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- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.

ANDREI ANDREEVICH SHKALIKOV

(to the 70th birthday)

Andrei Andreevich Shkalikov, corresponding member of the Russian Academy of Sciences, an outstanding mathematician with a wide range of interests, a remarkable person, professor of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University was born on November 19, 1949.

Andrei Andreevich is a leading specialist in the theory of operators and their applications, especially to problems of mechanics and mathematical physics. He is well known for his work in the theory of functions and in the theory of spaces with indefinite metrics. He is also a specialist in the theory of entire and meromorphic functions and its applications to operator theory.

Andrei Andreevich is known for solving a number of difficult problems that for many years remained unsolved. His work on the basic properties of systems of root functions of differential operators is well known worldwide. He gave a justification for Mandelstam's hypothesis about the existence of solutions satisfying the radiation principle. He solved, in general form, the Rayleigh problem on the reflection of waves from a periodic surface, obtained a solution to the Sobolev problem on the stability of the motion of a top with a cavity filled with liquid. His contribution to the construction of an abstract theory of the Orr-Sommerfeld problem is invaluable. He obtained a description of the limiting spectral portraits for a large class of functions describing the profiles of fluid flows. He is one of the founders of the modern theory of differential operators, coefficients of which are distributions, and inverse problems for such operators.

Andrei Andreevich has been a plenary speaker at many international conferences. He conducts fruitful scientific work and collaborates with many international mathematical research centers.

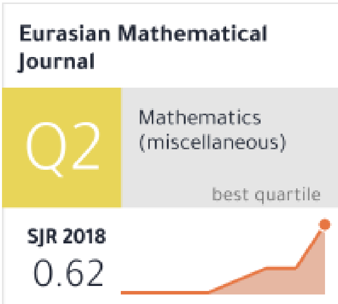
Andrei Andreevich is an author of more than 130 scientific publications. Among his pupils there are more than 20 Candidates of Sciences and 6 Doctors of Sciences. The results obtained by A.A. Shkalikov, his pupils, collaborators and followers gained worldwide recognition.

Professor Shkalikov is also an outstanding organizer. Under his supervision, many international conferences were held. In particular, conferences dedicated to the memory of I.G. Petrovsky, I.M. Gelfand, S.M. Nikol'skii, B.M. Levitan, anniversary conferences of V.A. Sadovnichy, and others.

Andrei Andreyevich is a deputy editor-in-chief of the journals Mathematical Notes, Moscow University Mathematics Bulletin, Moscow University Mechanics Bulletin, and a member of the editorial boards of the Russian Mathematical Surveys, Proceedings of the Moscow Mathematical Society and other journals, including the Eurasian Mathematical Journal.

The Editorial Board of the Eurasian Mathematical Journal cordially congratulates Andrei Andreevich on the occasion of his 70th birthday and wishes him good health, and new achievements in mathematics and mathematical education.

GOOD NEWS: EMJ IS NOW AN SJR Q2 JOURNAL



Recently the lists were published of all mathematical journals included in 2018 SCImago Journal Rank (SJR) quartiles Q1 (385 journals), Q2 (430 journals), Q3 (445 journals), and Q4 (741 journals), and Scopus CiteScore quartiles Q1 (443 journals), Q2 (375 journals), Q3 (348 journals), and Q4 (283 journals).

With great pleasure we inform our readers and authors that the Eurasian Mathematical Journal was included in the most popular scientific ranking database SJR in quartile Q2, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia. The SJR data for the Eurasian Mathematical Journal (2018) is as follows:

550. Eurasian Mathematical Journal (Kazakhstan), Q2, SJR=0.624.
(550 is the number in the list of all Q1 - Q4 journals.)

The SJR indicator is calculated by using the data of the Scopus Database of the Elsevier, the modern publishing business founded in 1880. It uses a sophisticated formula, taking into account various characteristics of journals and journals publications. This formula and related comments can be viewed on the web-page

http : //www.scimagojr.com/journalrank.php.

Some other SJR Q2 mathematical journals:
490. Studia Mathematica (Poland), SJR=0.706,
492. Comptes Rendus Mathematique (France), SJR=0.704,
522. Journal of Mathematical Physics (USA), SJR=0.667,
540. Doklady Mathematics (Russia), SJR=0.636,
570. Journal of Mathematical Sciences (Japan), SJR=0.602,
662. Journal of Applied Probability (UK), SJR=0.523,
733. Mathematical Notes (Russia), SJR=0.465,
791. Canadian Mathematical Bulletin (Canada), SJR=0.433.

