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EURASIAN MATHEMATICAL JOURNAL

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From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

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1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

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1.14. No fee for reviewing papers will be charged.

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2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

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- compliance of the paper with the scope of the EMJ;
- compliance of the title of the paper to its content;
- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);
- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);
- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.

VLADIMIR DMITRIEVICH STEPANOV

(to the 70th birthday)

Vladimir Dmitrievich Stepanov was born on December 13, 1949 in a small town Belovo, Kemerovo region. In 1966 he finished the Lavrentiev school of physics and mathematics at Novosibirsk academic town-ship and the same year he entered the Faculty of Mathematics of the Novosibirsk State University (NSU) from which he has graduated in 1971 and started to teach mathematics at the Khabarovsk Technical University till 1981 with interruption for postgraduate studies (1973-1976) in the NSU.

In 1977 he has defended the PhD dissertation and in 1985 his doctoral thesis "Integral convolution operators in Lebesgue spaces" in the S.L. Sobolev Institute of Mathematics. Scientific degree "Professor of Mathematics" was awarded to him in 1989. In 2000 V.D. Stepanov was elected a corresponding member of the Russian Academy of Sciences (RAS).

Since 1985 till 2005 V.D. Stepanov was the Head of Laboratory of Functional Analysis at the Computing Center of the Far Easten Branch of the Russian Academy of Science.

In 2005 V.D. Stepanov moved from Khabarovsk to Moscow with appointment at the Peoples Friendship University of Russia as the Head of the Department of Mathematical Analysis (retired in 2018). Also, he was hired at the V.A. Steklov Mathematical Institute of RAS at the Function Theory Department.

Research interests of V.D. Stepanov are: the theory of integral and differential operators, harmonic analysis in Euclidean spaces, weighted inequalities, duality in function spaces, approximation theory, asymptotic estimates of singular, approximation and entropy numbers of integral transformations, and estimates of the Schatten-Neumann type. Main achievements: the theory of integral convolution operators is constructed, the criteria for the boundedness and compactness of integral operators in function spaces are obtained, weighted inequalities and the behaviour of approximation numbers of the Volterra, Riemann-Liouville, Hardy integral operators are studied, etc.

Under his scientific supervision 15 candidate theses in Russia and 5 PhD theses in Sweden were successfully defended. Professor V.D. Stepanov has over 100 scientific publications including 3 monographs. Participation in scientific and organizational activities of V.D. Stepanov is well known. He is a member of the American Mathematical Society (since 1987) and a member of the London Mathematical Society (since 1996), Deputy Editor of the Analysis Mathematica, member of the Editorial Board of the Eurasian Mathematical Journal, invited speaker at many international conferences and visiting professor of universities in USA, Canada, UK, Spain, Sweden, South Korea, Kazakhstan, etc.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Vladimir Dmitrievich on the occasion of his 70th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF ANALYSIS, DIFFERENTIAL EQUATIONS AND ALGEBRA" (EMJ-2019), DEDICATED TO THE 10TH ANNIVERSARY OF THE EURASIAN MATHEMATICAL JOURNAL

From October 16 to October 19, 2019 at the L.N. Gumilyov Eurasian National University (ENU) the International Conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2019) was held. The conference was dedicated to the 10th anniversary of the Eurasian Mathematical Journal (EMJ).

The purposes of the conference were to discuss the current state of development of mathematical scientific directions, expand the number of potential authors of the Eurasian Mathematical Journal and further strengthen the scientific cooperation between the Faculty of Mechanics and Mathematics of the ENU and scientists from other cities of Kazakhstan and abroad.

The partner universities for the organization of the conference were the M.V. Lomonosov Moscow State University, the Peoples' Friendship University of Russia (the RUDN University, Moscow) and the University of Padua (Italy).

The conference was attended by more than 80 mathematicians from the cities of Almaty, Aktobe, Karaganda, Nur-Sultan, Shymkent, Taraz, Turkestan, as well as from several foreign countries: from Azerbaijan, Germany, Greece, Italy, Japan, Kyrgyzstan, Russia, Tajikistan and Uzbekistan.

The chairman of the International Programme Committee of the conference was Ye.B. Sydykov, rector of the ENU, co-chairmen were Chief editors of the EMJ: V.I. Burenkov, professor of the RUDN University, M. Otelbaev, academician of the National Academy of Sciences of the Republic of Kazakhstan (NAS RK), V.A. Sadovnichy, academician of the Russian Academy of Sciences (RAS), rector of the M.V. Lomonosov Moscow State University (MSU).

There were three sections at the conference: "Function Theory and Functional Analysis", "Differential Equations and Equations of Mathematical Physics" and "Algebra and Model Theory". 16 plenary presentations of 30 minutes each and more than 60 sectional presentations of 20 minutes each, devoted to contemporary areas of mathematics, were given.

It was decided to recommend selected reports of the participants for publication in the Eurasian Mathematical Journal and the Bulletin of the Karaganda State University (series "Mathematics").

Before the conference, a collection of abstracts of the participants' talks was published.

PROGRAMME OF THE INTERNATIONAL CONFERENCE EMJ-2019

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Conference Schedule:

16.10.2019
09.00 – 10.00 Registration
10.00 – 10.30 Opening of the conference
10.30 – 12.50 Plenary talks
12.50 – 14.00 Lunch
14.00 – 18.00 Session talks

17.10.2019
09.30 – 12.20 Plenary talks
12.20 – 14.00 Lunch
14.00 – 18.00 Session talks
18.00 – Dinner for participants of the conference

18.10.2019
09.30 – 13.00 Plenary talks
12.20 – 14.00 Lunch
14.00 – 17.00 Excursion around the city

19.10.2019
09.30 – 12.30 Plenary talks
12.30 – 13.00 Closing of the conference

At the opening ceremony welcome speeches were given by Ye.B. Sydykov, rector of the ENU, chairman of the Program Committee of the conference; V.I. Burenkov, professor of the RUDN University, editor-in-chief of the EMJ; L. Mukasheva, official representative of the international company Clarivate Analytics in the Central Asian region; A. Ospanova, official representative of Scopus.

