. In the case $\tau = 0$ we recover the original Besov spaces $B^s_{p,p}(\mathbb{R}^d)$. Here this gap disappears. In the $(1/p, s)$-diagram below, see Fig.1., we have tried to illustrate the result of Theorem 1.3. For convenience we have chosen $\tau = \frac{1}{p} - 1$ if $p < 1$.

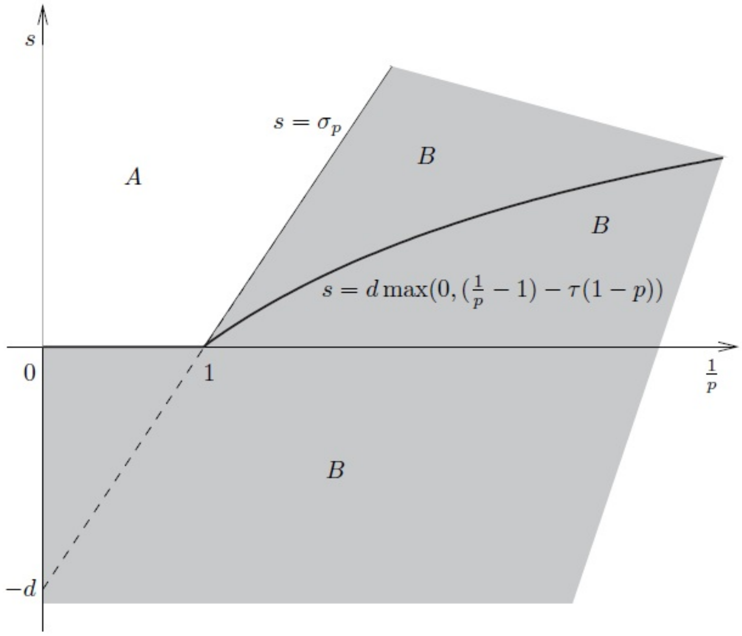

Fig. 1. $B^{s,\tau}_{p,p}(\mathbb{R}^d)$ and differences.

A: $\| \cdot \|_1$ describes $B^{s,\tau}_{p,p}(\mathbb{R}^d)$.

B: $\| \cdot \|_1$ does not describe $B^{s,\tau}_{p,p}(\mathbb{R}^d)$. On $[1,\infty)$ above of the thick line we have $B^{s,\tau}_{p,p}(\mathbb{R}^d) \subset L^{loc}_1(\mathbb{R}^d)$.

Of course we will not only look at the case $p = q$. So in the course of this paper we will prove the following result for possibly different p and q.

Theorem 1.4. *Let $0 < p < \infty$, $0 \le \tau < \frac{1}{p}$, $0 < q, v \le \infty$ and $N \in \mathbb{N}$. We suppose*

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\right) < s < N\,. \tag{1.3}$$

Then a function $f \in L^{loc}_p(\mathbb{R}^d)$ belongs to $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ if and only if $f \in L^{loc}_v(\mathbb{R}^d)$ and

$$\begin{aligned} \| f \|^{(v,1)} \;:=\; & \sup_{P\in\mathcal{Q}} \frac{1}{|P|^{\tau}} \Big(\int_P |f(x)|^p dx \Big)^{\frac{1}{p}} \\ & + \sup_{P\in\mathcal{Q}} \frac{1}{|P|^{\tau}} \Big(\int_0^1 t^{-sq} \Big(\int_P \Big(t^{-d} \int_{B(0,t)} |\Delta^N_h f(x)|^v dh \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \frac{dt}{t} \Big)^{\frac{1}{q}} \end{aligned}$$

is finite. In the case $q = \infty$ and/or $v = \infty$ the usual modifications are made. The quasi-norms $\| \cdot |B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and $\| \cdot \|^{(v,1)}$ are equivalent on $L^{loc}_p(\mathbb{R}^d)$.

In this paper we also will investigate the necessity of the restrictions in Theorem 1.4. For this purpose by $\mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$ we denote the collection of all functions $f \in L^{loc}_{\max(p,v)}(\mathbb{R}^d)$ that satisfy $\|f\|^{(v,1)} < \infty$.

Theorem 1.5. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \le \tau < \frac{1}{p}$, $0 < q, v \le \infty$ and $N \in \mathbb{N}$. Then $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$ if we are in one of the following cases:*

(i) $s \le 0$,

(ii) $s < d(\frac{1}{p} - 1) - d\tau(1-p)$ *and* $0 < p < 1$,

(iii) $s < d(\frac{1}{p} - \frac{1}{v}) - d\tau(1 - \frac{p}{v})$, $\max(p,1) < v < \infty$ *and* $B^{s,\tau}_{p,q}(B) \hookrightarrow L_1(B)$ *for some ball* $B \subset \mathbb{R}^d$,

(iv) $s \le d(\frac{1}{p} - \frac{1}{v})$ *and* $q = p \le v < \infty$,

(v) either $N < s$ *and* $0 < q \le \infty$ *or* $N = s$ *and* $0 < q < \infty$.

The additional condition $B^{s,\tau}_{p,q}(B) \hookrightarrow L_1(B)$ for some ball $B \subset \mathbb{R}^d$ in part (iii) is probably superfluous. At least, if $\tau = 0$ we know that $B^s_{p,q}(B) \hookrightarrow L_1(B)$ for some ball $B \subset \mathbb{R}^d$ is equivalent to $B^s_{p,q}(\mathbb{R}^d) \subset L_1^{loc}(\mathbb{R}^d)$ and this condition is needed to allow a characterization by differences. If we compare this result with Theorem 1.4 it becomes clear that we do not have a complete answer at this moment. For example, in the case $q \neq p < v$ with $1 \le v < \infty$ it is not clear what happens if

$$d\left(\frac{1}{p} - \frac{1}{v}\right) - d\tau\left(1 - \frac{p}{v}\right) \le s \le d\left(\frac{1}{p} - \frac{1}{v}\right).$$

Again we have tried to illustrate the situation in an $(1/p, s)$-diagram, see Fig.2. For simplicity we have chosen $v = 1$. In the diagram we assume $p \neq q$ for every p and $\tau = \frac{1}{p} - 1$ if $p < 1$. The influence of the parameter q is hidden.

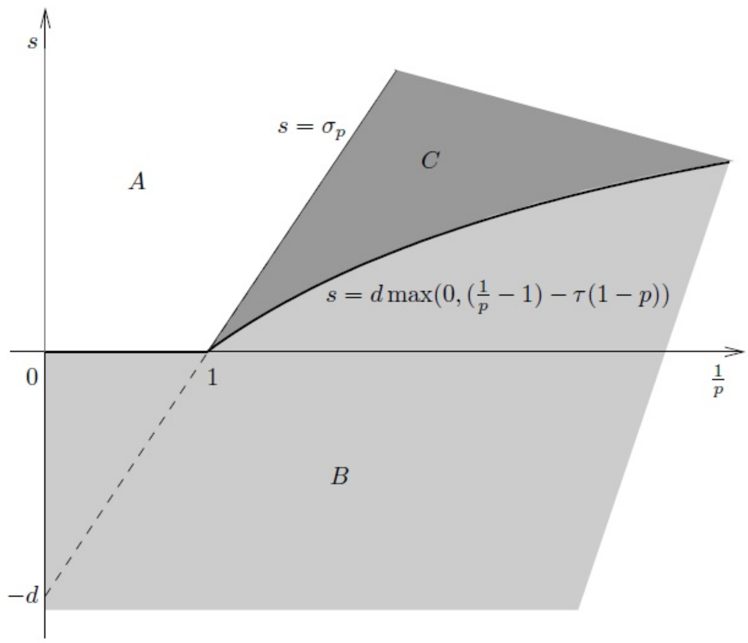

Fig. 2. Characterization in terms of differences for $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ with $q \neq p$ and $\tau = \frac{1}{p} - 1$ if $p < 1$.

A:
$B^{s,\tau}_{p,q}(\mathbb{R}^d) = \mathbf{B}^{s,\tau,N}_{p,q,1}(\mathbb{R}^d)$

B:
$B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N}_{p,q,1}(\mathbb{R}^d)$

C : open problem

The paper is organized as follows. In Section 2 we shall define the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. In addition we collect some useful properties of these classes. In Section 3 we shall prove Theorem 1.4. To do so, we make use of some ideas of Hedberg and Netrusov, see [18]. In Section 4 we will give some comments concerning the original Besov spaces. In Section 5 we deal with the necessity of some of our restrictions in the Theorems 1.3 and 1.5. Our main tool here is the method which has been developed by Christ and Seeger, see [9].

Notation

As usual $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0$ the set of all natural numbers and 0, $\mathbb{Z}$ the set of all integers and $\mathbb{R}$ the set of all real numbers. $\mathbb{R}^d$ denotes the d-dimensional Euclidean space. We put

$$B(x,t) := \{y \in \mathbb{R}^d : \quad |x-y| < t\}, \qquad x \in \mathbb{R}^d, \quad t > 0.$$

All functions are assumed to be complex-valued, i. e. we consider functions $f: \mathbb{R}^d \to \mathbb{C}$. Let $\mathcal{S}(\mathbb{R}^d)$ be the collection of all Schwartz functions on $\mathbb{R}^d$ endowed with the usual topology and denote by $\mathcal{S}'(\mathbb{R}^d)$ its topological dual, namely the space of all bounded linear functionals on $\mathcal{S}(\mathbb{R}^d)$ endowed with the weak $*$-topology. The symbol $\mathcal{F}$ refers to the Fourier transform and $\mathcal{F}^{-1}$ refers to its inverse transform. Both are defined on $\mathcal{S}'(\mathbb{R}^d)$. Almost all function spaces which we consider in this paper are subspaces of $\mathcal{S}'(\mathbb{R}^d)$, i. e. spaces of equivalence classes with respect to almost everywhere equality. However, if such an equivalence class contains a continuous representative, then usually we work with this representative and call also the equivalence class a continuous function. By $C_0^\infty(\mathbb{R}^d)$ we mean the set of all infinitely often continuously differentiable functions on $\mathbb{R}^d$ with compact support.

Given a quasi-Banach space X, the operator norm of a linear operator $T : X \to X$ is denoted by $\|T|\mathcal{L}(X)\|$. For two quasi-Banach spaces X and Y we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X into Y is continuous. For all $p \in (0,\infty]$ and $q \in (0,\infty]$ we write

$$\sigma_p := d \max\Big(0, \frac{1}{p} - 1\Big) \qquad \text{and} \qquad \sigma_{p,q} := d \max\Big(0, \frac{1}{p} - 1, \frac{1}{q} - 1\Big).$$

The symbols $C, C_1, c, c_1 \ldots$ denote positive constants that depend only on the fixed parameters d, s, τ, p, q and probably on auxiliary functions. Unless otherwise stated their values may vary from line to line. Sometimes we use the symbol "$\lesssim$" instead of "$\le$". The meaning of $A \lesssim B$ is given by: there exists a positive constant C such that $A \le C\, B$. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$. In this paper one important tool will be the differences of higher order. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a function. Then for $x, h \in \mathbb{R}^d$ we define the difference of the first order by $\Delta_h^1 f(x) := f(x+h) - f(x)$. Let $N \in \mathbb{N}$. Then we define the difference of order N by

$$\Delta_h^N f(x) := \left(\Delta_h^1 \left(\Delta_h^{N-1} f\right)\right)(x)\,, \qquad x \in \mathbb{R}^d.$$

2 Definition and basic properties of Besov-type spaces

2.1 Besov-type spaces

To define the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ we need a so-called smooth dyadic decomposition of the unity. Let $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$ be a non-negative function such that $\varphi_0(x) = 1$ if $|x| \le 1$ and $\varphi_0(x) = 0$ if $|x| \ge \frac{3}{2}$. For $k \in \mathbb{N}$ we define

$$\varphi_k(x) := \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad x \in \mathbb{R}^d.$$

Since

$$\sum_{k=0}^{\infty} \varphi_k(x) = 1\,, \qquad x \in \mathbb{R}^d\,,$$

and

$$\operatorname{supp} \varphi_k \subset \left\{x \in \mathbb{R}^d :\ 2^{k-1} \le |x| \le 3 \cdot 2^{k-1}\right\}, \qquad k \in \mathbb{N}\,,$$

we call the system $(\varphi_k)_{k\in\mathbb{N}_0}$ a smooth dyadic decomposition of the unity on $\mathbb{R}^d$. The Paley-Wiener-Schwarz theorem states that $\mathcal{F}^{-1}[\varphi_k\, \mathcal{F}f]$ with $k \in \mathbb{N}_0$ is a smooth function for any $f \in \mathcal{S}'(\mathbb{R}^d)$. Let $\mathcal{Q}$ be the collection of all dyadic cubes in $\mathbb{R}^d$, i.e.,

$$\mathcal{Q} := \{Q_{j,k} := 2^{-j}([0,1)^d + k)\ :\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}\,.$$

The symbol $l(P)$ denotes the side-length of a cube P and $j_P := -\log_2(l(P))$.

Definition 1. Let $s \in \mathbb{R}$, $0 \le \tau < \infty$ and $0 < p, q \le \infty$. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then the Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is defined to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \Bigg(\sum_{k=\max(j_P,0)}^{\infty} 2^{ksq} \Big(\int_P |\mathcal{F}^{-1}[\varphi_k\, \mathcal{F}f](x)|^p dx \Big)^{\frac{q}{p}} \Bigg)^{\frac{1}{q}} < \infty.$$

In the cases $p = \infty$ and/or $q = \infty$ the usual modifications are made.