Our journal ranks:

7726th place in the list of 31971 scientific journals, representing all subjects and all regions, included in this database (in the first 25% of journals of this category),

225th place in the list of 2519 scientific journals, representing all subjects, of the Asiatic region, included in this database (in the first 10% of journals of this category),

550th place in the list of 2011 mathematical journals, representing all regions, included in this database (in the first 30% of journals of this category),

19th place in the list of 165 mathematical journals of the Asiatic region, included in this database (in the first 15% of journals of this category).

On a separate page the SJR statistics for the Eurasian Mathematical Journal is attached.

Recall that the Eurasian Mathematical Journal started its work in 2010 (see [1]-[3]) and was first included in SJR indicator in 2014 (Q4, SJR=0.101, see [4], [5], [6]). So, the ambitious plan set in [6] was implemented and even essentially exceeded.

As for the Scopus CiteScore indicator, it uses another sophisticated formula, differently taking into account various characteristics journals publications. This formula and related comments can be viewed on the web-page

https : //ru.service.elsevier.com/app/answers/detail/a$_{i}$d/19266/supporthub/scopus.

In this indicator the Eurasian Mathematical Journal was included in quartile Q3. The CiteScore data for the Eurasian Mathematical Journal (2018) is as follows:

333. Eurasian Mathematical Journal (Kazakhstan), Q3, CiteScore = 0,41

(333 is the number in the list of only Q3 journals.)

Some other Scopus CiteScore Q3 mathematical journals:

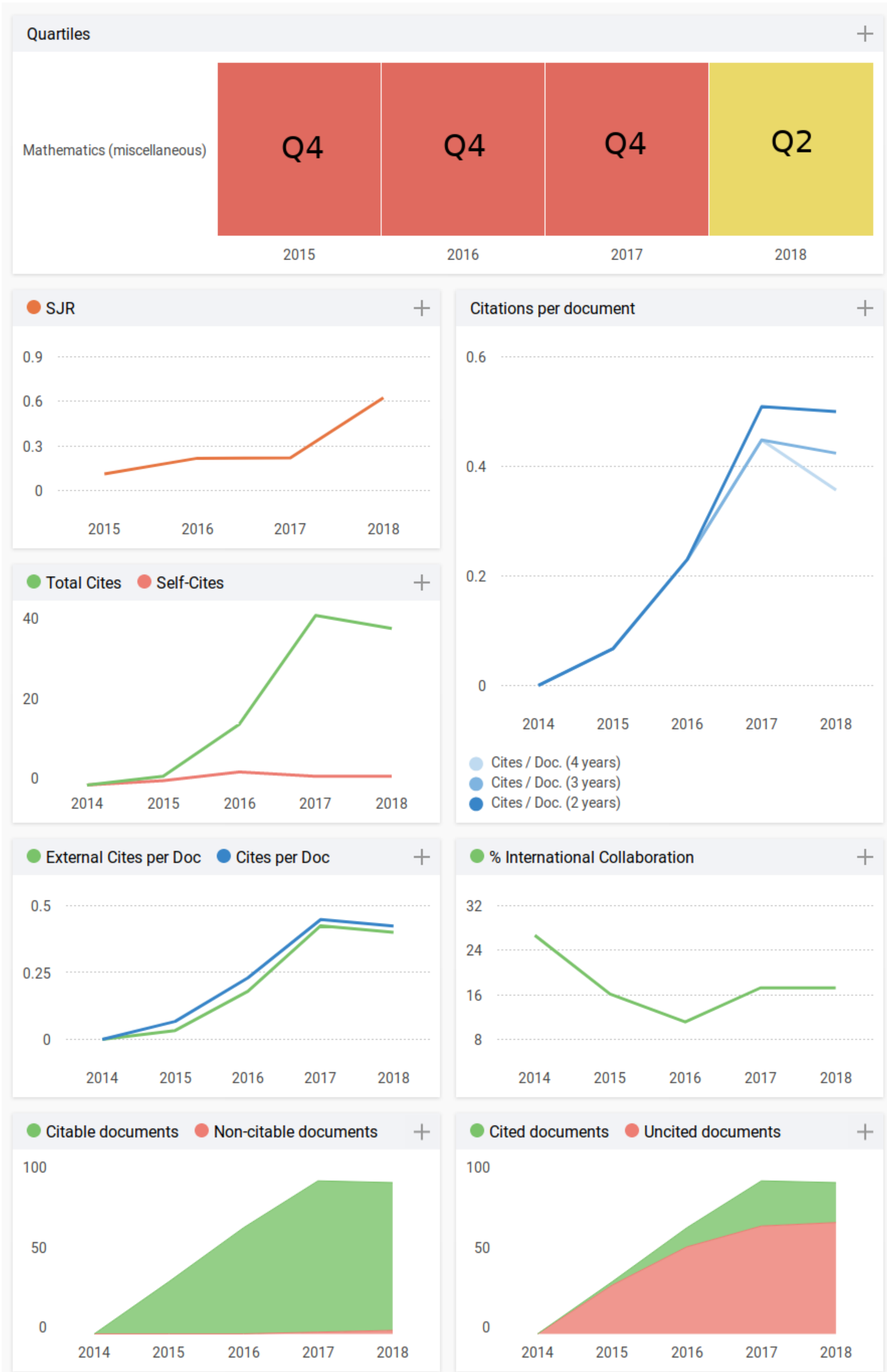
320. Czechoslovak Mathematical Journal (Czech Republic), CiteScore = 0.44,
321. Italian Journal of Pure and Applied Mathematics (Italy), CiteScore = 0.44,
323. Studia Scientiarum Mathematicarum Hungarica (Hungary), CiteScore = 0.44,
332. Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie (Romania), CiteScore = 0.41,
334. Indian Journal of Pure and Applied Mathematics (India), CiteScore = 0.41,
33. Transactions of the Moscow Mathematical Society (Russia), CiteScore = 0.41,
337. Illinois Journal of Mathematics (USA), CiteScore = 0.40,
339. Publications de l'Institut Mathematique (France), CiteScore = 0.40.