Plenary talks were given by

T.Sh. Kalmenov (Kazakhstan), M. Otelbaev and B.D. Koshanov (Kazakhstan), P.D. Lamberti and V. Vespri (Italy) – on 16.10.2019;

V.I. Burenkov (Russia), T. Ozawa (Japan), H. Begehr (Germany), M.A. Sadybekov and A.A. Dukenbaeva (Kazakhstan), D. Suragan (Kazakhstan) – on 17.10.2019;

M.L. Goldman (Russia), A. Bountis (Greece), A.K. Kerimbekov (Kyrgyzstan), S.N. Kharin (Kazakhstan), M.I. Dyachenko (Russia) – on 18.10.2019;

E.D. Nursultanov (Kazakhstan), M.A. Ragusa (Italy), P.D. Lamberti and V. Vespri (Italy), M.G. Gadoev (Russia) and F.S. Iskhokov (Tajikistan) – on 19.10.2019.

At the closing ceremony all participants unanimously congratulated the staff of the L.N. Gumilyov Eurasian National University and the Editorial Board of the Eurasian Mathematical Journal with the 10th anniversary of the journal and wished further creative successes.

They expressed hope that the journal will continue to play an important role in the development of mathematical science and education in Kazakhstan in the future.

V.I. Burenkov, K.N. Ospanov, A.M. Temirkhanova

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GASPARD MONGE (1746-1818): APPLICATION OF GEOMETRY TO ANALYSIS

M.T. Borgato, L. Pepe

Communicated by V.I. Burenkov

Key words: first-order PDEs, minimal surface equation, Monge-Ampère equation, optimal control problems, history of mathematics.

AMS Mathematics Subject Classification: 01A50, 35-03, 35J96, 49-03, 49J20.

Abstract. Two centuries after the death of the mathematician Gaspard Monge, his main results are presented, for the period preceding his political and organizational commitment during the French Revolution and the Napoleonic era in Italy and Egypt. After an examination of the fortune of his works in Russia, Monge's main contributions to the theory of partial differential equations of the first order, to the solution of the minimal surface equation (with his pupil Meusnier) and to the first study of the optimal transport problem (Monge-Ampère equation) are underlined. These subjects are well developed in the current mathematical research in analysis and in differential geometry.

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1 Introduction

Two hundred years after the death of Gaspard Monge (1746-1818) it seems right to honour one of the scientists who have left their mark on history as a result of their different activities, each of which deserves its share of glory. In Monge's case, however, the variety of his works did not increase his celebrity: his invention of descriptive geometry, his systematization of Carthesian geometry as a discipline, his contributions to the study of partial differential equations, his fundamental research on Lavoisier's new chemistry, his professional commitment in the defense of his country through his work on the manufacturing of cannons, the foundation of the *École Normale* and the *École Polytechnique* in Paris, his participation in the first Italian Campaign with Bonaparte, the establishment and the first constitutional laws of the Roman Republic of 1798, his expedition to Egypt and the creation of the *Institut d'Egypte*, and the defense of France as the Senator of Liège in 1814 [38, 41].

This paper is divided into three parts: the first presents Monge's main results relating to the partial differential equations of first order, with particular attention to the geometric aspects. The second part concerns Monge's contributions to the minimal surface equation that support the solution of this equation found by his pupil Meusnier. The third part concerns the problem of optimal transport which gave rise to the study of the Monge-Ampère equation and to important questions of differential geometry, linked to what is called the Gaussian curvature of a surface. The geometric aspects of these results will be highlighted, justifying our title *Application of geometry to analysis*, which reverses the name of *Application of analysis to geometry* given by Monge himself to the collection of his results published in 1807 and in 1809 for use by the Polytechnic school [48].

Gaspard Monge, born in Beaune on 9th May, 1746, to a family of modest means, studied firstly in the college of the Oratorians in his city and then in Lyon. At the age of eighteen, he entered the *École royale du génie de Mézières*, where mathematics was taught by the abbot, Charles Bossut, and physics by another abbot, Jean-Antoine Nollet, both of whom were celebrated scholars. In spite of his talent for mathematics, which had not escaped the notice of Bossut, who became his first protector, Monge could not aspire to a career as an officer in the Engineering Military Corps, since it was reserved for members of the aristocracy. He had to make do with a modest position in the same school as teaching assistant of draughtsmanship and stonecutting. It was from this point that he started the first research works which were to lead to the creation of his descriptive geometry, and thus began a long series of publications and gradual progress. In 1770, he published his first work in the proceedings of the Royal Society of Turin, and the following year he began to send some memoirs to the *Académie des Sciences*, where he attracted the attention of D'Alembert and Condorcet, and in 1772 he became Bossut's corresponding member (see [7, 14, 39, 43, 51]).

Monge's mathematical works greatly influenced the teaching of mathematics in universities, military schools, and fine art academies all over the world. In this sense, just as Euler may be considered the father of mathematical analysis, Monge can be seen as the father of modern geometry. Monge's treatises are well-known in Russia. The *Traité de statique* was very quickly published in translation (1803, 1825) [35], the *Application de l'analyse* was published in Russian in 1936 including comments and valuable notes [36], the *Géométrie descriptive* appeared in Russian in Moscow in 1947 [37]. Russian mathematicians were, of course, already familiar with these works, owing to the fact that since the beginning of the nineteenth century they had been reading (and writing) in Western languages, above all French and German, but also English and Italian.

Most of Monge's memoirs and his treatises, *Feuilles d'analyse* (1795) and *Application de l'analyse à la géométrie* (1807, 1809), derive from studies carried out in Mézières from 1764 to 1780 concerning the use of differential calculus and partial differential equations to study curves and surfaces. With Monge's work, these themes became, for the first time, a real discipline, engaging his best students: Meusnier, Malus, Lancret, and O. Rodrigues.