We collect a few basic facts.

Lemma 2.1. *Let $s \in \mathbb{R}$, $\tau \geq 0$, $0 < p, q \leq \infty$ and $\varepsilon > 0$. Then the following assertions are true.*

(i) The spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ are independent of the chosen smooth dyadic decomposition of the unity in the sense of equivalent quasi-norms.

(ii) The spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ are quasi-Banach spaces.

(iii) If $\theta := \min(1, p, q)$, then
$\|f + g|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^\theta \leq \|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^\theta + \|g|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^\theta$ *for all* $f, g \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$.

(iv) $\mathcal{S}(\mathbb{R}^d) \hookrightarrow B^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.

(v) The scale $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is monotone with respect to q, namely if $q_1 \leq q_2$, then $B^{s,\tau}_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^{s,\tau}_{p,q_2}(\mathbb{R}^d)$.

(vi) The scale $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is monotone with respect to s, namely for all $q_1, q_2 \in (0, \infty]$ we have $B^{s+\varepsilon,\tau}_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^{s,\tau}_{p,q_2}(\mathbb{R}^d)$.

(vii) $B^{s,0}_{p,q}(\mathbb{R}^d) = B^{s}_{p,q}(\mathbb{R}^d)$.

Proof. For most proofs we refer to [49]. In particular, (i) can be found in Corollary 2.1, (ii) and (iii) can be found in Lemma 2.1, (iv) is proved in Proposition 2.3. Parts (v) and (vi) can be found in Proposition 2.1. Part (vii) is obvious. □

In this paper we will concentrate on the case $0 \leq \tau < \frac{1}{p}$. The following lemma tells us that most of the other cases are less interesting anyway.

Lemma 2.2. *Let $s \in \mathbb{R}$, $0 \leq \tau < \infty$, $0 < p, q \leq \infty$. Let either $q \in (0, \infty)$ and $\tau \in (\frac{1}{p}, \infty)$ or $q = \infty$ and $\tau \in [\frac{1}{p}, \infty)$. Then $B^{s,\tau}_{p,q}(\mathbb{R}^d) = B^{s+d(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^d)$.*

Proof. We refer to [48]. □

In the case $0 \leq \tau < \frac{1}{p}$ the definition of the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ can be simplified.

Lemma 2.3. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \leq \tau < \frac{1}{p}$ and $0 < q \leq \infty$. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then the Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\sharp)} := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \Big(\sum_{k=0}^{\infty} 2^{ksq} \Big(\int_P |\mathcal{F}^{-1}[\varphi_k \mathcal{F}f](x)|^p dx \Big)^{\frac{q}{p}} \Big)^{\frac{1}{q}} < \infty.$$

Moreover $\| \cdot |B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and $\| \cdot |B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\sharp)}$ are equivalent quasi-norms on $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. In the case $q = \infty$ the usual modifications should be made.

Proof. A proof can be found in [36], see Proposition 3.1. □

Hereinafter we want to collect some further properties of the Besov-type spaces. Most of them will be used in proofs later. It is interesting to know under which restrictions $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ contains singular distributions and under which conditions it does not. The following result was proved in [17], see Theorem 3.6.

Lemma 2.4. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \leq \tau \leq \frac{1}{p}$, and $0 < q \leq \infty$. Then the following assertions are true.*

(i) Let either $s > 0$ and $p \geq 1$ or $s > d(\frac{1}{p} - 1) - d\tau(1-p)$ and $p < 1$. Then we have $B^{s,\tau}_{p,q}(\mathbb{R}^d) \subset L^{loc}_1(\mathbb{R}^d)$.

(ii) Let either $s < 0$ and $p \geq 1$ or $s < d(\frac{1}{p} - 1) - d\tau(1-p)$ and $p < 1$. Then we have $B^{s,\tau}_{p,q}(\mathbb{R}^d) \not\subset L^{loc}_1(\mathbb{R}^d)$.

There is also a result concerning the limiting cases, see Theorem 3.8. in [17].

Lemma 2.5. *Let $s = 0$, $0 < p < \infty$, $0 \leq \tau < \frac{1}{p}$ and $0 < q \leq \infty$. Then $B^{0,\tau}_{p,q}(\mathbb{R}^d) \not\subset L^{loc}_1(\mathbb{R}^d)$ in the following cases:*

(i) $p \geq 2$ and $q > 2$,

(ii) $1 \leq p < 2$ and $q > p \max\left(1, \frac{1}{d(1-p\tau)}\right)$.

This can be supplemented as follows.

Lemma 2.6. *Let $0 < q \leq p < 1$, $0 \leq \tau < \frac{1}{p}$ and $s = d(\frac{1}{p} - 1) - d\tau(1-p)$. Then there exists a constant $C > 0$ such that*

$$\| f \,|L_1(\mathbb{R}^d)\| \leq C \,\| f \,|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| \tag{2.1}$$

for all $f \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ satisfying $\operatorname{supp} f \subset [-1,1]^d$.

Proof. In [17, Theorem 3.8(i)] Haroske et al. showed that under the given restrictions $B^{s,\tau}_{p,q}(\mathbb{R}^d) \subset L^{loc}_1(\mathbb{R}^d)$. Looking into the details of their proof, we find that one can sharpen their result as stated above. □

Summarizing, the line $s = s(p,\tau)$, where

$$s(p,\tau) = \begin{cases} d\left(\frac{1}{p} - 1\right) - d\tau(1-p) & \text{if} \quad 0 < p < 1; \\ \\ 0 & \text{if} \quad 1 \leq p < \infty; \end{cases}$$

represents the barrier for singular distributions within the scale $B^{s,\tau}_{p,q}(\mathbb{R}^d)$.

Later on we shall also need Besov-type spaces on domains. We concentrate on smooth and bounded domains only. Let $\mathcal{D}'(\Omega)$ denote the usual space of distributions on $\Omega \subset \mathbb{R}^d$.

Definition 2. Let $s \in \mathbb{R}$, $0 \leq \tau < \infty$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $\Omega \subset \mathbb{R}^d$ be a bounded C^∞-domain. Then we define

$$B^{s,\tau}_{p,q}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) \ : \ \exists\, g \in B^{s,\tau}_{p,q}(\mathbb{R}^d) \quad \text{such that} \quad f = g \text{ in } \Omega \right\}.$$

We define

$$\|f|B^{s,\tau}_{p,q}(\Omega)\| := \inf \left\{ \|g|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| \ : \ f = g \text{ in } \Omega \right\}.$$

For our purposes the following results concerning Besov-type spaces on domains are of interest.

Lemma 2.7. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \leq \tau < \frac{1}{p}$ and $0 < q \leq \infty$. Let $\max(p,1) < v < \infty$. Let $\Omega \subset \mathbb{R}^d$ be a bounded C^∞-domain. Then the following assertions are true.*

(i) If $s > d(\frac{1}{p} - \frac{1}{v}) - d\tau(1 - \frac{p}{v})$, then $B^{s,\tau}_{p,q}(\Omega) \hookrightarrow L_v(\Omega)$.

(ii) If $s < d(\frac{1}{p} - \frac{1}{v}) - d\tau(1 - \frac{p}{v})$, then $B^{s,\tau}_{p,q}(\Omega) \not\subset L_v(\Omega)$.

Proof. We refer to [16]. □

For $m \in \mathbb{N}$ we denote by $C^m(\mathbb{R}^d)$ the space of all bounded continuous functions for which all classical derivatives up to order m are continuous and bounded. This space will be equipped with the norm

$$\| g \,| C^m(\mathbb{R}^d) \| := \sup_{|\alpha| \le m} \sup_{x \in \mathbb{R}^d} |D^\alpha g(x)| \,.$$

Lemma 2.8. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \le \tau < \frac{1}{p}$ and $0 < q \le \infty$. Let $m \in \mathbb{N}$ be sufficiently large. Then there exists a positive constant $C(m)$ such that for all $g \in C^m(\mathbb{R}^d)$ and all $f \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ we have*

$$\| f \cdot g \,| B^{s,\tau}_{p,q}(\mathbb{R}^d) \| \le C(m) \, \| g \,| C^m(\mathbb{R}^d) \| \, \| f \,| B^{s,\tau}_{p,q}(\mathbb{R}^d) \|.$$

Proof. A proof of this result can be found in [49, Theorem 6.1]. □

2.2 Smoothness spaces built on Morrey spaces

We shall need some further function spaces related to Morrey spaces, the so-called Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$. First we recall the definition of the Morrey spaces.

Definition 3. Let $0 < p \le u < \infty$. Then the Morrey space $\mathcal{M}^u_p(\mathbb{R}^d)$ is defined to be the set of all functions $f \in L^{loc}_p(\mathbb{R}^d)$ such that

$$\|f|\mathcal{M}^u_p(\mathbb{R}^d)\| := \sup_{y \in \mathbb{R}^d, r > 0} |B(y,r)|^{\frac{1}{u} - \frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty.$$

The Morrey spaces $\mathcal{M}^u_p(\mathbb{R}^d)$ are quasi-Banach spaces (Banach spaces for $p \ge 1$). Obviously we have $\mathcal{M}^p_p(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ for all p. Moreover, for $0 < p_2 \le p_1 \le u < \infty$ we have

$$L_u(\mathbb{R}^d) = \mathcal{M}^u_u(\mathbb{R}^d) \hookrightarrow \mathcal{M}^u_{p_1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}^u_{p_2}(\mathbb{R}^d) \,. \tag{2.2}$$

Another interesting feature of Morrey spaces is given by the fact that they are nonseparable if $u \neq p$. Notice that it is also possible to define the Morrey spaces using dyadic cubes instead of balls. Now we can define the Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$.

Definition 4. Let $0 < p \le u < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $(\varphi_k)_{k \in \mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then the Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ is defined to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| := \Big\| \Big(\sum_{k=0}^{\infty} 2^{ksq} |\mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x)|^q \Big)^{\frac{1}{q}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\| < \infty.$$

In the case $q = \infty$ the usual modifications should be made.

The spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ are related in the following way.

Lemma 2.9. *Let $0 < p \le u < \infty$ and $s \in \mathbb{R}$. Then $B^{s, \frac{1}{p} - \frac{1}{u}}_{p,p}(\mathbb{R}^d) \; = \; \mathcal{E}^s_{u,p,p}(\mathbb{R}^d)$.*

Proof. This result can be found in [49, Corollary 3.3]. □

2.3 Some more spaces

Probably with the paper of Kozono and Yamazaki [22] a refreshed interest in function spaces related to Morrey spaces has started. Kozono and Yamazaki considered in particular Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$, see also Mazzucato [25] in this context. As mentioned above, El Baraka [13, 14, 15] introduced the Besov-type spaces around 2002. The spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ have been introduced in 2005 by Tang and Xu, see [39]. Later, around 2008, Yang and Yuan [46, 47] introduced Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$, relatives of the classes $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$, and therefore generalizations of the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$, extensively studied in Triebel's series of monographs [40], [41], [42], [45], see also [5] and the recent monograph by Sawano [33]. In two further books Triebel introduced two more scales related to Morrey spaces, denoted by $\mathcal{L}^r B^s_{p,q}(\mathbb{R}^d)$ and $\mathcal{L}^r F^s_{p,q}(\mathbb{R}^d)$. If $r = d(\tau - \frac{1}{p})$, these spaces coincide locally with $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^d)$, respectively, see [50] and [44]. In the meanwhile there is an increasing number of papers dealing either with generalized Morrey spaces and smoothness spaces defined on their base or modifications of Morrey spaces. In the Introduction we have already refered to the monograph by Besov, Il'in, Nikol'skii [4, 5] and Netrusov [29]. There the authors modified the Morrey spaces by replacing

$$\sup_{y\in\mathbb{R}^d, r>0} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p dx\Big)^{\frac{1}{p}}$$

by

$$\sup_{y\in\mathbb{R}^d, r>0} \min\Big(1, |B(y,r)|\Big)^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p dx\Big)^{\frac{1}{p}},$$

see Definition 3. Another modification, the so-called local Morrey-type space, was investigated in the works by Burenkov, Nursultanov [6] and Burenkov, Chigambayeva, Nursultanov [7]. For generalized Morrey spaces, where $|B(y,r)|^{\frac{1}{u}-\frac{1}{p}}$ was replaced by a function $\varphi(y,r)$, we refer to the works by Mizuhara [26], Nakai [27] and Nakamura, Noi, Sawano [28].

2.4 Some important inequalities

Later on we will need some technical lemmas. A central role will be played by the Hardy-Littlewood maximal function $\mathbf{M}f(x)$. Recall, for $f \in L_1^{loc}(\mathbb{R}^d)$ we have

$$\mathbf{M}f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy, \qquad x \in \mathbb{R}^d .$$

Most important for us will be the following vector-valued maximal inequality.

Lemma 2.10. *Let $1 < p < \infty$, $0 \le \tau < \frac{1}{p}$ and $0 < q \le \infty$. Let $\{f_j\}_{j=0}^\infty$ be a sequence of local Lebesgue integrable functions on $\mathbb{R}^d$. Then there is a constant $C > 0$ independent of $\{f_j\}_{j=0}^\infty$, such that*

$$\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \Big(\sum_{j=0}^\infty \Big(\int_P |(\mathbf{M}(f_j))(x)|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \le C \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \Big(\sum_{j=0}^\infty \Big(\int_P |f_j(x)|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \tag{2.3}$$

(with the usual modifications in the case $q = \infty$).