Our main current aim is to preserve the status of an SJR Q2 journal and of a Scopus CiteScore Q3 journal.

We hope that all respected members of the international Editorial Board, reviewers, current authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

V.I. Burenkov, K.N. Ospanov, T.V. Tararykova, A.M. Temirkhanova.

Extract from http://www.scimagojr.com/journalrank.php
EURASIAN MATHEMATICAL JOURNAL



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THE SOLVABILITY RESULTS FOR THE THIRD-ORDER SINGULAR NON-LINEAR DIFFERENTIAL EQUATION

Zh.B. Yeskabylova, K.N. Ospanov, T.N. Bekjan

Communicated by M. Otelbaev

Key words: non-linear differential equation, intermediate term, solvability, estimates of solutions.

AMS Mathematics Subject Classification: 34A34, 34B40, 34C11.

Abstract. We give some conditions for solvability in $L_2(\mathbb{R})$ ($\mathbb{R} = (-\infty, +\infty)$) of the following singular non-linear differential equation:

$$ly \equiv -y'''(x) + q(x, y, y')\, y' + s(x, y, y')\, y = h(x).$$

We assume that q and s are real-valued unbounded functions and q does not obey the "potential" s. For the solution y we prove that

$$\|y'''\|_2 + \|q(\cdot, y, y')\, y'\|_2 + \|s(\cdot, y, y')\, y\|_2 < \infty,$$

where $\|\cdot\|_2$ is the norm in L_2. To establish these facts, we use coercive solvability results for the corresponding linear third-order differential equation obtained by us earlier.

DOI: https://doi.org/10.32523/2077-9879-2019-10-4-85-91

1 Introduction

We consider the following non-linear third-order differential equation:

$$ly \equiv -y'''(x) + q(x, y, y')\, y' + s(x, y, y')\, y = h(x), \tag{1.1}$$

where $q(x, u, w)$, $s(x, u, w)$ ($u, w \in \mathbb{R}$) and $h \in L_2 := L_2(\mathbb{R})$ are real-valued functions. We assume that q and s are, respectively, continuously differentiable and continuous. Using the coercive estimates for a solution of the linear differential equation

$$-y''' + qy' + sy = f(x)$$

in [4], we obtain the solvability result for (1.1).

We note that equation (1.1) contains the known generalized Korteweg-de Vries equation. Since 1980 the following quasilinear third-order equation was intensively studied

$$-y''' + g(x, y, y')y = f(x), \tag{1.2}$$

where $x \in \mathbb{R}$ and $g \geq 1$ (see [1], [2] and the references therein). Sufficient conditions for a coercive solvability of (1.2) with $f \in L_p(\mathbb{R})$ ($1 < p < +\infty$) and effective estimates of approximate characteristics (for example, the Kolmogorov diameters of the set of solutions, see [3]) were obtained. The properties of equation (1.1) are markedly differ from (1.2). Firstly in (1.2) the intermediate coefficient equal to zero. Secondly, in (1.2) $g \geq 1$, and in (1.1) the potential s can be the sign-variable function, or it may be not limited from below. In addition, if $s = 0$, and r is bounded, then equation (1.1) does not have a solution in L_2.