Gauss, who is considered the father of this new discipline, wrote in very respectful terms about Monge's works. Joseph Liouville (1809-1882), when reprinting the *Application de l'analyse à la géométrie* (Paris, Bachelier, 1850), added the Latin re-edition of Gauss' memoir *Disquisitiones generales circa superficies curvas* (1828), as a sort of complementary work to that of Monge.

At the end of the nineteenth century, the Italian mathematician, Luigi Bianchi, renamed the application of analysis to geometry as *differential geometry*, the name by which we know it today [3, 46].

It was in Mézières that Monge found up-to-date information about the calculus which stimulated his thoughts on possible applications; his sources were books and journals, above all the *Institutiones Calculi Integralis* (Saint Petersburg, Impensis Academiae Imperialis Scientiarum, vols. I-III, 1768-1770) by Euler, the third volume of which is devoted to the study of partial differential equations (hereinafter PDEs) [40]. Again in Mézières, most likely thanks to Abbot Nollet, who had been in Turin and had taught physics in the aforementioned school from 1761 to his death in 1770, Monge was introduced to the early memoirs of Lagrange. In those years, Lagrange was publishing various works on the proceedings of a new scientific society, founded in 1756, which had received royal recognition. Some of Monge's first memoirs on PDEs appeared in Turin, but Monge's first mathematical work was published in the *Journal encyclopédique* (June, 1769), in the form of a letter under the title *Sur les devéloppées des courbes à double courbure* [49].

In the past few decades Monge's name has above all been linked to problems of optimal control, brought to the attention of mathematicians by Leonid Vitaliyevich Kantorovich (1912-1986), Nobel Prize winner for economics in 1975. The first person to deal with such a matter, linked to the mass removal of land for the construction of fortifications, was, in fact, Monge, prompted by Condorcet, in a memoir printed in 1784. We will speak specifically about this memoir and successive developments carried out by Ampère, Appel, and by important exponents of the Russian school, such as Kantorovich (1942, 1948), A. D. Alexandrov (1942, 1950, 1958), A. Pogorelov and mathematicians of various countries [22].

Aleksandr Danilovich Aleksandrov (1912-1999) studied at the University of Leningrad where he obtained his doctorate and also taught. Awarded the Stalin Prize in 1942, he became the Rector of the Leningrad University in 1952. From 1964 to 1986, he directed the department of geometry at the University of Novosibirsk. His *Selected Papers* were also published in English (1996, 2005). Of great importance are also his studies on convex polyhedrons (*Convex polyedra*, Berlin, Springer, 2005). Aleksei Vasil'evich Pogorelov (1919-2002), a student of his, produced important results on the a priori estimates of the Monge-Ampère equation. Pogorelov also resolved Hilbert's fourth problem (1979): "To construct and study the geometries in which the segment constitutes the shortest connection between two points".

Some very important historical studies on partial differential equations were carried out by two Russian historians of mathematics, namely, Saltykov and Demidov.

Nikolay Ivanovich Saltykov (1872-1962), a student of both Aleksandr Mikhailovich Lyapunov (1857-1918) and Vladimir Andreevich Steklov (1863-1926) at the Kharkov University, was a teacher at that university up to the Revolution. As Mayor of Kharkow, he received the commander-in-chief of the Armed Forces of South Russia, Anton Ivanovich Denikin, in 1919. He then left Russia for Belgrade where he was to become one of the creators of the Serbian school of mathematics. He was highly thought of in the Western world and was the first to study and recognize the value of Paul Charpit's memoir, which had been presented at the Academy of Sciences in 1784 and was not rediscovered again until 1928 [6, 44].

Sergei Sergeevich Demidov has provided a very informative general review of the theory of partial differential equations of first degree from Lagrange to Sophus Lie [12].

2 First-order partial differential equations

The theory of PDEs was born in the middle of the eighteenth century, after the arrangement in the works of d'Alembert, Euler and Lagrange of the theory of ordinary differential equations. The first careful study on a PDE was the one that governs the propagation of waves (vibrating string equation). The success obtained in this study was still down to d'Alembert, Daniel Bernoulli, Euler and Lagrange with a series of works, at times polemical, which led to deep reflections on the concept of function itself (see [52]).

This promising beginning of the study of PDEs, which found their application in important problems of geometry and mechanics (think of Euler's equation of fluid dynamics), led the best scholars of the second half of the century to deal with the subject: Euler, Lagrange, Laplace, Monge, Condorcet, Legendre, and together with them other remarkable mathematicians: Charpit, Cousin, Lacroix, Brunacci, Paoli. However, the results obtained were far from the first successes. It was the analogy with the ordinary equations that oriented the studies towards a generality, which, as we have seen, could not be illustrated in its entirety. The ambitious studies of the mathematicians of the time were thus wrecked in the sea of the PDEs. Only by circumscribing the types of equations and connecting them with concrete problems of geometry and mechanics could good results be achieved. In the 70s and 80s of the eighteenth century the works that we must first of all remember are those related to the equations of the first order which, not surprisingly, are deeply linked to geometric issues.

In this order of ideas are to be found Gaspard Monge's numerous memoirs which occupy a large part of his scientific life before the French Revolution. In summary, it could be said that the path followed by Monge in all his works was not the application of the analysis to geometry, as the title of his monograph of 1807 indicates, but the application of geometry to analysis. In fact, we start from geometric questions: space curves (with double curvature), curvature of surfaces, envelopes of curves and surfaces, to go back to the study of PDEs that can describe them. These are the results presented by Monge, first fragmentarily in memoirs published in Turin, and in Paris as *Savant etranger*, i.e. not as a member of the *Académie des Sciences*, then in ponderous memoirs of the *Académie des Sciences*. First of all, they concern the PDEs of the first order, to which the notions of characteristic curves and the 'Monge cone' are bound, the minimal surface equation, whose main result is however linked to the name of his pupil Meusnier, and the equation of optimal transport or Monge-Ampère, whose non-linear part contains the current expression of this famous equation.