Proof. A proof of this result can be found in [52, Proposition 2.3]. In the case $q = \infty$ the proof is obvious. □

Remark 1. Observe that the maximal inequality stated in (2.3) is neither a consequence of

$$\| \mathbf{M} f \,|\mathcal{M}^u_p(\mathbb{R}^d)\| \le C \, \| f \,|\mathcal{M}^u_p(\mathbb{R}^d)\| \tag{2.4}$$

nor of

$$\bigg\| \Big(\sum_{j=0}^{\infty} |\mathbf{M} f_j(x)|^q \Big)^{\frac{1}{q}} \bigg| \mathcal{M}^u_p(\mathbb{R}^d) \bigg\| \le C \, \bigg\| \Big(\sum_{j=0}^{\infty} |f_j(x)|^q \Big)^{\frac{1}{q}} \bigg| \mathcal{M}^u_p(\mathbb{R}^d) \bigg\| , \tag{2.5}$$

at least if $q < \infty$. Inequality (2.4) was proved by Chiarenza, Frasca [8] and for (2.5) we refer to Tang, Xu [39].

Before we can write down the next result we have to introduce some additional notation. Let $\nu \in \mathbb{R}$. Then $H^\nu_2(\mathbb{R}^d)$ denotes a Bessel-potential space, defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ with

$$\|f|H^\nu_2(\mathbb{R}^d)\| = \|(1+|\xi|^2)^{\frac{\nu}{2}} (\mathcal{F} f)(\xi)|L_2(\mathbb{R}^d)\| < \infty \, .$$

Now we are able to quote a result obtained by Sawano and Tanaka [34, Theorem 2.4].

Lemma 2.11. *Let $0 < q \le \infty$, $0 < p \le u < \infty$, $\eta > 0$ and $\nu > \frac{1}{\eta} + \frac{d}{2}$. Let $h \in H^\nu_2(\mathbb{R}^d)$ and $R \in \mathbb{R}$ with $R > 0$. Let $f \in \mathcal{M}^u_p(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ with $\operatorname{supp} \mathcal{F} f \subset B(0,R)$. Then there is a constant $C > 0$ independent of R, h and f, such that*

$$\int_{\mathbb{R}^d} |\mathcal{F}^{-1} h(x-y) f(y)| dy \le C \, \|h(R\cdot)|H^\nu_2(\mathbb{R}^d)\| \cdot ((\mathbf{M}|f|^\eta)(x))^{\frac{1}{\eta}}$$

for all $x \in \mathbb{R}^d$.

3 The Hedberg-Netrusov approach to Besov-type spaces

In [18] Hedberg and Netrusov developed a general theory for function spaces that are related to Besov and Triebel-Lizorkin spaces. This approach can be applied to deduce a characterization of the Besov-type spaces in terms of differences. Below we will give a short summary of chapter 1 in [18]. Afterwards we shall show that Besov-type spaces represent a particular case within this theory. The starting point of the theory of Hedberg and Netrusov are quasi-Banach spaces of sequences of functions, denoted by the symbol E.

Definition 5. Let E be a quasi - Banach space of sequences of Lebesgue measurable functions on $\mathbb{R}^d$. Then on E we define a non-negative function $\|\cdot\|_E$, which should satisfy the following conditions:

(i) $\|\cdot\|_E$ has the same properties as a norm, except for the triangle inequality, which is replaced by the following property. There exist constants κ with $0 < \kappa \le 1$ and $C_E \ge 1$, such that for any family $\{F_i\}_{i=0}^j$ of elements in E and any $j \in \mathbb{N}$ one has the inequality

$$\Big\| \sum_{i=0}^{j} F_i \Big\|_E^\kappa \le C_E \sum_{i=0}^{j} \Big\| F_i \Big\|_E^\kappa .$$

(ii) The metric space $(E, \|\cdot\|_E)$ is complete.

(iii) If $\{f_i\}_{i=0}^\infty \in E$ and $\{g_i\}_{i=0}^\infty$ is a sequence of measurable functions such that $|g_i| \le |f_i|$ almost everywhere for all $i \in \mathbb{N}_0$, it follows that $\{g_i\}_{i=0}^\infty \in E$ and $\|\{g_i\}_{i=0}^\infty\|_E \le \|\{f_i\}_{i=0}^\infty\|_E$.

Based on this definition Hedberg and Netrusov introduced the classes $S(\epsilon_+,\epsilon_-,r)$ of spaces E with $\epsilon_+,\epsilon_-\in\mathbb{R}$ and $0<r<\infty$. To describe them we need some additional notation. For a sequence of functions $\{f_i\}_{i=0}^\infty$ we define the left shift S_+ and the right shift S_- by $S_+\left(\{f_i\}_{i=0}^\infty\right):=\{f_{i+1}\}_{i=0}^\infty$ and $S_-\left(\{f_i\}_{i=0}^\infty\right):=\{f_{i-1}\}_{i=0}^\infty$ with $f_{-1}=0$. Moreover, for $0<r<\infty$ and $t\geq 0$, we define the maximal function $M_{r,t}f$ and the operator $\hat{M}_{r,t}$ by

$$\hat{M}_{r,t}\left(\{f_i\}_{i=0}^\infty\right):=\{M_{r,t}f_i\}_{i=0}^\infty:=\left\{\sup_{a>0}\left(a^{-d}\int_{B(0,a)}\frac{|f_i(x+y)|^r}{(1+|y|)^{rt}}dy\right)^{\frac{1}{r}}\right\}_{i=0}^\infty.$$

Definition 6. Let $\epsilon_+,\epsilon_-\in\mathbb{R}$, $0<r<\infty$ and $t\geq 0$. The space E, satisfying $(i)-(iii)$ in Definition 5, belongs to the class $S(\epsilon_+,\epsilon_-,r,t)$, if the following conditions are satisfied.

(i) The linear operators S_+ and S_- are continuous on E and there are constants $C_1,C_2>0$ independent of j and $\{f_i\}_{i=0}^\infty$ such that for all $j\in\mathbb{N}$ we have

$$\|(S_+)^j|\mathcal{L}(E)\|\leq C_1 2^{-j\epsilon_+}\qquad\text{and}\qquad\|(S_-)^j|\mathcal{L}(E)\|\leq C_2 2^{j\epsilon_-}.$$

(ii) The operator $\hat{M}_{r,t}$ is bounded on E and there is a constant $C>0$, independent of $\{f_i\}_{i=0}^\infty$, such that $\left\|\{M_{r,t}f_i\}_{i=0}^\infty\right\|_E\leq C\left\|\{f_i\}_{i=0}^\infty\right\|_E$.

Set $S(\epsilon_+,\epsilon_-,r)=\bigcup_{t\geq 0}S(\epsilon_+,\epsilon_-,r,t)$.

Now we are able to define the function spaces denoted by $Y(E)$. Later it turns out that under particular conditions these spaces coincide with $B^{s,\tau}_{p,q}(\mathbb{R}^d)$.

Definition 7. Let $\epsilon_+,\epsilon_-\in\mathbb{R}$ and $r>0$. Let $E\in S(\epsilon_+,\epsilon_-,r)$. The space $Y(E)$ consists of all distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ which have a representation $f=\sum_{i=0}^\infty f_i$ converging in $\mathcal{S}'(\mathbb{R}^d)$ such that $\|\{f_i\}_{i=0}^\infty\|_E<\infty$, $\operatorname{supp}\mathcal{F}f_0\subset B(0,2)$ and $\operatorname{supp}\mathcal{F}f_i\subset B(0,2^{i+1})\setminus B(0,2^{i-1})$ for all $i\in\mathbb{N}$.

We put

$$\|f\|_{Y(E)}:=\inf\|\{f_i\}_{i=0}^\infty\|_E\ ,$$

where the infimum is taken over all admissible representations of f as described in Definition 7. Then $\|f\|_{Y(E)}$ is a quasi-norm and $Y(E)$ becomes a quasi-normed space. Now we are prepared to formulate an important result of Hedberg and Netrusov concerning differences. The following assertion is only one part of a more comprehensive result. For more details the reader may consult Theorem 1.1.14. in [18].

Proposition 3.1. *Let $\epsilon_+,\epsilon_->0$ and $E\in S(\epsilon_+,\epsilon_-,r)$. Let $0<r,v\leq\infty$, $N\in\mathbb{N}$ and suppose*

$$d\max\left(0,\frac{1}{r}-1,\frac{1}{r}-\frac{1}{v}\right)<\epsilon_+\qquad\textit{and}\qquad\epsilon_-<N\,.$$

Then a function $f\in L_r^{loc}(\mathbb{R}^d)$ belongs to $Y(E)$ if and only if $f\in L_v^{loc}(\mathbb{R}^d)$ and the functions

$$g_0(x)=\left(\int_{B(x,1)}|f(y)|^v dy\right)^{\frac{1}{v}}\quad\textit{and}\quad g_i(x)=2^{\frac{di}{v}}\left(\int_{B(0,2^{-i})}|\Delta_z^N f(x)|^v dz\right)^{\frac{1}{v}}\quad,i\in\mathbb{N},$$

satisfy $\|\{g_i\}_{i=0}^\infty\|_E<\infty$. The quasi-norms $\|f\|_{Y(E)}$ and $\|\{g_i\}_{i=0}^\infty\|_E$ are equivalent on $L_r^{loc}(\mathbb{R}^d)$. In the case $v=\infty$ the usual modifications should be made.

Proof. This result is just a combination of Theorem 1.1.14. and Proposition 1.1.12. from [18]. $\square$

Remark 2. A detailed study of the proof of Theorem 1.1.14. in [18] shows that it is possible to replace

$$g_0(x) = \Big(\int_{B(x,1)} |f(y)|^v dy \Big)^{\frac{1}{v}} \qquad \text{by} \qquad \tilde{g}_0(x) = |f(x)|$$

in the formulation of Proposition 3.1. To prove this fortunately almost everything in the proof of Theorem 1.1.14. in [18] can be retained unchanged. Only in the Steps 3 and 6 some minor modifications are necessary. The changes in Step 3 are trivial. In Step 6 a combination of Lemma 1.1.4. and Lemma 1.1.3. from [18] delivers the desired result.

In what follows we want to investigate how the theory of Hedberg and Netrusov is connected with the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. For that purpose we define the following space of sequences of functions.

Definition 8. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and $0 \leq \tau < \frac{1}{p}$. Let $\{f_j\}_{j=0}^\infty$ be a sequence of Lebesgue measurable functions on $\mathbb{R}^d$ which belong to $L_p^{loc}(\mathbb{R}^d)$. Then we define the space $\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)$ as the collection of all those sequences such that

$$\| \{f_j\}_{j=0}^\infty \|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \Big(\sum_{j=0}^\infty 2^{jsq} \Big(\int_P |f_j(x)|^p dx \Big)^{\frac{q}{p}} \Big)^{\frac{1}{q}} < \infty \, .$$

In the case $q = \infty$ the usual modifications are made.

Remark 3. It is not difficult to see that the spaces $\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)$ are examples for the space E in the theory of Hedberg and Netrusov. Of course the spaces $\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)$ are quasi-Banach spaces. Furthermore, with $\theta = \min(1,p,q)$ we have

$$\| \{f_j + g_j\}_{j=0}^\infty \|^\theta_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} \leq \| \{f_j\}_{j=0}^\infty \|^\theta_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} + \| \{g_j\}_{j=0}^\infty \|^\theta_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)}$$

for all sequences $\{f_j\}_{j=0}^\infty$ and $\{g_j\}_{j=0}^\infty$ of locally Lebesgue integrable functions. The proof of the lattice property is trivial.

Now we want to investigate under which conditions $\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d) \in S(\epsilon_+, \epsilon_-, r, t)$.

Lemma 3.1. *Let $s \in \mathbb{R}$, $0 < r < p < \infty$, $0 \leq \tau < \frac{1}{p}$, $0 < q \leq \infty$ and $t \geq 0$. Then $\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d) \in S(s,s,r,t)$.*

Proof. Step 1. For the shift operator S_+ we have to show that for all $j \in \mathbb{N}$ we get

$$\| (S_+)^j (\{f_i\}_{i=0}^\infty) \|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} \leq 2^{-js} \| \{f_i\}_{i=0}^\infty \|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} .$$

Since we can write

$$2^{isp} |f_{i+j}|^p = 2^{(i+j)sp} 2^{-jsp} |f_{i+j}|^p \, ,$$

this is simple. For the shift operator S_- we can argue analoguously.
Step 2. Now we want to deal with the maximal functions $M_{r,t} f_i$. We have

$$\begin{aligned} \| \{M_{r,t} f_i\}_{i=0}^\infty \|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} &= \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \Big(\sum_{i=0}^\infty 2^{isq} \Big(\int_P |M_{r,t} f_i(x)|^p dx \Big)^{\frac{q}{p}} \Big)^{\frac{1}{q}} \\ &\leq C_1 \Big[\sup_{P \in \mathcal{Q}} \frac{1}{|P|^{r\tau}} \Big(\sum_{i=0}^\infty \Big(\int_P |(\mathbf{M}(2^{is} f_i)^r)(x)|^{\frac{p}{r}} dx \Big)^{\frac{r}{p}\frac{q}{r}} \Big)^{\frac{r}{q}} \Big]^{\frac{1}{r}} . \end{aligned}$$

Here $\mathbf{M}f_i$ denotes the Hardy - Littlewood maximal function. Now we use Lemma 2.10. This is possible because of $0<r<p$ and $\tau<\frac{1}{p}$. We obtain

$$\begin{aligned}\|\{M_{r,t}f_i\}_{i=0}^{\infty}\|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} &\le C_2\left[\sup_{P\in\mathcal{Q}}\frac{1}{|P|^{r\tau}}\Big(\sum_{i=0}^{\infty}2^{isq}\Big(\int_P|f_i(x)|^p dx\Big)^{\frac{r}{p}\frac{q}{r}}\Big)^{\frac{r}{q}}\right]^{\frac{1}{r}}\\ &= C_2\,\|\{f_i\}_{i=0}^{\infty}\|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)}.\end{aligned}$$

The proof is complete. □

Proposition 3.2. *Let $s\in\mathbb{R}$, $0<r<p<\infty$, $0\le\tau<\frac{1}{p}$ and $0<q\le\infty$. Then we have*

$$Y(\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d))=B^{s,\tau}_{p,q}(\mathbb{R}^d).$$

Moreover $\|\cdot|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and $\|\cdot\|_{Y(\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d))}$ are equivalent quasi-norms.