Definition 1. A function $y \in L_2$ is called a solution of equation (1.1), if there exists a sequence $\{y_n\}_{n=1}^{\infty} \subseteq C_0^{(3)}(\mathbb{R})$ ($C_0^{(3)}(\mathbb{R})$ is the set of all thrice continuously differentiable functions on $\mathbb{R}$ with compact support) such that $\|\theta\,(y_n - y)\|_2 \to 0$ and $\|\theta\,(ly_n - h)\|_2 \to 0$ as $n \to \infty$ for any continuous function θ with compact support.

For real-valued continuous functions p and $v \neq 0$, we denote

$$\alpha_{p,v}(x,\zeta,\omega) = \left(\int_0^x |p(t,\zeta,\omega)|^2\,dt\right)^{\frac{1}{2}} \cdot \left(\int_x^{+\infty} v^{-2}(t,\zeta,\omega)\,dt\right)^{\frac{1}{2}},\ x > 0,$$

$$\beta_{p,v}(\tau,\zeta,\omega) = \left(\int_\tau^0 p^2(t,\zeta,\omega)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^\tau v^{-2}(t,\zeta,\omega)\,dt\right)^{\frac{1}{2}},\ \tau < 0,$$

$$\gamma_{p,v} = \max\left(\sup_{\{(x,\zeta,\omega):\ x>0,\,(\zeta,\,\omega)\in\mathbb{R}^2\}} \alpha_{p,v}(x),\ \sup_{\{(\tau,\zeta,\omega):\ \tau<0,\,(\zeta,\,\omega)\in\mathbb{R}^2\}} \beta_{p,v}(\tau)\right)$$

The following theorem is the main result of this work.

Theorem 1.1. *Assume that q and s are, respectively, continuously differentiable and continuous functions such that*

$$q(t,u,w) \geq C\left(1+t^2\right), \tag{1.3}$$

where $C > 0$ is independent of $t \in \mathbb{R}$,

$$\gamma_{s,q} < \infty, \tag{1.4}$$

and, for any positive number A,

$$\sup_{\substack{x,\eta\in\mathbb{R}:\\ |x-\eta|\leq 1}} \ \sup_{\substack{|C_i'|\leq A,\ |C_i''|\leq A,\\ |C_i'-C_i''|\leq A,\\ i=1,\,2}} \frac{q(x,C_1',C_2')}{q(\eta,C_1'',C_2'')} < \infty. \tag{1.5}$$

Then for any $h \in L_2$, equation (1.1) has a solution y. Moreover, the solution y satisfies

$$\|y'''\|_2 + \|q(\cdot,y,y')\,y'\|_2 + \|s(\cdot,y,y')\,y\|_2 < \infty. \tag{1.6}$$

Example 1. We consider the following nonlinear equation

$$-y''' + \left[13 + 2x^8 + y^6\sin^2 e^{x^2} + 2\,(y')^{10}\right]y' -$$

$$\left[x^3 + 2y^5 + (y')^2\right]y = f(x),\, x \in \mathbb{R}. \tag{1.7}$$

It is easy to check that all conditions of Theorem 1.1 are satisfied. Consequently, for any $f \in L_2$, exists a solution y of equation (1.7) and

$$\|y'''\|_2 + \left\|\left[13 + 2x^8 + y^6\sin^2 e^{x^2} + 2\,(y')^{10}\right]y'\right\|_2 + \left\|\left[x^3 + 2y^5 + (y')^2\right]y\right\|_2 < +\infty.$$

2 On linear singular differential equations

In this section we consider the following singular linear differential equation

$$-y''' + \tilde{q}(x)\, y' + \tilde{s}(x)\, y = u_0(x)\,, x \in \mathbb{R}, u_0 \in L_2. \tag{2.1}$$

We denote by $\tilde{l}$ the closure in L_2 of the differential operator $\tilde{l}_0 y \equiv -y''' + \tilde{q}(x)\, y' + \tilde{s}(x)\, y$ defined on $C_0^{(3)}(\mathbb{R})$. We say that $y \in L_2$ is a solution of equation (2.1), if $y \in D(\tilde{l})$ and $\tilde{l}y = u_0$. The following lemma follows from Theorem 1.1 in [4].