In the history of the theory of partial differential equations of the first order, from the mid-eighteenth century on, Demidov appropriately distinguishes four stages characterized by different techniques and methods, also related to the development of other disciplines, primarily geometry and mechanics. In the first period, concerning Euler and D'Alembert, analytical techniques were developed, that aimed to lead, with the multiplier method, the integration of PDEs in terms of total differentials (1760-1770). With Lagrange, we enter a second period in which the PDEs are studied as such. This period, which goes up to C. G. Jacobi and includes works by Legendre, Charpit, Cauchy, Pfaff, also sees the numerous and remarkable contributions of Monge, which highlight the geometric aspects of the theory. With Jacobi a third period begins, in which the engine of studies becomes the analytical mechanics of Poisson, Hamilton and Jacobi himself. Then, again in 1870, we return to questions of a geometric nature with the work of Sophus Lie [12].

In analogy with the ordinary differential equations, Euler had found that the complete solution of a PDE of order n had to include n arbitrary functions. Lagrange (1774) shows that the complete solution φ of a PDE of the first order:[1]

$$f(x, y, z, p, q) = 0 \qquad \left(p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}\right) \tag{2.1}$$

depends on two arbitrary constants a and b: $\varphi(x, y, a, b)$.

Explicating b with respect to a and placing $\dfrac{\partial z}{\partial a} = 0$ in $z = \varphi(x, y, a, b)$, he finds the general integral

$$z = \varphi(x, y, a, \psi(a)).$$

Then placing

$$\frac{\partial z}{\partial a} = 0, \frac{\partial z}{\partial b} = 0$$

in $\varphi(x, y, a, b)$, he finds a particular integral.

Lagrange links the general integral to a geometric fact: it corresponds to the set of envelopes of an arbitrary one parameter subfamily ($b = \psi(a)$) of the surfaces contained in the complete solution (see [24, 25]).

The geometric theory of first order PDEs in Lagrange was only outlined. To complete it, many works of Monge would follow, resumed and arranged in the memoir *Sur le calcul intégral des équations aux differences partielles* and then in the volume *Applications de l'analyse à la géométrie* (Paris, 1807, 1809). Paul Charpit, in an important memoir of 1784, which although unpublished was known to Lagrange (1793) and to Monge (1807), reduced the study of the equation (2.1) to the integration of a system of ordinary equations of the first order (see [44, 11]).

Even if Lagrange thought, very differently from Monge, that geometry was absorbed by analysis, we end with the testimonies of esteem that Lagrange expressed towards Monge from the beginning to the end of his mathematical life. They are reported in the eulogy of Monge by François Arago [2]:

[1]The letters p, q, r, s and t to represent the first and second partial derivatives are known as 'Monge's notations' [3].

The first Memoirs of Monge, relating to the search for the equations of the surfaces known by their mode of generation, were printed in the Collection of the Turin Royal Society, for the years 1770 to 1773. It will perhaps be curious to find, beside Monge's so modest appreciation of his work, Lagrange's judgment: "Persuaded," said Monge in the preamble of his memoir, "that an idea, sterile in the hands of an ordinary man, may become very profitable in those of a skillful geometer, I will inform you about my research at the Turin Academy." Here are the words of Lagrange in all their candour: "With his application of analysis to the representation of surfaces, this devil of man will be immortal!" [...] In one of the non-compulsory lessons of the former École polytechnique; in one of those lessons, now suppressed, which were intended to develop a taste for science among the first students, Monge applied his theory of lines of curvature to the ellipsoid. Several professors had hastened to listen to their confrere: in those days they showed each other these marks of deference. At the end of the session, Monge was surrounded and congratulated. Lagrange's comment has come down to us: "You have come, my dear colleague, to exhibit very elegant things; I wish I had done them." Monge admitted that no compliment had ever touched his heart so much.[2]

3 Minimal Surfaces

Both Monge and Condorcet lived in Paris for a long time and had the opportunity to meet in person. Their correspondence regarding PDEs is thus limited essentially to two periods. The first one being when Monge was at Mézières (1771-1772) at the beginning of his scientific production. At the origin of the correspondence there is a letter from Monge to d'Alembert dated 3rd March 1771, in which he explains some difficulties encountered in the study of minimal surfaces. Condorcet took charge of answering and the correspondence expanded to the study of other PDEs. In April 1772, Monge became Bossut's correspondent of the *Académie des Sciences* and an official connection with the Parisian scientific environment opened up for him. Monge quotes Euler as his reference author, and does not mention Lagrange's memoir of 1762 on minimal surfaces. The second group of letters of 1786 deals with the reduction of the integration of first-order PDEs to the form of total differentials (or 'ordinary differences' as they were then called). The two mathematicians were both academics and Condorcet did not missed the opportunity to give his support to his colleague, who instead encountered difficulties with analysts like Laplace and Legendre (see [13, 19, 47, 48, pp. 169-173, 182-186]).

The study of minimal area surfaces begins with the publication of Lagrange's memoir *Essai d'une*

[2]"Les premiers Mémoires de Monge, relatifs à la recherche des équations des surfaces connues par leur mode de génération, ont été imprimés dans le Recueil de l'Académie de Turin, pour les années 1770 à 1773. On sera peut-être curieux de trouver à côté de l'appréciation si franchement modeste que Monge faisait de son œuvre, le jugement qu'en portait Lagrange: «Persuadé, disait Monge dans le préambule de son Mémoire, qu'une idée, stérile entre les mains d'un homme ordinaire, peut devenir très-profitable entre celles d'un habile géomètre, je vais faire part de mes recherches à l'Académie de Turin».Voici maintenant les paroles de Lagrange dans toute leur naïveté: «Avec son application de l'analyse à la représentation des surfaces, ce diable d'homme sera immortel!» [...] Dans une des leçons, non obligatoires, de l'ancienne École polytechnique; dans une de ces leçons, aujourd'hui supprimées, qui étaient destinées à développer le goût des sciences chez les premiers élèves, Monge appliqua sa théorie des lignes de courbure à l'ellipsoïde. Plusieurs professeurs s'étaient empressés d'aller écouter leur confrère: ils se donnaient alors les uns les autres de ces marques de déférence. À l'issue de la séance, Monge fut entouré et comblé de félicitations. Celles qui sortirent de la bouche de Lagrange nous ont été conservées: «Vous venez, mon cher confrère, d'exposer des choses très-élégantes; je voudrais les avoir faites.» Monge avouait que jamais compliment n'alla plus droit à son cœur."

nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies [23]:[3]

$$\frac{\partial}{\partial x}\left(\frac{\frac{\partial u}{\partial x}}{\sqrt{1+|Du|^2}}\right)+\frac{\partial}{\partial y}\left(\frac{\frac{\partial u}{\partial y}}{\sqrt{1+|Du|^2}}\right)=0$$

if we place:

$$r=\frac{\partial^2 u}{\partial x^2},\quad s=\frac{\partial^2 u}{\partial x\partial y},\quad t=\frac{\partial^2 u}{\partial y^2}$$

the minimal surface equation becomes:

$$\frac{\partial}{\partial x}\left(\frac{p}{\sqrt{1+p^2+q^2}}\right)+\frac{\partial}{\partial y}\left(\frac{q}{\sqrt{1+p^2+q^2}}\right)=0$$

that is:

$$(1+q^2)r-2pqs+(1+p^2)t=0$$

With his method of variations, Lagrange is able to treat, for the first time, the case of functions of several variables and obtain the minimal surface equation. Lagrange, however, does not provide any solution to the equation. No progress is made until a pupil of Monge at the *École de Mézières*, Jean-Baptiste Meusnier de la Place (1754-1793), presents a memoir to the *Académie des Sciences* in 1776, printed a few years later: *Sur la courbure des surfaces*. Meusnier's starting point is Euler's memoir on the curvature of a surface at one point, in which he defines the mean curvature as semi-sum of the smallest and largest curvature of curves passing through the point (see [23, 15, 28]).

Meusnier and Charles Tinseau d'Amondans de Gennes (1748-1822) were Monge's pupils of the nobility whose scientific activity most benefited from his presence in Mézières. Tinseau studied the twisted curves and surfaces as well as the solids limited by latter. Meusnier produced a fundamental memoir in which it is recognized that the minimal surfaces are of zero mean curvature, and he, moreover, gave the first examples of surfaces of minimum area, the catenoid, that is the rotation surface around an axis, generated by the catenary:

$$u(x)=\cosh x$$

already obtained by Euler as a minimum of the functional:

$$F(u)=2\pi\int_{-1}^{1}u(x)\sqrt{1+u'(x)^2}\,dx$$

Meusnier had started from the equation studied by Monge:

$$q^2r-2pqs+p^2t=0$$

with the condition $r+t=0$, finding, just as a solution to the minimal surface equation, the helicoid:

$$u(x,y)=k\arctan\frac{y}{x}.$$

We have to wait fifty years to discover a third surface of minimal area, the surface of Scherk (1835), starting from the memoirs of Monge and Meusnier [45]:[4]

$$u(x,y)=\log\left(\frac{\cos x}{\cos y}\right).$$

[3]Lagrange had already written the minimal area equation in one of his first letters to Euler [5].

[4]Heinrich Ferdinand Scherk (1798-1885) had been a student of W. Bessel in Könisberg, he succeeded J.F. Pfaff in the chair of mathematics at Halle.

At the base of these results, there was an attempt by Monge to study the minimal surface equation, an attempt that highlights the merits and defects of his approach to the theory of partial differential equations [32].

The solution provided by Monge was criticized by Legendre who claimed to have been invited by Monge himself to study the issue he was debating with Laplace. According to Legendre, Monge's error was due to unclear metaphysical principles on which the mathematicians did not agree. Instead of Monge's solution, Legendre provided another one by making a very elegant change in the variable (Legendre transformation) (see [27, 21]).

Monge had also attributed to Borda the first study of the equation of the minimal area obtained previously by Lagrange (and already contained in a letter to Euler of 1756). In those years, Legendre found his necessary condition for the minimum of a functional (see [4, 26, 50]).

The limits of Monge's 'geometric' point of view in the study of PDEs, already detected by Laplace and Legendre, were then brought to light by Cauchy, in his report to the *Académie des sciences* on a memoir of Poncelet (1820). This is the excessive trust that is placed in the so-called 'geometric intuition', which is so strong and, in general, so exact, in Monge. Cauchy was not less strict in judging, for example, the principle of continuity, the legacy of German dogmatic philosophy, than he was in the critique of the foundations of analysis:

> This principle is, strictly speaking, only a strong induction, by means of which we extend theorems first established under certain restrictions, in cases where these same restrictions no longer exist [...]. We think that it cannot be generally accepted and applied indistinctly to all sorts of questions in geometry, or even in analysis. By giving it too much confidence, one could sometimes fall into obvious errors. We know, for example. that in the determination of definite integrals, and consequently in the evaluation of lengths, surfaces, and volumes, we find a great number of formulas, which are true only to the extent that the values of the quantities which they contain remain within certain limits.[5]

4 The Monge-Ampère Equation

The Monge-Ampère Equation originates from Monge's memoir presented in 1776 and published in 1784 under the title *Sur la théorie des déblais et des remblais*. It begins with a concrete problem:

> When land is to be moved from one place to another, it is customary to give the name of *déblai* (excavation) to the volume of land to be transported, and the name of *remblai* (embankment) to the space which they must occupy after transportation. The price of the transport of a molecule being, all other things equal, proportional to its weight and the space it is made to travel, and consequently the product of the total transport proportional to the sum of the products of the molecules multiplied by the space traveled, it follows that the excavation and the embankment being given of figure and position, it is not indifferent that such molecule of the excavation is transported in this or that other place of the embankment, but that there is a certain distribution to be made of the molecules from the first to the second, according to which the sum of these

[5]"Ce principe n'est à proprement parler, qu'une forte induction, à l'aide de laquelle on étend des théorèmes établis d'abord à la faveur de certaines restrictions, aux cas où ces même restrictions n'existent plus [...] nous pensons qu'il ne saurait être admis généralement et appliqué indistinctement à toutes sortes de questions en géométrie, ni même en analyse. En lui accordant trop de confiance, on pourrait quelquefois tomber dans des erreurs manifestes. On sait, par exemple, que, dans la détermination des intégrales définies, et par suite, dans l'évaluation des longueurs, des surfaces et des volumes, on rencontre un grand nombre de formules, qui ne sont vraies qu'autant que les valeurs des quantités qu'elles renferment restent comprises entre certaines limites" [42, 2, p. 557].