Proof. Step 1. At first we prove $B^{s,\tau}_{p,q}(\mathbb{R}^d)\hookrightarrow Y(\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d))$. For this purpose we take $f\in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and show that all properties, which can be found in Definition 7, are fulfilled. Due to part (iv) of Lemma 2.1 we have $f\in\mathcal{S}'(\mathbb{R}^d)$. If $(\varphi_j)_{j\in\mathbb{N}_0}$ is a smooth dyadic decomposition of the unity we get $f=\sum_{j=0}^{\infty}\mathcal{F}^{-1}[\varphi_j\cdot\mathcal{F}f]$ with convergence in $\mathcal{S}'(\mathbb{R}^d)$. Now, because of the properties of the functions $(\varphi_j)_{j\in\mathbb{N}_0}$, Lemma 2.3 and Definition 1 we obtain

$$\|f\|_{Y(\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d))}\le\Big\|\big\{\mathcal{F}^{-1}[\varphi_j\cdot\mathcal{F}f]\big\}_{j=0}^{\infty}\Big\|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)}=\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\sharp)}\le C\,\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|.$$

It follows $f\in Y(\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d))$.
Step 2. Now we prove $Y(\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d))\hookrightarrow B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Let $f\in Y(\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d))$. So we have a representation $f=\sum_{i=0}^{\infty}f_i$ that satisfies all the properties written down in Definition 7. We take $\theta=\min(1,p,q)$. Then we conclude

$$\begin{aligned}\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{\theta}&\le\left[\sup_{P\in\mathcal{Q}}\frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty}2^{ksq}\Big(\int_P|\mathcal{F}^{-1}[\varphi_k\,\mathcal{F}f](x)|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}\\ &=\left[\sup_{P\in\mathcal{Q}}\frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty}2^{ksq}\Big(\int_P|\mathcal{F}^{-1}[\varphi_k\,\mathcal{F}\sum_{i=0}^{\infty}f_i](x)|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}\\ &\le C_1\max_{i\in\{-1,0,1\}}\left[\sup_{P\in\mathcal{Q}}\frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty}2^{ksq}\Big(\int_P\Big|\int_{\mathbb{R}^d}\mathcal{F}^{-1}\varphi_k(x-y)f_{k+i}(y)dy\Big|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}.\end{aligned}$$

Now we use Lemma 2.11. Choose $0<\eta<p$ and $\nu>\frac{1}{\eta}+\frac{d}{2}$. Since $\varphi_k\in\mathcal{S}(\mathbb{R}^d)$ we have $\varphi_k\in H^{\nu}_2(\mathbb{R}^d)$ for all $k\in\mathbb{N}_0$. Moreover, because of $\|\{f_k\}_{k=0}^{\infty}\|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)}<\infty$ we find $f_k\in\mathcal{M}^u_p(\mathbb{R}^d)$ for all $k\in\mathbb{N}_0$ and $\tau=\frac{1}{p}-\frac{1}{u}$. We obtain

$$\begin{aligned}&\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{\theta}\\ &\le C_2\max_{i\in\{-1,0,1\}}\left[\sup_{P\in\mathcal{Q}}\frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty}2^{ksq}\Big(\int_P\Big(\|\varphi_k(2^{k+2}\cdot)|H^{\nu}_2(\mathbb{R}^d)\|\cdot((\mathbf{M}|f_{k+i}|^{\eta})(x))^{\frac{1}{\eta}}\Big)^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}\\ &\le C_2\sup_{j\in\mathbb{N}}\|\varphi_j(2^{j+2}\cdot)|H^{\nu}_2(\mathbb{R}^d)\|^{\theta}\\ &\qquad\times\max_{i\in\{-1,0,1\}}\left[\sup_{P\in\mathcal{Q}}\frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty}2^{ksq}\Big(\int_P\Big(((\mathbf{M}|f_{k+i}|^{\eta})(x))^{\frac{1}{\eta}}\Big)^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}.\end{aligned}$$

By definition we have $\varphi_k(\xi) = \varphi_1(2^{-k+1}\xi)$, $k \in \mathbb{N}$, $\xi \in \mathbb{R}^d$. This implies that $\sup_{j\in\mathbb{N}_0} \|\varphi_j(2^{j+2}\cdot)|H_2^\nu(\mathbb{R}^d)\| < \infty$. Now since $\eta < p$ we can use Lemma 2.10 and obtain

$$\begin{aligned}\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^\theta &\le C_3 \max_{i\in\{-1,0,1\}} \Big[\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\sum_{k=0}^\infty 2^{ksq}\Big(\int_P |f_{k+i}(x)|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\Big]^\theta \\ &\le C_4 \,\|\{f_k\}_{k=0}^\infty\|^\theta_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)}.\end{aligned}$$

Therefore, we get $f \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$, since $\|\{f_k\}_{k=0}^\infty\|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} < \infty$. Moreover the calculations we did are correct for any admissible representation $f = \sum_{i=0}^\infty f_i$ and thus also for the infimum over all such representations. Hence we get $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| \le C\,\|f\|_{Y(\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^d))}$. □

If we combine Proposition 3.1 and Proposition 3.2 we obtain a characterization of $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ that uses differences.

Proposition 3.3. *Let $0 < p < \infty$, $0 \le \tau < \frac{1}{p}$, $0 < q, v \le \infty$, $N \in \mathbb{N}$ and*

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\right) < s < N\,.$$

Then a function $f \in L_p^{loc}(\mathbb{R}^d)$ belongs to $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ if and only if $f \in L_v^{loc}(\mathbb{R}^d)$ and

$$\begin{aligned}\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\clubsuit)} \quad &:= \quad \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big[\Big(\int_P\Big(\int_{B(x,1)}|f(y)|^v dy\Big)^{\frac{p}{v}}dx\Big)^{\frac{q}{p}} \\ &+ \quad \sum_{j=1}^\infty 2^{jsq}\Big(\int_P 2^{\frac{djp}{v}}\Big(\int_{B(0,2^{-j})}|\Delta_z^N f(x)|^v dz\Big)^{\frac{p}{v}}dx\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}} < \infty\end{aligned}$$

(modify if $q = \infty$ and/or $v = \infty$). The quasi-norms $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\clubsuit)}$ are equivalent for $f \in L_p^{loc}(\mathbb{R}^d)$.

Proof. This result is just a combination of Proposition 3.1 and Proposition 3.2. Firstly, we get the above proposition not for functions $f \in L_p^{loc}(\mathbb{R}^d)$ but for $f \in L_r^{loc}(\mathbb{R}^d)$ with $\max(\frac{d}{s+d}, \frac{d}{s+\frac{d}{v}}) < r < p$. Clearly, $L_p^{loc}(\mathbb{R}^d) \subset L_r^{loc}(\mathbb{R}^d)$. Hence the claimed result follows. □

Remark 4. In Proposition 3.3 we may replace the quasi-norm $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\clubsuit)}$ by

$$\begin{aligned}\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)} &:= \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big[\Big(\int_P |f(x)|^p dx\Big)^{\frac{q}{p}} \\ &+ \sum_{j=1}^\infty 2^{jsq}\Big(\int_P 2^{\frac{djp}{v}}\Big(\int_{B(0,2^{-j})}|\Delta_z^N f(x)|^v dz\Big)^{\frac{p}{v}}dx\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}}.\end{aligned}$$

To prove this, we have to combine Remark 2, Proposition 3.1 and Proposition 3.2.

Now we are able to prove Theorem 1.4. For that purpose we have to modify the quasi-norm showing up in Proposition 3.3.

Proposition 3.4. *Let p, τ, q, v, N and s as in Proposition 3.3. Then a function $f \in L_p^{loc}(\mathbb{R}^d)$ belongs to $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ if and only if $f \in L_v^{loc}(\mathbb{R}^d)$ and*

$$\begin{aligned}\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)} \quad &:= \quad \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P |f(x)|^p dx\Big)^{\frac{1}{p}} \\ &+ \quad \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_0^1 t^{-sq}\Big(\int_P\Big(t^{-d}\int_{B(0,t)}|\Delta_h^N f(x)|^v dh\Big)^{\frac{p}{v}}dx\Big)^{\frac{q}{p}}\frac{dt}{t}\Big)^{\frac{1}{q}} < \infty\end{aligned}$$

(with standard modifications if $q = \infty$ and/or $v = \infty$). The quasi-norms $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)}$ are equivalent on $L^{loc}_p(\mathbb{R}^d)$.

Proof. Step 1. At first we prove that there exists a constant $C > 0$, independent of $f \in L^{loc}_p(\mathbb{R}^d)$, such that $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)} \le C\,\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)}$. Applying the monotonicity of $\int_{B(0,t)} |\Delta^N_z f(x)|^v dz$ in t we get

$$\begin{aligned}
&\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big[\sum_{j=1}^{\infty} 2^{jsq}\Big(\int_P 2^{\frac{djp}{v}}\Big(\int_{B(0,2^{-j})} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}} \\
&\qquad \le C_1 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big[\int_0^1 t^{-sq}\Big(\int_P \Big(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}} \frac{dt}{t}\Big]^{\frac{1}{q}}.
\end{aligned}$$

This implies

$$\begin{aligned}
\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)} &\le C_2 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P |f(x)|^p dx\Big)^{\frac{1}{p}} \\
&\quad + C_2 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big[\int_0^1 t^{-sq}\Big(\int_P \Big(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}} \frac{dt}{t}\Big]^{\frac{1}{q}}.
\end{aligned}$$

Step 2. Secondly we will prove that for $f \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ there is a constant $C > 0$ independent of f such that $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)} \le C\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)}$. By Proposition 3.3 and Remark 4 we know $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)} < \infty$. Again we shall apply the monotonicity of $\int_{B(0,t)} |\Delta^N_h f(x)|^v dh$ in t. This yields

$$\begin{aligned}
&\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_0^1 t^{-sq}\Big(\int_P \Big(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}} \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\quad \le C_1 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big[\sum_{j=0}^{\infty} 2^{jsq}\Big(\int_P 2^{\frac{djp}{v}}\Big(\int_{B(0,2^{-j})} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}} \\
&\quad \le C_2\,\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)} + C_2 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P \Big(\int_{B(0,1)} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

Recall that

$$\Delta^N_h f(x) = \sum_{k=0}^{N} (-1)^{N-k}\binom{N}{k} f(x+kh)\,, \qquad x \in \mathbb{R}^d\,. \tag{3.1}$$

We use this in combination with a transformation of the coordinates and obtain

$$\begin{aligned}
&\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P \Big(\int_{B(0,1)} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}} \\
&\quad \le C_3\,\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)} + C_3 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P \Big(\int_{B(x,N)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

Next we want to cover the ball $B(x,N)$ with $(2N+1)^d$ small balls with radius one. Let $i \in \{1,2,\ldots,(2N+1)^d\}$ and w_i be appropriate displacement vectors such that

$$\bigcup_{i=1}^{(2N+1)^d} B(x+w_i,1) \supset B(x,N). \tag{3.2}$$

Then due to the translation-invariance of the Morrey quasi-norm we get

$$\begin{aligned}
\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P & \Big(\int_{B(x,N)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}} \\
& \le C_4 \sum_{i=1}^{(2N+1)^d} \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P \Big(\int_{B(x+w_i,1)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}} \\
& \le C_5 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P \Big(\int_{B(x,1)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}} \\
& \le C_6 \, \|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\clubsuit)} \le C_7 \, \|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)},
\end{aligned}$$

where in the last step we used Proposition 3.3 and Remark 4. □

This can be slightly generalized.

Theorem 3.1. *Let $0 < p < \infty$, $0 \le \tau < \frac{1}{p}$, $0 < q, v \le \infty$, $N \in \mathbb{N}$ and*

$$d \max\Big(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\Big) < s < N \, .$$

Let $1 \le a \le \infty$. Then a function $f \in L^{loc}_p(\mathbb{R}^d)$ belongs to $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ if and only if $f \in L^{loc}_v(\mathbb{R}^d)$ and

$$\begin{aligned}
\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,a)} \;:=\; & \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P |f(x)|^p dx\Big)^{\frac{1}{p}} \\
+ \; & \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_0^a t^{-sq}\Big(\int_P \Big(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}} \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty
\end{aligned}$$

(with the usual modifications if $q = \infty$ and/or $v = \infty$). The quasi-norms $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,a)}$ are equivalent on $L^{loc}_p(\mathbb{R}^d)$.

Remark 5. The letters v and a in the symbol (v, a) indicate the dependence of the concrete quasi-norm on these parameters.