Lemma 2.1. *Let $\tilde{q} \geq 1$ be a continuously differentiable function, and $\tilde{s}$ be a continuous function such that*

$$\gamma_{|\tilde{s}|+1,\tilde{q}} =$$

$$\max \left(\sup_{x>0} \left[\int_0^x (|\tilde{s}(t)| + 1)^2 \, dt \right]^{1/2} \left[\int_x^{+\infty} \tilde{q}^{-2}(t) dt \right]^{1/2}, \right.$$

$$\left. \sup_{\tau<0} \left[\int_\tau^0 (|\tilde{s}(t)| + 1)^2 \, dt \right]^{1/2} \left[\int_{-\infty}^\tau \tilde{q}^{-2}(t) dt \right]^{1/2} \right) < +\infty \tag{2.2}$$

and

$$\sup_{\substack{\forall x,\, \eta \in \mathbb{R}: \\ |x-\eta| \leq 1}} \frac{\tilde{q}(x)}{\tilde{q}(\eta)} < \infty.$$

Then for any $u_0 \in L_2$ there exists a unique solution y of equation (2.1). Moreover, y satisfies the following coercive estimate:

$$\|y'''\|_2 + \|\tilde{q}\, y'\|_2 + \|\tilde{s}\, y\|_2 \leq C \left(1 + 2\gamma_{|\tilde{s}|+1,\, \tilde{q}}\right) \|u_0\|_2\,,$$

where $C > 0$ is independent of y.

3 Proof of Theorem 1.1

Let $W_2^2(\mathbb{R})$ be the completion of $C_0^{(2)}(\mathbb{R})$ with the norm

$$\|z\|_{W_2^2(\mathbb{R})} = \left(\|z''\|_2^2 + \|z'\|_2^2 + \|z\|_2^2 \right)^{1/2}.$$

Assume that ε and A are positive numbers, and $S_A = \left\{ v \in W_2^2(\mathbb{R}) : \|v\|_{W_2^2(\mathbb{R})} \leq A \right\}$. We consider the following linear differential equation:

$$\tilde{l}_{0,v,\varepsilon} y = -y''' + \left[q(x,\, v(x),\, v'(x)) + \varepsilon \left(1 + x^2\right)^4 \right] y'$$

$$+ \left[s\left(x,\, v(x),\, v'(x)\right) \right] y = h(x), \tag{3.1}$$

where $v \in S_A$. Let $\tilde{l}_{v,\varepsilon}$ be the closure in L_2 of the differential operator $\tilde{l}_{0,v,\varepsilon} y$ defined on $C_0^{(3)}(\mathbb{R})$. We say that $y \in L_2$ is a solution of (3.1), if $y \in D\left(\tilde{l}_{v,\varepsilon}\right)$ and $\tilde{l}_{v,\varepsilon} y = h$.

We show that $q_{\nu,\varepsilon}(x) := q(x,\, v(x),\, v'(x)) + \varepsilon\,(1 + x^2)^4 \geq 1$ and $s_\nu(x) := s\,(x,\, v,\, v')$ satisfy conditions of Lemma 2.1.

(2.2) follows from conditions (1.3) and (1.4).

Let $x,\, \eta \in \mathbb{R} : |x-\eta| \le 1$. Then

$$\frac{q_{v,\,\varepsilon}\left(x\right)}{q_{v,\,\varepsilon}\left(\eta\right)} \le \frac{q\left(x,\, v\left(x\right),\, v'\left(x\right)\right)}{q\left(\eta,\, v\left(\eta\right),\, v'\left(\eta\right)\right)} + \frac{\varepsilon\left(1+x^2\right)^4}{\varepsilon\left(1+\eta^2\right)^4}$$

$$\le \frac{q\left(x,\, v\left(x\right),\, v'\left(x\right)\right)}{q\left(\eta,\, v\left(\eta\right),\, v'\left(\eta\right)\right)} + 81. \tag{3.2}$$

For $v \in S_A$ the following estimates hold:

$$|v(x)-v(\eta)| \le A, \;\; |\nu'\left(x\right)-\nu'\left(\eta\right)| \le A. \tag{3.3}$$

Indeed, we have

$$|\nu\left(x\right)-\nu\left(\eta\right)| \le \sqrt{x-\eta}\,\|v\|_{W_2^1(\mathbb{R})} \le \sqrt{x-\eta}\,\|v\|_{W_2^2(\mathbb{R})} \le A$$

and

$$|\nu'\left(x\right)-\nu'\left(\eta\right)| \le \sqrt{x-\eta}\,\|v\|_{W_2^2(\mathbb{R})} \le A.$$

Let $v\left(x\right)=C_1^{'}$, $v'\left(x\right)=C_2^{'}$, $v\left(\eta\right)=C_1^{''}$ and $v'\left(\eta\right)=C_2^{''}$. By (3.3), we have that

$$\frac{q(x,\, v(x),\, v'(x))}{q(\eta,\, v(\eta),\, v'(\eta))} \le \sup_{\substack{\left|C_i^{'}\right| \le A,\; \left|C_i^{''}\right| \le A, \\ \left|C_i^{'}-C_i^{''}\right| \le A \\ i=1,\,2}} \frac{q(x, C_1^{'},\, C_2^{'})}{q(\eta, C_1^{''},\, C_2^{''})}.$$