> products will be the least possible, and the price of the total transport will be a minimum. [...] It is the solution of this question that I propose to give here.[6]

Monge recognized that the solution to this problem was far from being applicable in practice, but it would open the way to considerations about new general properties of curved surfaces. However, he put his memoir in the drawer from 1776 to 1784 and allowed it to be printed only on the insistence of Condorcet, then secretary of the *Académie des sciences*, who presented it with these words:

> Thus, one sees in the Sciences, sometimes brilliant but long useless theories, suddenly become the foundation of the most important applications, and sometimes very simple applications in appearance, to give rise to the idea of abstract theories which one did not yet need, to direct, towards the theories, the works of the Geometers, and to open them to a new career.[7]

The equation to which Monge arrives takes the following form:

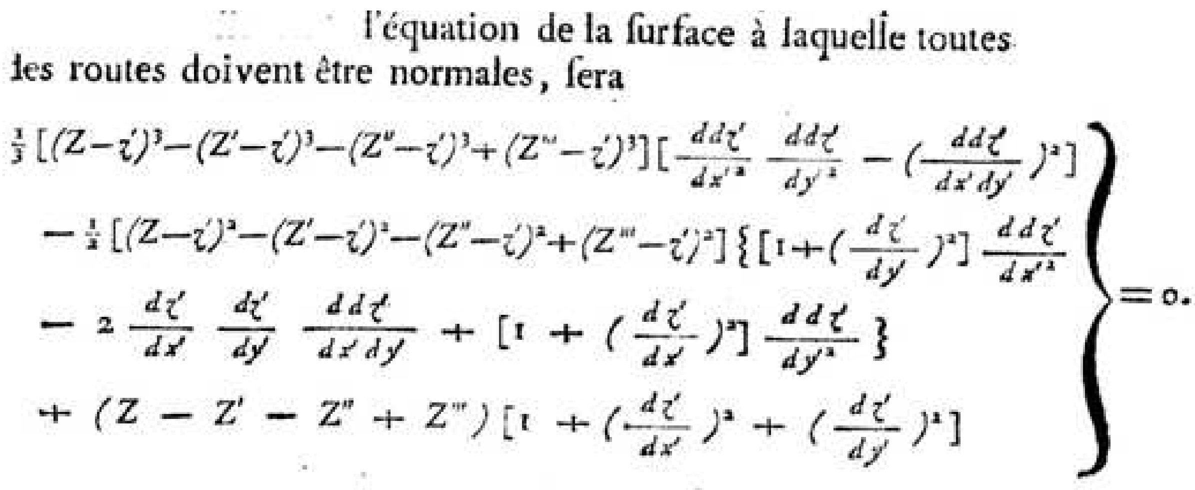
$$A(rt-s^2)+B[(1+q^2)r-2pqs+(1+p^2)t]+C(1+p^2+q^2)=0$$

In Monge's memoir it appears as:

$$\begin{aligned}
&\frac{1}{3}\left[(Z-z')^3-(Z'-z')^3-(Z''-z')^3+(Z'''-z')^3\right]\left[\frac{ddz'}{dx'^2}\frac{ddz'}{dy'^2}-\left(\frac{ddz'}{dx'dy'}\right)^2\right]+\\
&-\frac{1}{2}[(Z-z')^2-(Z'-z')^2-(Z''-z')^2+(Z'''-z')^2]\cdot\\
&\cdot\left\{\left[1+\left(\frac{dz'}{dy'}\right)^2\right]\frac{ddz'}{dx'^2}-2\frac{dz'}{dx'}\frac{dz'}{dy'}\frac{ddz'}{dx'dy'}+\left[1+\left(\frac{dz'}{dx'}\right)^2\right]\frac{ddz'}{dy'^2}\right\}+\\
&+(Z-Z'-Z''+Z''')\left[1+\left(\frac{dz'}{dx'}\right)^2+\left(\frac{dz'}{dy'}\right)^2\right]=0
\end{aligned}$$

[6]"Lorsqu'on doit transporter des terres d'un lieu dans un autre, on a coutume de donner le nom de déblai au volume des terres que l'on doit transporter, et le nom de remblai à l'espace qu'elles doivent occuper aprés le transport. Le prix du transport d'une molécule étant, toutes choses d'ailleurs égales, proportionnel à son poids et à l'espace qu'on lui fait parcourir, et par conséquent le produit du transport total devant être proportionnel à la somme des produits des molécules multipliées par l'espace parcouru, il s'ensuit que le déblai et le remblai étant donnés de figure et de position, il n'est pas indifférent que telle molécule du déblai soit transportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine distribution à faire des molécules du premier dans le second, d'après laquelle la somme de ces produits sera la moindre possible, et le prix du transport total sera un minimum.[...] C'est la solution de cette question que je me propose de donner ici" [29, p. 666].