Proof. It will be enough to deal with $a = \infty$. Then the case $1 < a < \infty$ will become a consequence of Proposition 3.4 and $a = \infty$. Clearly, for all $f \in L^{loc}_p(\mathbb{R}^d)$ we have $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)} \le \|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,\infty)}$. So we have to deal with the reverse inequality only. Let $f \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Proposition 3.4 implies that $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)} < \infty$. Hence, we have to estimate

$$I := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_1^\infty t^{-sq}\Big(\int_P \Big(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}} \frac{dt}{t}\Big)^{\frac{1}{q}}.$$

Again we use the monotonicity of $\int_{B(0,t)} |\Delta^N_h f(x)|^v dh$ in t and obtain

$$I \le C_1 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\sum_{j=1}^\infty 2^{-jsq} 2^{-jd\frac{q}{v}}\Big(\int_P \Big(\int_{B(0,2^j)} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}.$$

Next we use formula (3.1). It follows

$$\begin{aligned}
I \le\, & C_2 \, \|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)} \\
& + C_2 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\sum_{j=1}^\infty 2^{-jsq} 2^{-jd\frac{q}{v}}\Big(\int_P \Big(\int_{B(x,N2^j)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}.
\end{aligned}$$

The ball $B(x, N\cdot 2^j)$ can be covered by $(2N\cdot 2^j+1)^d$ small balls with radius one. Let $i \in \{1, 2, \ldots, (2N\cdot 2^j + 1)^d\}$ and w_i be appropriate displacement vectors such that

$$\bigcup_{i=1}^{(2N\cdot 2^j+1)^d} B(x + w_i, 1) \supset B(x, N \cdot 2^j).$$

We obtain

$$\begin{aligned} I &\le C_2 \, \|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)} \\ &\quad + C_2 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big(\int_P \Big(\sum_{i=1}^{(2N\cdot 2^j+1)^d} \int_{B(x+w_i,1)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}. \end{aligned}$$

Now we put $\mu := \min(p, v)$. In what follows we shall use the triangle inequality as well as

$$\Big(\sum_{i=1}^{M} a_i\Big)^{\alpha} \le \sum_{i=1}^{M} a_i^{\alpha}\,.$$

Here $\alpha \in (0, 1)$ and $a_i \ge 0$ for all i. It will be important that $\frac{p}{\mu} \ge 1$ and $\frac{\mu}{v} \le 1$. At the end we reach

$$\begin{aligned} &\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big(\int_P \Big(\sum_{i=1}^{(2N\cdot 2^j+1)^d} \int_{B(x+w_i,1)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \\ &\quad\le C_3 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big[\sum_{i=1}^{(2N\cdot 2^j+1)^d} \Big(\int_P \Big(\int_{B(x+w_i,1)} |f(z)|^v dz\Big)^{\frac{p\mu}{v\mu}} dx\Big)^{\frac{\mu}{p}}\Big]^{\frac{q}{\mu}}\Big)^{\frac{1}{q}} \\ &\quad\le C_3 \Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big[\sum_{i=1}^{(2N\cdot 2^j+1)^d} \sup_{P\in\mathcal{Q}} \frac{1}{|P|^{\tau\mu}}\Big(\int_P \Big(\int_{B(x+w_i,1)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{\mu}{p}}\Big]^{\frac{q}{\mu}}\Big)^{\frac{1}{q}} \\ &\quad\le C_4 \Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} 2^{jd\frac{q}{\mu}} \Big[\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P \Big(\int_{B(x,1)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}}\Big]^{q}\Big)^{\frac{1}{q}}, \end{aligned}$$

where we used the translation invariance of the Morrey norm. Since $\mu = \min(p, v)$ and $s > d\max(0, \frac{1}{p} - \frac{1}{v})$ the series converges. Finally we get

$$\begin{aligned} &\Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} 2^{jd\frac{q}{\mu}} \Big[\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P \Big(\int_{B(x,1)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}}\Big]^{q}\Big)^{\frac{1}{q}} \\ &\qquad\qquad \le C_5 \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P \Big(\int_{B(x,1)} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}} \\ &\qquad\qquad \le C_6 \, \|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\clubsuit)} \le C_7 \, \|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)}. \end{aligned}$$

In the last step we used Proposition 3.3 and Proposition 3.4. $\square$

In the literature there exist some more characterizations of $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ in terms of differences. They look slightly different. For convenience of the reader we recall a special case, see Theorem 4.7. in [49] and Theorem 4.2. in [12]. We shall use the following notation. We say that a Lebesgue measurable function f belongs to $L^\tau_p(\mathbb{R}^d)$ if

$$\|\, f \,|L^\tau_p(\mathbb{R}^d)\| := \sup_{P\in\mathcal{Q},\ |P|\ge 1} \frac{1}{|P|^\tau}\Big(\int_P |f(x)|^p dx\Big)^{\frac{1}{p}} < \infty\,. \tag{3.3}$$

Proposition 3.5. *Let $0 < q \le \infty$ and $N \in \mathbb{N}$.*
(i) *Let $1 \le p < \infty$, $0 \le \tau < \frac{1}{p}$ and $0 < s < N$. Then a function $f \in L_p^{loc}(\mathbb{R}^d)$ belongs to $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ if and only if $f \in L_p^\tau(\mathbb{R}^d)$ and*

$$\begin{aligned} \| f | B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^* &:= \| \, f \, | L_p^\tau(\mathbb{R}^d)\| \\ &+ \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \Big(\int_0^{2\min(l(P),1)} t^{-sq} \sup_{|h|\le t} \Big\| \Delta_h^N f(\,\cdot\,) \Big| L_p(P)\Big\|^q \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty \end{aligned}$$

(with standard modifications if $q = \infty$). The quasi-norms $\| f | B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and $\| f | B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^*$ are equivalent on $L_p^\tau(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$.
(ii) *Let $0 < p < 1$, $0 \le \tau < \frac{1}{p}$ and $d \max(0, \frac{1}{p} - 1) < s < N$. Let $\sigma_p < s_0 < s$. Then a function $f \in L_p^{loc}(\mathbb{R}^d)$ belongs to $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ if and only if*

$$\sup_{P\in\mathcal{Q},\ |P|\ge 1} \frac{\| \, f \, | B^{s_0}_{p,\infty}(\mathbb{R}^d)\|}{|P|^\tau} < \infty$$

and $\| f | B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^* < \infty$ (with standard modifications if $q = \infty$). The quasi-norms $\| f | B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and $\| f | B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^* + \sup_{P\in\mathcal{Q},\ |P|\ge 1} \frac{\| \, f \, | B^{s_0}_{p,\infty}(\mathbb{R}^d)\|}{|P|^\tau}$ are equivalent on $L_p^\tau(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$.

Remark 6. Proposition 3.5(i) is in some sense satisfactory. It is quite close to the classical characterization of $B^s_{p,q}(\mathbb{R}^d)$ by means of the modulus of smoothness. Part (ii) was understood as a first attempt to characterize Besov-type spaces by differences in the case $p < 1$. Here, in this paper, we chose the version given in Theorem 3.1, since in this variant we have a better understanding of the necessary conditions. It is an interesting open problem to find necessary and sufficient conditions for the validity of the characterizations in Proposition 3.5.

4 Besov spaces and differences

We shall prove Theorem 1.2.

Proof. The equivalence of the Fourier-analytic definition with the characterization by differences can be found in many works. We refer to [40, Theorem 2.5.12] and [41, Theorem 3.5.3]. Concerning necessity we mention the following easy explanations for the restrictions with respect to s.

- Let $0 < p < 1$ and $s < d(\frac{1}{p} - 1)$. Then the Dirac-Delta distribution belongs to $B^s_{p,q}(\mathbb{R}^d)$, we refer to [40, Remark 2.5.3/3] or [32, Remark 2.2.4/3], and a characterization by differences makes no sense. The same argument applies to the limiting case $0 < p \le 1$, $s = d(\frac{1}{p} - 1)$ and $q = \infty$.

- Let $0 < p < 1$, $0 < q < \infty$ and $s = d(\frac{1}{p} - 1)$. Then a detailed investigation of the behavior of the operator norms of the mappings $T_k : \ f \to f(2^k \cdot)$, $k \in \mathbb{N}$, with respect to Besov spaces yields a different picture than for the norm given in (1.2). More exactly,

$$\lim_{k\to\infty} \frac{\| \, T_k \, | \mathbf{B}^s_{p,q,N}(\mathbb{R}^d) \to \mathbf{B}^s_{p,q,N}(\mathbb{R}^d)\|}{\| \, T_k \, | B^s_{p,q}(\mathbb{R}^d) \to B^s_{p,q}(\mathbb{R}^d)\|} = \infty \, ,$$

  see the work by Schneider [35].

- The case $s = 0$, $N = 1$ and $1 \le p, q < \infty$ has been investigated by Besov [3]. He proved that the class $\mathbf{B}^0_{p,q,1}(\mathbb{R}^d)$ is a proper subspace of $B^0_{p,q}(\mathbb{R}^d)$. This can be extended to higher order differences as well, see Lemma 5.2 below. In addition let us mention that $B^0_{p,\infty}(\mathbb{R}^d)$, $1 \le p \le \infty$, contains singular distributions, see [37], and therefore

$$\mathbf{B}^0_{p,q,N}(\mathbb{R}^d) \neq B^0_{p,q}(\mathbb{R}^d)\,, \qquad N \in \mathbb{N}\,, \quad 1 \le p \le \infty\,, \quad 0 < q \le \infty\,.$$

The proof is complete. □

We have left open the case $s = N$ for some $N \in \mathbb{N}$. Here we were not able to close the gap completely. However, some partial results are known.

- Let $s = N$ and $0 < q < \infty$. It is not difficult to see, that there exists a function $f \in C_0^\infty(\mathbb{R}^d)$ such that $\| f \,|\mathbf{B}^s_{p,q,N}(\mathbb{R}^d)\| = \infty$, see Proposition 5.5 below for further details. Hence $\mathbf{B}^N_{p,q,N}(\mathbb{R}^d) \neq B^N_{p,q}(\mathbb{R}^d)$.

- Let $s = N$ and $q = \infty$. It is well-known that

$$\mathbf{B}^N_{p,\infty,N}(\mathbb{R}^d) = W^N_p(\mathbb{R}^d)\,, \qquad 1 < p < \infty\,, \tag{4.1}$$

see Nikol'skii [31, Theorem 4.8] or Stein [38, Chapter 5, Section 3, Proposition 3]. However the Sobolev space $W^N_p(\mathbb{R}^d)$ is a proper subspace of $B^N_{p,\infty}(\mathbb{R}^d)$. Hence we have $\mathbf{B}^N_{p,\infty,N}(\mathbb{R}^d) \neq B^N_{p,\infty}(\mathbb{R}^d)$ for $1 < p < \infty$.

- Let $p = 1$ and $N = 1$. Then one also knows that $\mathbf{B}^1_{1,\infty,1}(\mathbb{R})$ does not coincide with $B^1_{1,\infty}(\mathbb{R})$. We refer to [31, Theorem 4.8.2] or [11, Chapter 2, Theorem 9.3].

5 Besov-type spaces and differences: necessary conditions

In Theorem 1.4 we have the following restrictions for s:

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\right) < s < N\,. \tag{5.1}$$

In this section our main goal is to investigate whether these conditions are not only sufficient but also necessary. For this purpose we will define the following function spaces.

Definition 9. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, $0 \le \tau < \frac{1}{p}$, $0 < v \le \infty$ and $N \in \mathbb{N}$. Then $\mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$ is the collection of all $f \in L^{loc}_{\max(p,v)}(\mathbb{R}^d)$ such that $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)}$ is finite.

Let us mention that the classes $\mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$ are quasi-Banach spaces. In what follows we will investigate in which cases we have $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$. To answer this question we will use quite a number of different techniques. Therefore it seems to be reasonable to examine each condition separately.

5.1 The necessity of $s > d\left(\frac{1}{p} - \frac{1}{v}\right)$

We start with some preliminaries.

Proposition 5.1. *Let $s \in \mathbb{R}$, $0 < p < 1$, $0 < q \leq \infty$, $0 \leq \tau < \frac{1}{p}$, $0 < v \leq \infty$, $N \in \mathbb{N}$ with $N > s$ and*

$$s < d\left(\frac{1}{p} - 1\right) - d\tau(1-p).$$

Then $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$.

Proof. In the case $s < d(\frac{1}{p} - 1) - d\tau(1-p)$ with $0 < p < 1$ the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ contain singular distributions, see Lemma 2.4. So a characterization of $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ in terms of differences is not possible. Therefore $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$. $\square$

In the case $\max(p,1) < v < \infty$ this result can be improved. For that purpose we have to work with Besov-type spaces on domains.

Proposition 5.2. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $0 \leq \tau < \frac{1}{p}$, $N \in \mathbb{N}$ and $s < N$. Let $\max(p,1) < v < \infty$. Furthermore we assume that*

$$s < d\left(\frac{1}{p} - \frac{1}{v}\right) - d\tau\left(1 - \frac{p}{v}\right)$$

and $B^{s,\tau}_{p,q}(B) \hookrightarrow L_1(B)$ for some ball $B \subset \mathbb{R}^d$. Then $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$ follows.