Consequently, by (1.5) and (3.2), we obtain that

$$\sup_{x,\eta\in R:\,|x-\eta|\le 1} \frac{q_{v,\,\varepsilon}\left(x\right)}{q_{v,\,\varepsilon}\left(\eta\right)} \le$$

$$\sup_{x,\eta\in\mathbb{R}:\,|x-\eta|\le 1} \sup_{\substack{\left|C_i^{'}\right| \le A,\; \left|C_i^{''}\right| \le A, \\ \left|C_i^{'}-C_i^{''}\right| \le A \\ i=1,\,2}} \frac{q(x, C_1^{'},\, C_2^{'})}{q(\eta, C_1^{''},\, C_2^{''})} + 81 < \infty.$$

So, the coefficients $q_{\nu,\varepsilon}$ and s_ν of equation (3.1) satisfy the conditions of Lemma 2.1. Then, for any $h \in L_2$ there exists a unique solution $y_{v,\,\varepsilon} \in L_2$ of equation (3.1), and for $y_{v,\,\varepsilon}$ the following estimate holds:

$$\left\|y_{v,\,\varepsilon}^{'''}\right\|_2 + \left\|q_{v,\,\varepsilon}y_{v,\,\varepsilon}^{'}\right\|_2 + \|s_v y_{v,\,\varepsilon}\|_2 \le C_3\,\|h\|_2\,,$$

where $C_3>0$ is independent of $y_{v,\,\varepsilon}$. Taking into account condition (1.3) and the following obvious inequality

$$\left\|\sqrt{1+x^2}y_{v,\,\varepsilon}^{''}\right\|_2 \le 3\left(\left\|y_{v,\,\varepsilon}^{'''}\right\|_2 + \left\|\left(1+x^2\right)y_{v,\,\varepsilon}^{'}\right\|_2\right),$$

we obtain that a solution $y_{v,\,\varepsilon}$ of (3.1) satisfies the following estimate:

$$\left\|y_{v,\,\varepsilon}^{'''}\right\|_2 + \left\|\sqrt{1+x^2}y_{v,\,\varepsilon}^{''}\right\|_2 + \left\|q_{v,\,\varepsilon}y_{v,\,\varepsilon}^{'}\right\|_2 + \|s_v y_{v,\,\varepsilon}\|_2 \le C_4\,\|h\|_2\,, \tag{3.4}$$

where $C_4=4C_3$.

We assume $A = 2C_4 \|h\|_2$. Let P_ε be the operator given by $P_\varepsilon(v) = \tilde{l}_{v,\varepsilon}^{-1} h$, where $v \in S_A$. The operator P_ε maps the set S_A into itself. Indeed, by (3.4), P_ε maps the set S_A into the following set

$$Q_A = \left\{ y : \left\| y''' \right\|_2 + \left\| \sqrt{1+x^2} y'' \right\|_2 + \left\| q_{v,\varepsilon} y' \right\|_2 + \| s_{v,\varepsilon} y \|_2 \le C_4 \left\| \tilde{h} \right\|_2 \right\},$$

according to (1.3), $Q_A \subseteq S_A$.

P_ε is a compact operator. Indeed, by (3.4), for any $y \in Q_A$ and $\sigma \neq 0$, we have

$$\int_{-\infty}^{+\infty} |y''(x+\sigma) - y''(x)|^2 dx + \int_{-\infty}^{+\infty} |y'(x+\sigma) - y'(x)|^2 dx +$$

$$\int_{-\infty}^{+\infty} |y(x+\sigma) - y(x)|^2 dx \le$$

$$C_5 |\sigma|^2 \left[\int_{-\infty}^{+\infty} |y'''(t)|^2 dt + \int_{-\infty}^{+\infty} |y''(t)|^2 dt + \int_{-\infty}^{+\infty} |y'(t)|^2 dt \right] \le$$

$$C_4 C_5 |\sigma|^2 \left\| \tilde{h} \right\|_2^2 \to 0 \, (\sigma \to 0).$$

If $N > 0$, then there exists $\mu \in (0,\ 1/2]$ such that $(y \in Q_A)$

$$\int_{\mathbb{R} \setminus [-N,\, N]} \left[|y''(t)|^2 + |y'(t)|^2 + |y(t)|^2 \right] dt \le$$

$$\le \frac{1}{(1+N^2)^\mu} \left[\left\| \sqrt{1+t^2} y'' \right\|_2 + \left\| (1+t^2) y' \right\|_2 + \left\| (1+t^2)^\mu y \right\|_2 \right] \le$$

$$\frac{C_4 \left\| \tilde{h} \right\|_2}{(1+N^2)^\mu} \to 0 \, (N \to \infty).$$

So, by the Riesz-Kolmogorov theorem, Q_A is compact in $W_2^2(\mathbb{R})$.