[7]"Ainsi, l'on voit dans les Sciences, tantôt [...] des théories brillantes, mais longtemps inutiles, devenir tout à coup le fondement des applications les plus importantes, et tantôt des applications très simples en apparence, faire naître l'Idée de théories abstraites dont on n'avoit pas encore le besoin, diriger vers les théories des travaux des Géomètres, et leur ouvrir une carrière nouvelle" [8, pp. 34-38].


l'équation de la ſurface à laquelle toutes les routes doivent être normales, ſera

$$\left.\begin{aligned}&\tfrac{1}{3}[(Z-\zeta')^3-(Z'-\zeta')^3-(Z''-\zeta')^3+(Z'''-\zeta')^3]\left[\frac{dd\zeta'}{dx'^2}\frac{dd\zeta'}{dy'^2}-\left(\frac{dd\zeta'}{dx'dy'}\right)^2\right]\\&-\tfrac{1}{2}[(Z-\zeta')^2-(Z'-\zeta')^2-(Z''-\zeta')^2+(Z'''-\zeta')^2]\left\{\left[1+\left(\frac{d\zeta'}{dy'}\right)^2\right]\frac{dd\zeta'}{dx'^2}\right.\\&\left.-2\frac{d\zeta'}{dx'}\frac{d\zeta'}{dy'}\frac{dd\zeta'}{dx'dy'}+\left[1+\left(\frac{d\zeta'}{dx'}\right)^2\right]\frac{dd\zeta'}{dy'^2}\right\}\\&+(Z-Z'-Z''+Z''')\left[1+\left(\frac{d\zeta'}{dx'}\right)^2+\left(\frac{d\zeta'}{dy'}\right)^2\right]\end{aligned}\right\}=0.$$

Fig. 1. Monge's original equation

It is a non-linear partial differential equation of the second order, which in some cases can be treated as an elliptic equation. It recalls the Laplace equation where, in place of the trace of the Hessian, its determinant appears. This is not a simple problem. It was not resolved by Monge, it was resumed without significant results by Ampère and Dupin and then long abandoned. It re-emerged as the theme of the Bordin Award for 1885, assigned to a long memoir by Paul Appell, which however did not resolve the issue. Indeed Appell returned to it several years later without substantial improvements (see [16, 18, 1]).

The Monge-Ampère Equation is today one of the most fruitful topics of contemporary research on partial differential equations, connected to the theory of optimal transport (Gauss curvature, weak solutions, regularity, partial regularity, uniqueness etc.). Two Italian scholars, Luigi Ambrosio and Alessio Figalli, have made significant contributions to the study of these equations in a generalized form. Figalli, also for these studies, was awarded a Fields medal for 2018 (see [20, 10, 22, 17]).

Let us close by returning to Russia again. The Finnish mathematician Anders Johann Lexell (1740-1784), after having taught at Uppsala, moved to Saint Petersburg to collaborate with Euler, to whom he succeeded in 1783. He made a trip to France, Germany and England, leaving us interesting observations. In a letter to Euler from Paris, 7th January 1781, he wrote:

> Monge is a very skilful man, he is a new addition to the Academy. He gave a memoir on the bodies whose surfaces can be developed in one plane and another on the partial differences that are highly valued. His appearance is not attractive, he is very brown, dark eyebrows and curved upper lip. Besides, he has a fairly high opinion of himself because the first time I saw him he told me that he was the only one in the country who treated Geometry.[8]

That Monge was the only one in 1781 in France, among creative mathematicians, to deal with advanced questions of geometry is true, and it is also true that within a few years geometry became fashionable again thanks mainly to him.

[8] "Monge est un homme très habile, il est nouvellement agrégé à l'Académie. Il a donné un mémoire sur les corps dont les surfaces peuvent être développées dans un plan et un autre sur les différences partielles qu'on estime beaucoup. Sa figure n'est pas prévenante, il est très noir, les sourcils foncés et la lèvre supérieure recourbée. D'ailleurs il s'estime assez soi-même, car la première fois que je l'ai vû il m'a dit que c'étoit lui seul qui traitoit la Géométrie dans ce païs-ci" [48, p. 193].

References

[1] P. Appell, *Le problème géométrique des déblais et remblais.* Mémorial des sciences mathématiques, 27 (1928), Paris, Gauthier-Villars, 32 pp.

[2] F. Arago, *Gaspard Monge, biographie lue par extraits en séance publique de l'académie des sciences, le 11 mai 1846.* Oeuvres complètes de François Arago, secrétaire perpétuel de l'académie des sciences, 1854, 2, 427-592.

[3] L. Bianchi, *Lezioni di geometria differenziale.* Third edition, vol. 1, Bologna, Zanichelli, 1922, 221-223.

[4] J. Ch. Borda, *Eclaircissements sur les méthodes de trouver les courbes qui jouissent des quelques propriétés du maximum ou du minimum.* Hist. Mem. Acad. Sci 1767 (1770), 551-563.

[5] M.T. Borgato, L. Pepe, *Lagrange a Torino (1750-1759) e le sue lezioni inedite nelle R. Scuole di Artiglieria.* Boll. storia sci. mat., 7 (1987) n. 2, 3-43.

[6] D.V. Breslavski , A.A. Larin, V.B. Konovalova, *The scientific and social activity of Professor N. N. Saltykov in Russia in 1894-1919.* Journal of Theoretical and Applied Mechanics, 49 (2014), 179-187.

[7] H. Cartan, *Gaspard Monge, sa vie, son œuvre.* Paris, Université de Paris, 1948.

[8] N. Condorcet, *Sur les déblais et les remblais.* Hist. Mem. Acad. Sci. 1781 (1784), 34-38.

[9] G. Darboux, *Leçons sur la théorie générale des surfaces,* première partie. Paris, Gauthier Villars, 1914, 319-331.

[10] G. De Philippis, A. Figalli, *The Monge Ampère equation and its link to optimal transportation.* Bull. Amer. Math. Soc., 51 (2014), 527-580.

[11] J. Delcourt, *Monge et les équations différentielles "ordinaires"*, in S. Féry (ed.), Aventures de l'analyse de Fermat à Borel, Nancy, PUN-Editions Universitaires de Lorraine, 2012, 439-471.

[12] S.S. Demidov, *The study of partial differential equations of the first order in the 18th and the 19th centuries.* Arch. History Exact Sci. 26 (1982), 325-350.

[13] J.C. Domingues, *Lacroix and the calculus.* Basel, Birkhäuser, 2008.

[14] C. Dupin, *Essai historique sur les services et les travaux scientifiques de Gaspard Monge.* Paris, 1819.

[15] L. Euler, *Recherches sur la courbure des surfaces.* Mém. Acad. Berlin, 1760 (1767), 119-143.