Proof. In this proof we will apply some ideas from [20, 5.3]. We will argue by contradiction. Our first assumption is $B^{s,\tau}_{p,q}(\mathbb{R}^d) = \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$ as sets. Then $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ can not contain singular distributions. Hence, $B^{s,\tau}_{p,q}(\mathbb{R}^d) \subset L_1^{loc}(\mathbb{R}^d)$ follows. Our second assumption is a sharpening of the first one. We assume that the identity $Id : \mathbf{B}^{s,\tau,N}_{p,q,v}(B(0,\frac{1}{8N})) \to B^{s,\tau}_{p,q}(B(0,\frac{1}{8N}))$ is a continuous operator. Here $\mathbf{B}^{s,\tau,N}_{p,q,v}(B(0,\frac{1}{8N}))$ is defined as the set of all $f \in \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$ satisfying $\operatorname{supp} f \subset B(0,\frac{1}{8N})$. First we will disprove the second assumption, afterwards the first assumption.

Step 1. Let f be a function satisfying $f \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $\operatorname{supp} f \subset B(0,\frac{1}{4N})$. We will prove that there is a constant $C > 0$ independent of f such that

$$\| f | L_v(\mathbb{R}^d) \| \leq C \, \| f | B^{s,\tau}_{p,q}(\mathbb{R}^d) \|. \tag{5.2}$$

Our assumption implies

$$\begin{aligned} \| f | B^{s,\tau}_{p,q}(\mathbb{R}^d) \| &\geq C_1 \, \| f | B^{s,\tau}_{p,q}(\mathbb{R}^d) \|^{(v,1)} \\ &\geq C_1 \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \Big(\int_0^1 t^{-sq} \Big(\int_P \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \frac{dt}{t} \Big)^{\frac{1}{q}}. \end{aligned}$$

Next we replace the supremum by selecting a dyadic cube P^* as small as possible such that $B(0,\frac{N+1}{4}) \subset P^*$. Then we obtain

$$\begin{aligned} \| f | B^{s,\tau}_{p,q}(\mathbb{R}^d) \| &\geq C_1 \frac{1}{|P^*|^\tau} \Big(\int_0^1 t^{-sq} \Big(\int_{P^*} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \frac{dt}{t} \Big)^{\frac{1}{q}} \\ &\geq C_2 \Big(\int_0^1 t^{-sq} \Big(\int_{B(0,\frac{N+1}{4})} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \frac{dt}{t} \Big)^{\frac{1}{q}} \\ &\geq C_2 \Big(\int_{\frac{N+2}{4N}}^1 t^{-sq} \Big(\int_{\frac{N}{4} < |x| < \frac{N+1}{4}} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \frac{dt}{t} \Big)^{\frac{1}{q}}. \end{aligned}$$

Now we use that for $\frac{N}{4} < |x| < \frac{N+1}{4}$ and $t \geq \frac{N+2}{4N}$ we have $B(\frac{-x}{N}, \frac{1}{4N}) \subset B(0,t)$. Hence we get

$$\begin{aligned}
\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| &\geq C_2\Big(\int_{\frac{N+2}{4N}}^1 t^{-sq}\Big(\int_{\frac{N}{4}<|x|<\frac{N+1}{4}} \Big(t^{-d}\int_{B(\frac{-x}{N},\frac{1}{4N})} |\Delta_h^N f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}} \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\geq C_2\Big(\int_{\frac{N+2}{4N}}^1 t^{-sq-d\frac{q}{v}-1} dt \Big(\int_{\frac{N}{4}<|x|<\frac{N+1}{4}} \Big(\int_{B(\frac{-x}{N},\frac{1}{4N})} |\Delta_h^N f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \\
&\geq C_3\Big(\int_{\frac{N}{4}<|x|<\frac{N+1}{4}} \Big(\int_{B(\frac{-x}{N},\frac{1}{4N})} |\Delta_h^N f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

Recall that

$$\Delta_h^N f(x) = \sum_{k=0}^N (-1)^{N-k}\binom{N}{k} f(x+kh).$$

With $\frac{N}{4} < |x| < \frac{N+1}{4}$ and $h \in B(\frac{-x}{N}, \frac{1}{4N})$, taking also into account $\operatorname{supp} f \subset B(0, \frac{1}{4N})$, we conclude that $f(x+kh) = 0$ for $k \in \{0, 1, \ldots, N-1\}$. This yields

$$\begin{aligned}
\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| &\geq C_4\,\Big(\int_{\frac{N}{4}<|x|<\frac{N+1}{4}} \Big(\int_{B(\frac{-x}{N},\frac{1}{4N})} |f(x+Nh)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}} \\
&\geq C_5\,\Big(\int_{\frac{N}{4}<|x|<\frac{N+1}{4}} \Big(\int_{B(0,\frac{1}{4})} |f(z)|^v dz\Big)^{\frac{p}{v}} dx\Big)^{\frac{1}{p}} \\
&\geq C_6\,\|f|L_v(\mathbb{R}^d)\|.
\end{aligned}$$

Hence, inequality (5.2) is proved.
Step 2. We shall need Besov-type spaces on smooth and bounded domains, more exactly on $B(0, \frac{1}{8N})$, see Definition 2. We claim

$$B^{s,\tau}_{p,q}\Big(B(0,\frac{1}{8N})\Big) \hookrightarrow L_v\Big(B(0,\frac{1}{8N})\Big). \tag{5.3}$$

Let $f \in B^{s,\tau}_{p,q}(B(0,\frac{1}{8N}))$. By definition, there is a function $g \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ with $f = g$ on $B(0, \frac{1}{8N})$. Since $\|\,g\,|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)} < \infty$, locally on $B(0,\frac{1}{8N})$, we can understand f as a pointwise defined function. Now we take a sequence $(h_l)_{l=1}^\infty \subset B^{s,\tau}_{p,q}(\mathbb{R}^d)$ with $h_l = f$ on $B(0, \frac{1}{8N})$ for every $l \in \mathbb{N}$ such that

$$\lim_{l\to\infty} \|h_l|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| = \|f|B^{s,\tau}_{p,q}(B(0,\frac{1}{8N}))\|.$$

Next we define a smooth cut-off function $\Psi \in C_0^\infty(\mathbb{R}^d)$ such that $\Psi(x) = 1$ on $B(0, \frac{1}{8N})$ and $\Psi(x) = 0$ for every x with $|x| \geq \frac{1}{4N}$. Then for every $l \in \mathbb{N}$ we get

$$\|\,f\,|L_v(B(0,\frac{1}{8N}))\| \leq \|\,h_l\cdot\Psi\,|L_v(\mathbb{R}^d)\|\,.$$

By Lemma 2.8 we know that $h_l \cdot \Psi \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Moreover $\operatorname{supp}(h_l \cdot \Psi) \subset B(0, \frac{1}{4N})$. Now we apply formula (5.2) with respect to the product $h_l \cdot \Psi \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and get

$$\|f|L_v(B(0,\frac{1}{8N}))\| \leq C_1\|h_l\cdot\Psi|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|.$$

Here C_1 is the same constant as in (5.2) and therefore independent of f, h_l and Ψ. Lemma 2.8 yields

$$\begin{aligned}
\|f|L_v(B(0,\frac{1}{8N}))\| &\leq C_2\,\|\Psi|C^m(\mathbb{R}^d)\|\;\|h_l|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| \\
&\leq C_3\,\|h_l|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|
\end{aligned}$$

with $m \in \mathbb{N}$ sufficiently large. If l tends to infinity we find

$$\| f | L_v(B(0, \frac{1}{8N}))\| \le C_3 \, \| f | B^{s,\tau}_{p,q}(B(0, \frac{1}{8N}))\|$$

as claimed.
Step 3. Lemma 2.7 implies $B^{s,\tau}_{p,q}(B(0, \frac{1}{8N})) \not\subset L_v(B(0, \frac{1}{8N}))$ in the case $s < d(\frac{1}{p} - \frac{1}{v}) - d\tau(1 - \frac{p}{v})$ which is in contradiction with (5.3). Hence, our assumption on the continuity of the identity must be wrong.

Now let $(f_j)_j$ be a convergent sequence in $\mathbf{B}^{s,\tau,N}_{p,q,v}(B(0, \frac{1}{8N}))$ with limit $f \in \mathbf{B}^{s,\tau,N}_{p,q,v}(B(0, \frac{1}{8N}))$. In addition we assume $\lim_{j\to\infty} f_j = g$ in $B^{s,\tau}_{p,q}(B(0, \frac{1}{8N}))$. Our first assumption implies convergence in $L_p(\mathbb{R}^d)$, see Theorem 3.1 and Definition 9. This yields convergence almost everywhere for an appropriate subsequence $(f_{j_\ell})_\ell$. The second assumption, applied to this subsequence, yields existence of extensions h_{j_ℓ} of $f_{j_\ell} - g$ such that

$$\lim_{\ell\to\infty} \| \, h_{j_\ell} \, | B^{s,\tau}_{p,q}(\mathbb{R}^d)\| = 0 \, .$$

Without loss of generality we may assume $\operatorname{supp} h_{j_\ell} \subset [-1,1]^d$. Because of our assumption $B^{s,\tau}_{p,q}(B) \hookrightarrow L_1(B)$ for some ball $B \subset \mathbb{R}^d$ we find

$$\lim_{\ell\to\infty} \| \, f_{j_\ell} - g \, | L_1(B(0, \frac{1}{8N}))\| = 0.$$

Here we have used that $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $L_1(\mathbb{R}^d)$ have translation invariant quasi-norms and that a scaling $f \mapsto f(\lambda\,\cdot\,)$ will not influence the membership in these spaces as long as $0 < c \le \lambda \le C < \infty$ for some fixed c, C. By switching to a further subsequence if necessary we conclude $f = g$ almost everywhere. Thus, we have proved that the identity $Id : \mathbf{B}^{s,\tau,N}_{p,q,v}(B(0, \frac{1}{8N})) \to B^{s,\tau}_{p,q}(B(0, \frac{1}{8N}))$ is a closed linear operator. The Closed Graph Theorem, which remains to hold for quasi-Banach spaces, yields that Id must be continuous. This contradicts our previous conclusion. Therefore our assumption concerning the equality of the sets also must be wrong. This proves $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$ as claimed. $\square$

In the special case $p = q$ there is an essential improvement of Proposition 5.2. For that purpose we will use ideas of Christ and Seeger, see [9] and [10] as well.

Proposition 5.3. *Let $s \in \mathbb{R}$, $0 < q = p \le v < \infty$ and $0 \le \tau < \frac{1}{p}$. Let $N \in \mathbb{N}$ with $N > s$. Furthermore we have*

$$s \le d\left(\frac{1}{p} - \frac{1}{v}\right).$$

Then $B^{s,\tau}_{p,p}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N}_{p,p,v}(\mathbb{R}^d)$.

Proof. Our argument relies on [9, 6.2], see also [10, 6.2]. However, Christ and Seeger concentrated on the special case $v = \infty$ and gave only a comment on $0 < v < \infty$. For our purpose almost everything what Christ and Seeger did there can be taken over unchanged. Only very few modifications were necessary. Therefore we will be rather rough here. Without a further reading of [9, 10] and [20, 5.4] (where the adaption to Morrey-type spaces is described) the following few lines will maybe not be understandable.
Step 1. In their proof Christ and Seeger constructed a random function and made estimations for the expected value of the different quasi-norms of this function. Here we will work with the same random function. Moreover, we will use the same notation as there. Let $\eta \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp} \mathcal{F}\eta \subset \{\xi \in \mathbb{R}^d \, : \, \frac{1}{2} < |\xi| < 1\}$ and $|\eta(x)| \ge 1$ for $|x| \le 2^{-M+d+2}$ with $M \in \mathbb{N}$. Next we need a function $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \phi \subset [-2^{-2M-4}, 2^{-2M-4}]^d$ such that

$$|\phi * \eta(z)| \ge C(M) > 0 \qquad \text{if} \quad |z| \le 2^{-M+d+1}. \tag{5.4}$$

Let $R \in \mathbb{N}$ be sufficiently large and choose a large number $W \in \mathbb{N}$. For $k \in \mathbb{N}$ we define $n_k := kR$ and $r_k := 2^{n_k - M}$. Furthermore we define the functions

$$\phi_k(x) := r_k^d\, \phi(r_k x) \qquad \text{and} \qquad \eta_k(x) := r_k^d\, \eta(r_k x)\,, \qquad x \in \mathbb{R}^d.$$

Put $\alpha := 2^{-Wd}$. For $n \in \mathbb{N}_0$ let $\mathcal{Q}(n)$ be the set of all dyadic cubes with side length 2^{-n} that are located in $[0,1)^d$. For every cube Q let χ_Q be the corresponding indicator function. Let Ω be a probability space with a probability measure μ. There is a family $\{\theta_{Q,\alpha}\}$ of independent random variables, indexed by the dyadic cubes Q and having the following property: each of these random variables takes the value 1 with the probability α and the value 0 with the probability $1-\alpha$. We consider random functions

$$h_k^{\omega,\alpha}(x) := \sum_{Q \in \mathcal{Q}(n_k)} \theta_{Q,\alpha}(\omega) \chi_Q(x)\,, \qquad x \in \mathbb{R}^d\,.$$

These functions are supported in $[0,1]^d$ and for all x we have $h_k^{\omega,\alpha}(x) \in \{0,1\}$. Now we define

$$g_k^{\omega,\alpha}(x) := (\eta_k * h_k^{\omega,\alpha})(x)\,,\ G_k^{\omega,\alpha}(x) := 2^{-n_k s} g_k^{\omega,\alpha}(x) \quad \text{and} \quad G^{\omega,\alpha}(x) := \sum_{k=1}^{2^{Wd}} G_k^{\omega,\alpha}(x).$$

For more details concerning the used notation we refer to the Sections 2 and 4 and Subsection 6.2. from [10].