Now, we show that $P_\varepsilon(v)$ is a continuous operator. Let $\{v_n\}_{n=1}^{+\infty} \subseteq S_A$ and $\|v_n - v\|_{W_2^2(\mathbb{R})} \to 0$ as $n \to \infty$, and $y_{v_n,\varepsilon}$ and $y_{v,\varepsilon}$ be such that $\tilde{l}_{v_n,\varepsilon} y_{v_n,\varepsilon} = h(x)$ and $\tilde{l}_{v,\varepsilon} y_{v,\varepsilon} = h(x)$. We have

$$\tilde{l}_{v_n,\varepsilon} (y_{v_n,\varepsilon} - y_{v,\varepsilon}) = [q_{v,\varepsilon} - q_{v_n,\varepsilon}] y'_{v,\varepsilon} + [s_v - s_{v_n}] y_{v,\varepsilon}. \tag{3.5}$$

The functions $q_{v,\varepsilon}$,$q_{v_n,\varepsilon}$, s_v,s_{v_n} $(n = 1, 2, ...)$ are continuous. By (3.4), there exists the number N_0 such that $[q_{v,\varepsilon} - q_{v_n,\varepsilon}] y'_{v,\varepsilon} + [s_v - s_{v_n}] y_{v,\varepsilon} \in L_2$ $\forall n \ge N_0$. By Lemma 2.1 and (3.5), we get

$$y_{v_n,\varepsilon} - y_{v,\varepsilon} = \tilde{l}_{v_n,\varepsilon}^{-1} \left([q_{v,\varepsilon} - q_{v_n,\varepsilon}] y'_{v,\varepsilon} + [s_v - s_{v_n}] y_{v,\varepsilon} \right). \tag{3.6}$$

Let $a > 0$. Since $v(x),\ v_n(x) \in S_A$ $(n = 1, 2, ...)$,

$$\max \left[\max_{x \in [-a,\, a]} |q_{v_n,\varepsilon}(x) - q_{v,\varepsilon}(x)| \ , \ \max_{x \in [-a,\, a]} |s_{v_n}(x) - s_v(x)| \right] \to 0$$

as $n \to \infty$. By (3.6) and Lemma 2.1, we obtain

$$\|y_{v_n,\varepsilon} - y_{v,\varepsilon}\|_{W_2^2[-a,\, a]} \le C \left(\max_{x \in [-a,\, a]} |q_{v,\varepsilon}(x) - q_{v_n,\varepsilon}(x)| \left\| y'_{v,\varepsilon} \right\|_{L_2[-a,\, a]} \right.$$

$$\left. + \max_{x \in [-a,\, a]} |s_v(x) - s_{v_n}(x)| \left\| y_{v,\varepsilon} \right\|_{L_2[-a,\, a]} \right) \to 0 \quad (n \to \infty),$$

where $C>0$ is independent of $y_{v,\varepsilon}$. We recall that $y_{v_n,\varepsilon}(x), y_{v,\varepsilon}(x) \in W_2^2(\mathbb{R})$. So, for each $\bar{\varepsilon}>0$, we can choose $a=a(\bar{\varepsilon})>0$ and $N=N(\bar{\varepsilon})$ such that for any $n \geq N$ the following inequalities

$$\|y_{v_n,\varepsilon}-y_{v,\varepsilon}\|_{W_2^2[-a,a]}<\frac{\bar{\varepsilon}}{3}, \quad \|y_{v_n,\varepsilon}\|_{W_2^2(\mathbb{R}\setminus[-a,a])}<\frac{\bar{\varepsilon}}{3}, \quad \|y_{v,\varepsilon}\|_{W_2^2(R\setminus[-a,a])}<\frac{\bar{\varepsilon}}{3}$$

hold. Hence,

$$\|P_\varepsilon(v_n)-P_\varepsilon(v)\|_{W_2^2(\mathbb{R})}=\|y_{v_n,\varepsilon}-y_{v,\varepsilon}\|_{W_2^2(\mathbb{R})}<\bar{\varepsilon}, \quad \forall n \geq N.$$

So, P_ε is continuous.

