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From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

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- compliance of the title of the paper to its content;

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- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

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At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.

ANDREI ANDREEVICH SHKALIKOV

(to the 70th birthday)

Andrei Andreevich Shkalikov, corresponding member of the Russian Academy of Sciences, an outstanding mathematician with a wide range of interests, a remarkable person, professor of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University was born on November 19, 1949.

Andrei Andreevich is a leading specialist in the theory of operators and their applications, especially to problems of mechanics and mathematical physics. He is well known for his work in the theory of functions and in the theory of spaces with indefinite metrics. He is also a specialist in the theory of entire and meromorphic functions and its applications to operator theory.

Andrei Andreevich is known for solving a number of difficult problems that for many years remained unsolved. His work on the basic properties of systems of root functions of differential operators is well known worldwide. He gave a justification for Mandelstam's hypothesis about the existence of solutions satisfying the radiation principle. He solved, in general form, the Rayleigh problem on the reflection of waves from a periodic surface, obtained a solution to the Sobolev problem on the stability of the motion of a top with a cavity filled with liquid. His contribution to the construction of an abstract theory of the Orr-Sommerfeld problem is invaluable. He obtained a description of the limiting spectral portraits for a large class of functions describing the profiles of fluid flows. He is one of the founders of the modern theory of differential operators, coefficients of which are distributions, and inverse problems for such operators.

Andrei Andreevich has been a plenary speaker at many international conferences. He conducts fruitful scientific work and collaborates with many international mathematical research centers.

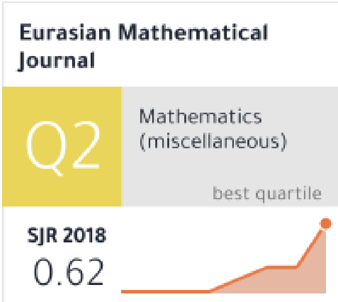
Andrei Andreevich is an author of more than 130 scientific publications. Among his pupils there are more than 20 Candidates of Sciences and 6 Doctors of Sciences. The results obtained by A.A. Shkalikov, his pupils, collaborators and followers gained worldwide recognition.

Professor Shkalikov is also an outstanding organizer. Under his supervision, many international conferences were held. In particular, conferences dedicated to the memory of I.G. Petrovsky, I.M. Gelfand, S.M. Nikol'skii, B.M. Levitan, anniversary conferences of V.A. Sadovnichy, and others.

Andrei Andreyevich is a deputy editor-in-chief of the journals Mathematical Notes, Moscow University Mathematics Bulletin, Moscow University Mechanics Bulletin, and a member of the editorial boards of the Russian Mathematical Surveys, Proceedings of the Moscow Mathematical Society and other journals, including the Eurasian Mathematical Journal.

The Editorial Board of the Eurasian Mathematical Journal cordially congratulates Andrei Andreevich on the occasion of his 70th birthday and wishes him good health, and new achievements in mathematics and mathematical education.

GOOD NEWS: EMJ IS NOW AN SJR Q2 JOURNAL



Recently the lists were published of all mathematical journals included in 2018 SCImago Journal Rank (SJR) quartiles Q1 (385 journals), Q2 (430 journals), Q3 (445 journals), and Q4 (741 journals), and Scopus CiteScore quartiles Q1 (443 journals), Q2 (375 journals), Q3 (348 journals), and Q4 (283 journals).

With great pleasure we inform our readers and authors that the Eurasian Mathematical Journal was included in the most popular scientific ranking database SJR in quartile Q2, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia. The SJR data for the Eurasian Mathematical Journal (2018) is as follows:

550. Eurasian Mathematical Journal (Kazakhstan), Q2, SJR=0.624.
(550 is the number in the list of all Q1 - Q4 journals.)

The SJR indicator is calculated by using the data of the Scopus Database of the Elsevier, the modern publishing business founded in 1880. It uses a sophisticated formula, taking into account various characteristics of journals and journals publications. This formula and related comments can be viewed on the web-page

http : //www.scimagojr.com/journalrank.php.

Some other SJR Q2 mathematical journals:
490. Studia Mathematica (Poland), SJR=0.706,
492. Comptes Rendus Mathematique (France), SJR=0.704,
522. Journal of Mathematical Physics (USA), SJR=0.667,
540. Doklady Mathematics (Russia), SJR=0.636,
570. Journal of Mathematical Sciences (Japan), SJR=0.602,
662. Journal of Applied Probability (UK), SJR=0.523,
733. Mathematical Notes (Russia), SJR=0.465,
791. Canadian Mathematical Bulletin (Canada), SJR=0.433.

Our journal ranks:

7726th place in the list of 31971 scientific journals, representing all subjects and all regions, included in this database (in the first 25% of journals of this category),

225th place in the list of 2519 scientific journals, representing all subjects, of the Asiatic region, included in this database (in the first 10% of journals of this category),

550th place in the list of 2011 mathematical journals, representing all regions, included in this database (in the first 30% of journals of this category),

19th place in the list of 165 mathematical journals of the Asiatic region, included in this database (in the first 15% of journals of this category).

On a separate page the SJR statistics for the Eurasian Mathematical Journal is attached.

Recall that the Eurasian Mathematical Journal started its work in 2010 (see [1]-[3]) and was first included in SJR indicator in 2014 (Q4, SJR=0.101, see [4], [5], [6]). So, the ambitious plan set in [6] was implemented and even essentially exceeded.