Step 2. Now let $u = \frac{1}{\frac{1}{p}-\tau}$. We will prove that there is a constant $C_1 > 0$, independent of R and W, such that

$$\Big(\int_\Omega \|G^{\omega,\alpha} | B^{s,\tau}_{p,p}(\mathbb{R}^d)\|^u d\mu(\omega)\Big)^{\frac{1}{u}} < C_1.$$

Therefore we can use Lemma 6.2.1. in [10] which we now recall.

Lemma 5.1. *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q < \infty$. Then there is a constant $C > 0$ which only depends on p, q, N and d such that*

$$\Big(\int_\Omega \|G^{\omega,\alpha} | F^{s}_{p,q}(\mathbb{R}^d)\|^p d\mu(\omega)\Big)^{\frac{1}{p}} \le C.$$

Since $u = \frac{1}{\frac{1}{p}-\tau}$ we have $\tau = \frac{1}{p} - \frac{1}{u}$ and $p \le u$. By Lemma 2.9 and (2.2) we obtain

$$F^s_{u,p}(\mathbb{R}^d) = \mathcal{E}^s_{u,u,p}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^s_{u,p,p}(\mathbb{R}^d) = B^{s,\tau}_{p,p}(\mathbb{R}^d).$$

Thus, we get

$$\Big(\int_\Omega \|G^{\omega,\alpha} | B^{s,\tau}_{p,p}(\mathbb{R}^d)\|^u d\mu(\omega)\Big)^{\frac{1}{u}} \le C_1 \Big(\int_\Omega \|G^{\omega,\alpha} | F^{s}_{u,p}(\mathbb{R}^d)\|^u d\mu(\omega)\Big)^{\frac{1}{u}} \le C_2.$$

Step 3. Next we prove that for large $W \in \mathbb{N}$ there is a $C > 0$ such that

$$\Big(\int_\Omega (\|G^{\omega,\alpha} | B^{s,\tau}_{p,p}(\mathbb{R}^d)\|^{(v,1)})^u d\mu(\omega)\Big)^{\frac{1}{u}} \ge C \max\Big(2^{W(-s-\frac{d}{v}+\frac{d}{p})}, W^{\frac{1}{p}}\Big)\,.$$

At first we will replace the supremum with respect to all dyadic cubes by choosing the specific dyadic cube $P^* = [0,1)^d$. Then we find

$$\begin{aligned}
&\Big(\int_\Omega (\|G^{\omega,\alpha} | B^{s,\tau}_{p,p}(\mathbb{R}^d)\|^{(v,1)})^u d\mu(\omega)\Big)^{\frac{1}{u}} \\
&\ge \Big(\int_\Omega \Big(\frac{1}{|P^*|^\tau} \Big(\int_0^1 t^{-sp} \int_{P^*} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N G^{\omega,\alpha}(x)|^v dh \Big)^{\frac{p}{v}} dx \frac{dt}{t} \Big)^{\frac{1}{p}} \Big)^u d\mu(\omega)\Big)^{\frac{1}{u}} \\
&\ge \Big(\int_\Omega \Big(\int_{[\frac14,\frac34]^d} \int_0^1 t^{-sp} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N G^{\omega,\alpha}(x)|^v dh \Big)^{\frac{p}{v}} \frac{dt}{t} dx \Big)^{\frac{u}{p}} d\mu(\omega)\Big)^{\frac{1}{u}}.
\end{aligned}$$

Next, because of $\frac{u}{p} \geq 1$, we can use Hölder's inequality and obtain

$$\Big(\int_\Omega (\| G^{\omega,\alpha} | B^{s,\tau}_{p,p}(\mathbb{R}^d)\|^{(v,1)})^u d\mu(\omega) \Big)^{\frac{1}{u}}$$
$$\geq C_2 \Big(\int_\Omega \int_{[\frac14,\frac34]^d} \int_0^1 t^{-sp} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N G^{\omega,\alpha}(x)|^v dh \Big)^{\frac{p}{v}} \frac{dt}{t}\, dx\, d\mu(\omega) \Big)^{\frac{1}{p}}.$$

Now we are exactly in the same situation as in Step 2 of the proof of Proposition 10 in [20]. So we can argue as there omitting further details. □

Remark 7. *Notice that for $p = q$ Proposition 5.3 is much stronger than Proposition 5.2. At this moment we do not know whether the condition $s > d(\frac{1}{p} - \frac{1}{v})$ is also necessary in the case $p \neq q$.*

5.2 The necessity of $s > 0$

To see that it is impossible to describe $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ by differences we follow Besov [3] and [21]. We have to study the oszillations of linear combinations of indicator functions.

Lemma 5.2. *Let $0 < p < \infty$, $0 < q < \infty$, $0 \leq \tau < \frac{1}{p}$, $0 < v \leq \infty$ and $N \in \mathbb{N}$. Then there exists a sequence $(g_j)_j$ of functions with the following properties:*
(i) $\operatorname{supp} g_j \subset [0,1]^d$ *for all* $j \in \mathbb{N}$,
(ii) $\|g_j|L_\infty(\mathbb{R}^d)\| \leq 1$ *for all* $j \in \mathbb{N}$,
(iii) $\sup_{j\in\mathbb{N}} \| g_j \,|\mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d)\| = \infty$.

Proof. Let $\mathcal{X}$ denote the characteristic function of the unit cube $[0,1)^d$. We put

$$g_j(x) := \sum_{k_1=0}^{2^{j-1}} \ldots \sum_{k_d=0}^{2^{j-1}} \frac{1-(-1)^{K(k)}}{2}\, \mathcal{X}(2^j x - k)\,, \qquad x \in \mathbb{R}^d\,, \quad j \in \mathbb{N}\,, \tag{5.5}$$

where the function K is defined as

$$K(k) := K(k_1, \ldots, k_d) = \sum_{i=1}^d k_i\,, \qquad k \in \mathbb{Z}^d.$$

In the case $d = 2$ a picture of this function looks like a checkerboard (if those parts where g_j has value 1 are printed in black). Obviously $\operatorname{supp} g_j \subset [0,1]^d$, the coefficients are either 1 or 0 and hence

$$\| g_j \,|L_\infty(\mathbb{R}^d)\| \leq 1 \qquad \text{for all} \quad j \in \mathbb{N}\,. \tag{5.6}$$

It remains to prove (iii). Therefore we recall that some properties already were investigated in [21]. There exist sets $X_j \subset [0,1]^d$ and $H_j \subset \mathbb{R}^d$ such that

$$\Delta_h^N g_j(x) = 1 \qquad \text{if} \quad (x,h) \in X_j \times H_j\,. \tag{5.7}$$

Furthermore, $|X_j| \asymp 1$ and

$$|B(0,t) \cap H_j| \asymp t^d\,, \qquad 2^{-j} < t < 1\,,$$

where the hidden constants depend on N and d only. Based on these properties it is now easy to derive the following estimate

$$\begin{aligned} \| g_j \,|\mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d)\| &\geq \Big(\int_0^1 \Big(\int_{X_j} \Big(t^{-d} \int_{B(0,t)\cap H_j} |\Delta_h^N g_j(x)|^v dh \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \frac{dt}{t} \Big)^{\frac{1}{q}} \\ &\geq \Big(\int_{2^{-j}}^1 \frac{dt}{t} \Big)^{\frac{1}{q}} \asymp j^{1/q}\,. \end{aligned}$$

This proves the claim. □

Let $L^*_\infty(\mathbb{R}^d)$ be the set of all functions $g \in L_\infty(\mathbb{R}^d)$ such that $\operatorname{supp} g \subset [0,1]^d$. Then, as a conclusion of Lemma 5.2 and the Theorem of Banach-Steinhaus (in the variant of the uniform boundedness principle, valid also with target space being a quasi-Banach space, see [19] for an appropriate version), we obtain

$$L^*_\infty(\mathbb{R}^d) \not\subset \mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d).$$

Obviously we have

$$L^*_\infty(\mathbb{R}^d) \hookrightarrow \mathcal{M}^u_p(\mathbb{R}^d).$$

For $u = \frac{1}{\frac{1}{p}-\tau}$ this can be complemented by

$$\mathcal{M}^u_p(\mathbb{R}^d) = \mathcal{E}^0_{u,p,2}(\mathbb{R}^d) \hookrightarrow B^{0,\tau}_{p,\max(p,2)}(\mathbb{R}^d)$$

if $1 < p < \infty$, see [36]. Next we shall use the wavelet characterization of $B^{0,\tau}_{p,q}(\mathbb{R}^d)$ obtained in [23], [24]. For functions having support in $[0,1]^d$ it follows the monotonicity of the quasi-norm with respect to p. So for $p_0 \le p_1$ we have

$$\| f \,|B^{0,\tau}_{p_0,q}(\mathbb{R}^d)\| \lesssim \| f \,|B^{0,\tau}_{p_1,q}(\mathbb{R}^d)\| \,,$$

see [23]. Alltogether we have obtained

$$L^*_\infty(\mathbb{R}^d) \hookrightarrow B^{0,\tau}_{p,\max(p,2)}(\mathbb{R}^d)\,, \qquad 0 < p < \infty\,. \tag{5.8}$$

In case $L^*_\infty(\mathbb{R}^d) \subset B^{0,\tau}_{p,q}(\mathbb{R}^d)$ we obtain $B^{0,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d)$. To cover the other values of q as well we need a further modification of our test functions. Instead of (5.5) we consider

$$\tilde{g}_j(x) := \sum_{k_1=0}^{2^{j-1}} \ldots \sum_{k_d=0}^{2^{j-1}} \frac{1-(-1)^{K(k)}}{2}\, \Psi(2^j x - k)\,, \qquad x \in \mathbb{R}^d\,, \quad j \in \mathbb{N}\,, \tag{5.9}$$

where $\Psi(x) := \prod_{j=1}^d \psi(x_j)$, $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$, and ψ denotes the Haar wavelet that looks like

$$\psi(t) := \begin{cases} 1 & \text{if} \quad 0 \le t < \frac{1}{2}; \\ -1 & \text{if} \quad \frac{1}{2} \le t < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the support properties are not changed under this modification.

Lemma 5.3. *Let $0 < p < \infty$, $0 < q < \infty$, $0 \le \tau < \frac{1}{p}$, $0 < v \le \infty$ and $N \in \mathbb{N}$. Then the sequence $(\tilde{g}_j)_j$ has the following properties:*
(i) $\operatorname{supp} \tilde{g}_j \subset [0,1]^d$ *for all* $j \in \mathbb{N}$,
(ii) $\|\tilde{g}_j|L_\infty(\mathbb{R}^d)\| \le 1$ *for all* $j \in \mathbb{N}$,
(iii) $\sup_{j\in\mathbb{N}} \| \tilde{g}_j \,|\mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d)\| = \infty$,
(iv) $\sup_{j\in\mathbb{N}} \| \tilde{g}_j \,|B^{0,\tau}_{p,q}(\mathbb{R}^d)\| < \infty$.

Proof. We use the notation from the proof of Lemma 5.2. The proof of properties (i)-(iii) remains unchanged compared to Lemma 5.3. It will be enough to prove (iv). Therefore we shall use the wavelet characterizations of $B^{0,\tau}_{p,q}(\mathbb{R}^d)$ obtained in [24], [23], [43] and [51]. From the characterization by smooth and compactly supported wavelets and the property (i) it follows the monotonicity of the quasi-norm with respect to p. So for $p_0 \le p_1$ we have

$$\| \tilde{g}_j \,|B^{0,\tau}_{p_0,q}(\mathbb{R}^d)\| \lesssim \| \tilde{g}_j \,|B^{0,\tau}_{p_1,q}(\mathbb{R}^d)\| \,,$$

see [23], [24]. Hence, it will be enough to deal with $1 < p < \infty$ in (iv). In this situation we may use the Haar wavelet characterization from [51] (alternatively one may use [43] and take into account that the spaces $\mathcal{L}^r B^0_{p,q}(\mathbb{R}^d)$ with $r = d(\tau - \frac{1}{p})$ and $B^{0,\tau}_{p,q}(\mathbb{R}^d)$ coincide locally, i.e., for functions with property (i), see [50] and [44]). For $k \in \mathbb{Z}^d$ we put $\mathcal{X}_{0,k}(x) := \mathcal{X}(x-k)$. Let $\tilde{\mathcal{X}}$ denote the characteristic function of the interval $[0,1)$. Then we put

$$h_{i,j,k}(x) := 2^{\frac{jd}{2}} \Big(\prod_{n \in I_1} \tilde{\mathcal{X}}(2^j x_n - k_n) \Big) \Big(\prod_{n \in I_2} \psi(2^j x_n - k_n) \Big) , \quad x \in \mathbb{R}^d .$$

Here I_1, I_2 depend on $i \in \{1, \ldots, 2^d - 1\}$ and have the following properties. $I_1 \cup I_2 = \{1, 2, \ldots, d\}$, $I_1 \cap I_2 = \emptyset$ and $I_2 \neq \emptyset$. This yields $2^d - 1$ possibilities. $\langle f, \mathcal{X}_{0,k} \rangle$ and $\langle f, h_{i,j,k} \rangle$ denote scalar products (Fourier coefficients of f with respect to the Haar system). For all sequences $t := \{t_{i,j,m}\}_{i \in \{1,\ldots,2^d-1\}, j \in \mathbb{N}_0, m \in \mathbb{Z}^d} \subset \mathbb{C}$ a further abbreviation which will be used is given by

$$\|t \,|b^{0,\tau}_{p,q}(\mathbb{R}^d)\| := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \Bigg(\sum_{j=\max(j_P,0)}^{\infty} 2^{j(\frac{d}{2} - \frac{d}{p})q} \sum_{i=1}^{2^d-1} \Bigg[\sum_{m:\ Q_{j,m} \subset P} |t_{i,j,m}|^p \Bigg]^{\frac{q}{p}} \Bigg)^{\frac{1}{q}} .$$