[16] L. Euler, *De solidis quorum superficiem in planum explicare licet.* Nov. Comm. Acad. Petrop. 16, 1771 (1772), 3-34.

[17] A. Figalli, *On the Monge-Ampère equation.* Séminaire Bourbaki, n. 1148, Juin 2018, 26 pp.

[18] C.F. Gauss, *Disquisitiones generales circa superficies curvas.* Gottingae, Typis Dieterichianis, 1828, 50 pp.

[19] C. Gilain, *Le calcul intégral dans la correspondance mathématique de Condorcet*, in A. Cogliati (ed.), La matematica nella storia. Saggi in onore di Umberto Bottazzini, Milan, Egea Editore, 2019, 31-58.

[20] E. Ghys, *Gaspard Monge. Le mémoire sur les déblais et les remblais.* 20 Janvier 2012. http://images.math.cnrs.fr/Gaspard-Monge,1094.html?lang=fr

[21] E. Goursat, *Sur le problème de Monge.* Bull. Soc. Math. Fr. 33 (1905), 201-210.

[22] C.E. Gutierrez, *The Monge Ampère equation.* Basel, Birkhäuser, 2016.

[23] J.L. Lagrange, *Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies.* Mém. Turin 1760-61, II p. 2 (1762), 173-195.

[24] J.L. Lagrange, *Sur l'intégration des équations partielles du premier ordre.* Nouv. Mém. Acad. Berlin 1772 (1774), 353-372.

[25] J.L. Lagrange, *Sur les intégrales particulières des équations différentielles.* Nouv. Mém. Acad. Berlin 1774 (1776), 197-275.

[26] A.M. Legendre, *Sur la manière de distinguer les maxima et les minima dans le calcul des variations.* Mém. Acad, Sci. 1786 (1788), pp. 7-37. *Additions.* Hist. Mem. Acad. Sci, 1787 (1789), 348-351.

[27] A.M. Legendre, *Sur l'intégration de quelques équations aux différences partielles.* Mém. Acad, Sci. 1787 (1789), II, 309-351.

[28] J.B. Meusnier, *Sur la courbure des surfaces.* Mém. Div. Sav. 10 (1785), 477-510.

[29] G. Monge, *Sur la théorie des déblais et des remblais.* Hist. Mem. Acad. Sci. 1781 (1784), 666-704.

[30] G. Monge, *Sur l'intégration de l'équation aux différences partielles:* $L\frac{ddx}{dx^2} + M\frac{ddz}{dxdy} + N\frac{ddz}{dy^2}$ *dans laquelle les coefficiens* L, M, N *sont quelconques.* Mém. Acad. Sci. Turin, I, 1784-1785, (1786), p. I, 31-33.

[31] G. Monge, *Sur une méthode d'intégrer les équations aux différences ordinaires.* Hist. Mem. Acad. Sci. 1783 (1786), 719-724.

[32] G. Monge, *Sur l'expression analytique de la génération des surfaces courbes.* Mém. Acad. Sci. 1784 (1787), 85-117.

[33] G. Monge, *Sur le calcul intégral des équations aux différences partielles.* Mém. Acad. Sci. 1784 (1787), 118-192.

[34] G. Monge, *Supplément où l'on fait voir que les équations aux différences ordinaires etc.* Mém. Acad. Sci. 1784 (1787), 502-576.

[35] G. Monge, *Traité élémentaire de statique.* (Paris, Musier 1788), Saint Petersburg, 1803.

[36] G. Monge, *Application de l'analyse à la géométrie.* (Paris, Bernard 1807), Mosca 1936, notes and comments by N. I. Yougodki.

[37] G. Monge, *Géométrie descriptive.* (Paris, 1799), Moscow 1947, notes and comments by D. I. Karguin.

[38] G. Monge, *Dall'Italia, (1796-1798).* S. Cardinali and L. Pepe (eds.), Palermo, Sellerio, 1993.

[39] F. Pairault, *Gaspard Monge. Le fondateur de Polytechnique.* Paris, Tallandier, 2000.

[40] L. Pepe, *La formazione della biblioteca dell'Ecole Polytechnique. Il contributo involontario del Belgio e dell'Italia.* Boll. Sto. Sci. Mat., 16 (1996), 155-198.

[41] L. Pepe, *Istituti azionali, accademie e società scientifiche nell'Europa di Napoleone.* Firenze, Olschki, 2005.

[42] J.V. Poncelet, *Applications d'analyse et de géométrie*, vol. 2, Paris, Mallet-Bachelier, 1862-1864.

[43] J.D. Reuss, *Repertorium commentationum a societatibus litterariis editarum secondum disciplinarum ordinem, tom. VII.* Gottingae, Apud Henricum Dieterich, 1808.

[44] N. Saltykov, *Méthodes classiques d'intégration des équations aux dérivées partielles du premier ordre à une fonction inconnue.* Mémorial des sciences mathématiques, 50 (1931), Paris, Gauthier-Villars, 72 pp.

[45] H.F. Scherk, *Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen.* J. R. Angew. Math. 13 (1835), 185-208.

[46] D.J. Struik, *Classical differential geometry.* New York, Dover, 1961.

[47] R. Taton, *Une correspondance mathématique inédite de Monge.* La Revue scientifique, 85 (1947), 963-989.

[48] R. Taton, *L'œuvre scientifique de Gaspard Monge.* Paris, Presses universitaires de France 1951.

[49] R. Taton, *La première note mathématique de Gaspard Monge (juin 1769).* Revue d'histoire des sciences et de leurs applications, 19, no. 2 (1966), 143-149.

[50] I. Todhunter, *A history of the progress of the calculus of variations during the nineteenth century.* Cambridge, Macmillan 1861.

[51] *Un savant en son temps: Gaspard Monge (1746-1818).* Bulletin de la Société des Amis de la Bibliothèque de l'Ecole polytechnique, 41 (2007).

[52] A.P. Yushkevich, *The concept of function up to the middle of the 19th century.* Arch. Hist. Exact Sci. 16 (1976), 37-85.

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