By the Schauder theorem, there exists $y_\varepsilon \in S_A$ $(\varepsilon>0)$ such that $P_\varepsilon(y_\varepsilon)=y_\varepsilon$. It is clear that y_ε is a solution of the following equation

$$\tilde{l}_{y_\varepsilon,\varepsilon} y=-y'''+q_{y_\varepsilon,\varepsilon} y'+s_{y_\varepsilon} y=h(x).$$

By Lemma 2.1, y_ε satisfies the following estimate:

$$\left\|y_\varepsilon'''\right\|_2+\left\|\sqrt{1+x^2} y_\varepsilon''\right\|_2+\left\|q_{y_\varepsilon,\varepsilon} y_\varepsilon'\right\|_2+\|s_{y_\varepsilon} y_\varepsilon\|_2 \leq C_4\|h\|_2,$$

where $C_4>0$ is independent of y_ε.

Let $\{\varepsilon_j\}_{j=1}^{+\infty}$ be a positive sequence such that $\{\varepsilon_j\}_{j=1}^{+\infty} \to 0$ as $n \to \infty$ and

$$\tilde{l}_{y_{\varepsilon_j},\varepsilon_j} y_{\varepsilon_j}=-y_{\varepsilon_j}'''+q_{y_{\varepsilon_j},\varepsilon_j} y_{\varepsilon_j}'+s_{y_{\varepsilon_j}} y_{\varepsilon_j}=h(x).$$

By Lemma 2.1, the following estimate

$$\left\|y_{\varepsilon_j}'''\right\|_2+\left\|\sqrt{1+x^2} y_{\varepsilon_j}''\right\|_2+\left\|q_{y_{\varepsilon_j},\varepsilon_j} y_{\varepsilon_j}'\right\|_2+\left\|s_{y_{\varepsilon_j}} y_{\varepsilon_j}\right\|_2 \leq C_4\|h\|_2 \tag{3.7}$$

holds. Let $[a, b] \subset \mathbb{R}$. We have that $y_{\varepsilon_j} \in W_2^3[a, b]$, and by (3.7), there exists $y \in W_2^2[a, b]$ such that $\left\|y_{\varepsilon_j}-y\right\|_{W_2^2[a, b]} \to 0$ as $j \to \infty$. In particular,

$$\left\|y_{\varepsilon_j}-y\right\|_{L_2[a, b]} \to 0 \ \ (j \to \infty). \tag{3.8}$$

By (3.7),

$$\begin{aligned} \left\|l y_{\varepsilon_j}-h\right\|_{L_2[a, b]} &= \left\|\tilde{l}_{y_{\varepsilon_j},\varepsilon_j} y_{\varepsilon_j}-h-\varepsilon_j\left(1+x^2\right)^4 y_{\varepsilon_j}'\right\|_{L_2[a, b]} \\ \leq \left\|\tilde{l}_{y_{\varepsilon_j},\varepsilon_j} y_{\varepsilon_j}-h\right\|_{L_2[a, b]} &+\varepsilon_j\left\|\left(1+x^2\right)^4 y_{\varepsilon_j}'\right\|_{L_2[a, b]} \to 0 \ \ (j \to \infty). \end{aligned}$$

Taking into account (3.8), it shows that y is a solution of the equation $ly=h$.

We show that for this solution y estimate (1.6) holds. We denote by $W_{2,u,\theta}^3(\mathbb{R})$ the completion of $C_0^{(3)}(\mathbb{R})$ with respect to the norm $\|y\|_{W,u,\theta}=\|y'''\|_2+\|uy'\|_2+\|\theta y\|_2$. By (3.7),

$$\left\|y_{\varepsilon_i}'''-y_{\varepsilon_j}'''\right\|_2+\left\|q_{y_{\varepsilon_i},\varepsilon_i} y_{\varepsilon_i}'-q_{y_{\varepsilon_j},\varepsilon_j} y_{\varepsilon_j}'\right\|_2+\left\|s_{y_{\varepsilon_i}} y_{\varepsilon_i}-s_{y_{\varepsilon_j}} y_{\varepsilon_j}\right\|_2 \to 0$$

as $i, j \to \infty$. Since $W_{2,u,w}^3(\mathbb{R})$ is a Banach space,

$$\left\|y_{\varepsilon_i}'''-y'''\right\|_2+\left\|q_{y_{\varepsilon_i},\varepsilon_i} y_{\varepsilon_i}'-q\, y'\right\|_2+\left\|s_{y_{\varepsilon_i}} y_{\varepsilon_i}-s\, y\right\|_2 \to 0$$

as $i \to \infty$. Consequently, (3.7) implies (1.6). □

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