Proposition 5.4. *Let $1 < p < \infty$, $0 < q \le \infty$ and $0 \le \tau < \frac{1}{p}$. Let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then $f \in B^{0,\tau}_{p,q}(\mathbb{R}^d)$ if and only if f can be represented in $\mathcal{S}'(\mathbb{R}^d)$ as*

$$f = \sum_{k \in \mathbb{Z}^d} \langle f, \mathcal{X}_{0,k} \rangle \, \mathcal{X}_{0,k} + \sum_{i=1}^{2^d-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \langle f, h_{i,j,k} \rangle \, h_{i,j,k} \tag{5.10}$$

with convergence in $\mathcal{S}'(\mathbb{R}^d)$ and

$$\|\mu(f) \,|b^{0,\tau}_{p,q}(\mathbb{R}^d)\| := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \Bigg(\sum_{m:\ Q_{0,m} \subset P} |\langle f, \mathcal{X}_{0,m} \rangle|^p \Bigg)^{\frac{1}{p}} + \big\| \{\langle f, h_{i,j,m} \rangle\}_{i,j,m} \big| b^{0,\tau}_{p,q}(\mathbb{R}^d) \big\| < \infty .$$

Moreover, the mapping

$$J : \ f \mapsto \{\langle f, \mathcal{X}_{0,m} \rangle\}_m \cup \{\langle f, h_{i,j,k} \rangle\}_{i,j,k}$$

is an isomorphic map of $B^{0,\tau}_{p,q}(\mathbb{R}^d)$ onto $\ell^p \times b^{0,\tau}_{p,q}(\mathbb{R}^d)$. In other words $\|\mu(f) \,|b^{0,\tau}_{p,q}(\mathbb{R}^d)\|$ is equivalent to $\|f|B^{0,\tau}_{p,q}(\mathbb{R}^d)\|$.

Now it is easy to see that $\tilde{g}_j$ is given by its Fourier-Haar series. The frequency level is j and it has 2^{dj} non-zero coefficients which are all equal to $2^{-\frac{jd}{2}}$. This yields for all $j \in \mathbb{N}$

$$\begin{aligned} \big\| \{\langle \tilde{g}_j, h_{i,n,m} \rangle\}_{i,n,m} \big| b^{0,\tau}_{p,q}(\mathbb{R}^d) \big\| & \le \sup_{\substack{P \in \mathcal{Q} \\ P \subset [0,1]^d, |P| \ge 2^{-jd}}} \frac{1}{|P|^\tau} 2^{j(\frac{d}{2} - \frac{d}{p})} \Bigg[\sum_{m:\ Q_{j,m} \subset P} |\langle \tilde{g}_j, h_{i_0,j,m} \rangle|^p \Bigg]^{\frac{1}{p}} \\ & \le \max_{\ell = 0,1,\ldots,j} 2^{\ell d \tau} 2^{j(\frac{d}{2} - \frac{d}{p})} 2^{-\frac{jd}{2}} 2^{(j-\ell)\frac{d}{p}} \le 1 , \end{aligned}$$

since $\tau < \frac{1}{p}$. This proves Lemma 5.3. □

Corollary 5.1. *Let $0 < p < \infty$, $0 < q < \infty$, $0 \le \tau < \frac{1}{p}$, $0 < v \le \infty$ and $N \in \mathbb{N}$. Then $B^{0,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d)$.*

Proof. In the case $q \geq \max(p,2)$ we are done, see (5.8). Let $1 \leq q < \max(p,2)$. Then we apply the Theorem of Banach-Steinhaus (in the variant of the uniform boundedness principle) and Lemma 5.3. Then we conclude $B^{0,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d)$. For $0 < q < 1$ we will argue by the Closed Graph Theorem, but first we exclude those cases where $B^{0,\tau}_{p,q}(\mathbb{R}^d)$ contains singular distributions. Now we assume $B^{0,\tau}_{p,q}(\mathbb{R}^d) = \mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d)$. Let $(f_j)_j \subset B^{0,\tau}_{p,q}(\mathbb{R}^d)$ be a convergent sequence with limit f. In addition we assume that $(f_j)_j$ converges in $\mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d)$ to some function g. Convergence in $B^{0,\tau}_{p,q}(\mathbb{R}^d)$ implies convergence in $\mathcal{S}'(\mathbb{R}^d)$ and therefore

$$\lim_{j\to\infty} \int_{Q_{0,m}} |f(x) - f_j(x)|\, dx = 0 \qquad \text{for all} \quad m \in \mathbb{Z}^d\,.$$

Hence, a subsequence converges almost everywhere. Convergence in $\mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d)$ implies convergence in $L^\tau_p(\mathbb{R}^d)$. Again there is a subsequence which must converge almost everywhere. This yields $f = g$ almost everywhere, hence the embedding $Id : \mathbf{B}^{0,\tau,N}_{p,q,v}(\mathbb{R}^d) \to B^{0,\tau}_{p,q}(\mathbb{R}^d)$ is a closed operator. The Closed Graph Theorem yields continuity of this embedding, but by Lemma 5.3 this is wrong and hence the assumption was wrong. □

It remains to check the case $q = \infty$.

Corollary 5.2. *Let $0 < p < \infty$, $0 \leq \tau < \frac{1}{p}$, $0 < v \leq \infty$ and $N \in \mathbb{N}$. Then $B^{0,\tau}_{p,\infty}(\mathbb{R}^d) \neq \mathbf{B}^{0,\tau,N}_{p,\infty,v}(\mathbb{R}^d)$.*

Proof. This is an immediate consequence of Lemma 2.5. □

5.3 The necessity of $s < N$

Proposition 5.5. *Let $s \geq 0$, $0 < p < \infty$, $0 \leq \tau < \frac{1}{p}$ and $N \in \mathbb{N}$. Let $0 < v \leq \infty$. If either*

$$N < s \ \text{ and } \ 0 < q \leq \infty \qquad \text{or} \qquad N = s \ \text{ and } \ 0 < q < \infty\,,$$

then $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$.

Proof. To prove this result we will follow the ideas from [20, 5.6]. We work with a function $f \in C_0^\infty(\mathbb{R}^d)$ that has its support in $B(0, 3N+3)$. In $B(0, 2N+2)$ this function looks like

$$f(x_1, x_2, \ldots, x_d) = e^{x_1 + x_2 + x_3 + \ldots + x_d}. \tag{5.11}$$

Then, because of Lemma 2.1, we have $f \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$. We want to prove that $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)} = \infty$. Let $0 < \varepsilon < 1$. We define

$$H^d_+ = \Big\{ h = (h_1, h_2, \ldots, h_d) \in \mathbb{R}^d :\ h_1 \geq 0, h_2 \geq 0, \ldots, h_d \geq 0 \Big\}\,.$$

At first, instead of the supremum, we choose a dyadic cube P^* with $B(0,1) \subset P^*$ that is as small as possible. Then we obtain

$$\begin{aligned}
\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(v,1)} &\geq \frac{1}{|P^*|^\tau} \Big(\int_0^1 t^{-sq} \Big(\int_{P^*} \Big(t^{-d} \int_{B(0,t)} |\Delta^N_h f(x)|^v dh \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \frac{dt}{t} \Big)^{\frac{1}{q}} \\
&\geq C_1 \Big(\int_\varepsilon^1 t^{-sq} \Big(\int_{B(0,1)} \Big(t^{-d} \int_{(B(0,t) \setminus B(0,\frac{t}{2})) \cap H^d_+} |\Delta^N_h f(x)|^v dh \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \frac{dt}{t} \Big)^{\frac{1}{q}}
\end{aligned}$$

for an appropriate positive constant C_1 and any $\varepsilon \in (0,1)$. Observe that $h \in (B(0,t) \setminus B(0,\frac{t}{2})) \cap H_+^d$ yields $|h| \geq \frac{t}{2} \geq \frac{\varepsilon}{2} > 0$. Due to the Mean Value Theorem in several variables there exists a $\zeta \in \mathbb{R}^d$ on the line that connects x and $x+h$ such that

$$\begin{aligned} |\Delta_h^N f(x)| &= |\Delta_h^{N-1} f(x+h) - \Delta_h^{N-1} f(x)| \\ &= \Big| \frac{\partial \Delta_h^{N-1} f}{\partial y_1}(\zeta) h_1 + \frac{\partial \Delta_h^{N-1} f}{\partial y_2}(\zeta) h_2 + \ldots + \frac{\partial \Delta_h^{N-1} f}{\partial y_d}(\zeta) h_d \Big|. \end{aligned}$$

We have $|\zeta + Nh| \leq 2 + N$ and so $\zeta + Nh \in B(0, 2N+2)$. Taking into account (5.11), for $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_d)$, we get

$$\begin{aligned} \frac{\partial \Delta_h^{N-1} f}{\partial y_k}(\zeta) &= \frac{\partial}{\partial y_k} \Big(\sum_{l=0}^{N-1} (-1)^{N-1-l} \binom{N-1}{l} e^{y_1+y_2+\ldots+y_d+lh_1+lh_2+\ldots+lh_d} \Big)(\zeta) \\ &= \sum_{l=0}^{N-1} (-1)^{N-1-l} \binom{N-1}{l} e^{\zeta_1+\zeta_2+\ldots+\zeta_d+lh_1+lh_2+\ldots+lh_d} \\ &= \Delta_h^{N-1} f(\zeta) \end{aligned}$$

for any $k \in \{1, 2, \ldots, d\}$. Next because of $h \in H_+^d$ we obtain

$$|\Delta_h^N f(x)| = \Big| \Delta_h^{N-1} f(\zeta) h_1 + \Delta_h^{N-1} f(\zeta) h_2 + \ldots + \Delta_h^{N-1} f(\zeta) h_d \Big| \geq \big| \Delta_h^{N-1} f(\zeta) \big| \; |h|.$$

By iteration we can find an $\eta \in B(0, N+1)$ such that

$$|\Delta_h^N f(x)| \geq |f(\eta)| \; |h|^N \geq |h|^N.$$

By means of these observations we obtain

$$\begin{aligned} \| f | B_{p,q}^{s,\tau}(\mathbb{R}^d) \|^{(v,1)} &\geq C_2 \Big(\int_\varepsilon^1 t^{-sq} \Big(\int_{B(0,1)} \Big(t^{-d} \int_{(B(0,t) \setminus B(0,\frac{t}{2})) \cap H_+^d} |h|^{Nv} dh \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \frac{dt}{t} \Big)^{\frac{1}{q}} \\ &\geq C_3 \Big(\int_\varepsilon^1 t^{q(N-s)-1} dt \Big)^{\frac{1}{q}}. \end{aligned}$$

For $s \geq N$ it follows that the right-hand side tends to infinity if ε tends to zero. Hence $\| f | B_{p,q}^{s,\tau}(\mathbb{R}^d) \|^{(v,1)} = \infty$ as claimed. This proves that $B_{p,q}^{s,\tau}(\mathbb{R}^d) \neq \mathbf{B}_{p,q,v}^{s,\tau,N}(\mathbb{R}^d)$. □

Remark 8. Notice that the proof of Proposition 5.5 does not work in the case $s = N$ and $q = \infty$.

5.4 The proof of the main results

Proof of Theorem 1.3. First we consider sufficiency of the conditions in Theorem 1.5. Therefore it is enough to apply Proposition 3.4 with $v = 1$. Then we turn to necessity of the given conditions. For $s < 0$ we know that $B_{p,q}^{s,\tau}(\mathbb{R}^d)$ contains singular distributions, see the Lemmas 2.4 and 2.5. The range $s = 0$ is excluded by using Corollary 5.1. For $s = N$ we make use of Propositon 5.5. Finally, Proposition 5.3 applied with $v = 1$, excludes $s = d\,(\frac{1}{p} - 1)$.

Proof of Theorem 1.4. This follows from Proposition 3.4.

Proof of Theorem 1.5. We collect Lemmas 2.4, 2.5, Corollaries 5.1, 5.2 and Propositions 5.1, 5.2, 5.3, 5.5.

Open problems

Open Problem 1
A comparison of Theorem 1.4 and Theorem 1.5 shows that there is a gap between sufficient and necessary conditions. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $0 \leq \tau < \frac{1}{p}$, $N \in \mathbb{N}$ with $N \geq s$ and $0 < v \leq \infty$. Then we do not know whether we have $B^{s,\tau}_{p,q}(\mathbb{R}^d) = \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$ or $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N}_{p,q,v}(\mathbb{R}^d)$ in the following cases:

(i) $d(\frac{1}{p} - \frac{1}{v}) - d\tau(1 - \frac{p}{v}) \leq s \leq d(\frac{1}{p} - \frac{1}{v})$ and $q \neq p < v < \infty$ with $v > 1$,

(ii) $d(\frac{1}{p} - 1) - d\tau(1 - p) \leq s \leq d(\frac{1}{p} - 1)$ with $0 < p < 1$ and $0 < v < 1$,

(iii) $N = s$ and $q = \infty$.

Open Problem 2
The second open problem is related to Proposition 3.5. Find necessary and sufficient conditions for the validity of the characterizations given there.

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