As for the Scopus CiteScore indicator, it uses another sophisticated formula, differently taking into account various characteristics journals publications. This formula and related comments can be viewed on the web-page

https : //ru.service.elsevier.com/app/answers/detail/a_id/19266/supporthub/scopus.

In this indicator the Eurasian Mathematical Journal was included in quartile Q3. The CiteScore data for the Eurasian Mathematical Journal (2018) is as follows:

333. Eurasian Mathematical Journal (Kazakhstan), Q3, CiteScore = 0,41
(333 is the number in the list of only Q3 journals.)

Some other Scopus CiteScore Q3 mathematical journals:

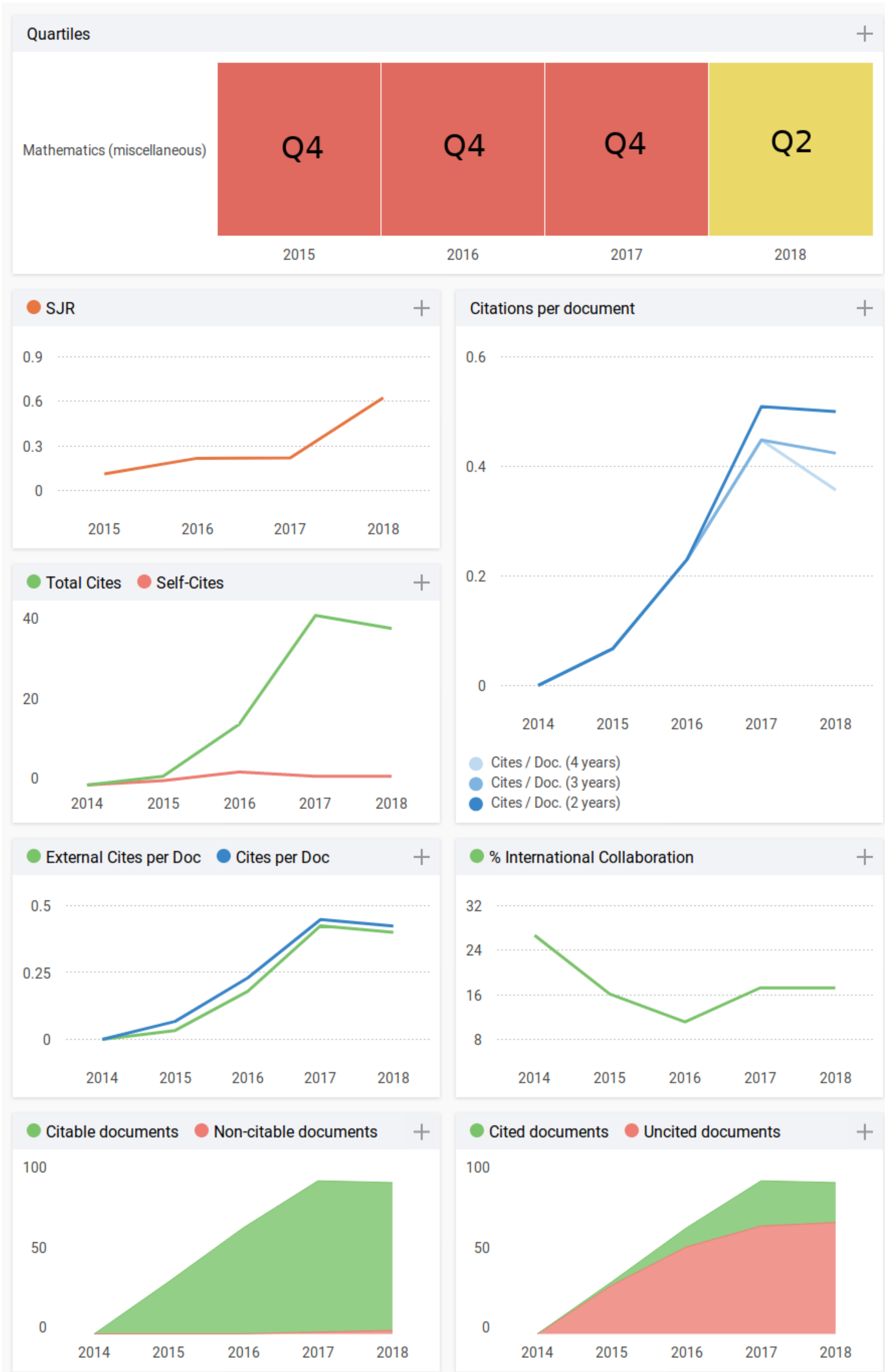
320. Czechoslovak Mathematical Journal (Czech Republic), CiteScore = 0.44,
321. Italian Journal of Pure and Applied Mathematics (Italy), CiteScore = 0.44,
323. Studia Scientiarum Mathematicarum Hungarica (Hungary), CiteScore = 0.44,
332. Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie (Romania), CiteScore = 0.41,
334. Indian Journal of Pure and Applied Mathematics (India), CiteScore = 0.41,
33. Transactions of the Moscow Mathematical Society (Russia), CiteScore = 0.41,
337. Illinois Journal of Mathematics (USA), CiteScore = 0.40,
339. Publications de l'Institut Mathematique (France), CiteScore = 0.40.

Our main current aim is to preserve the status of an SJR Q2 journal and of a Scopus CiteScore Q3 journal.

We hope that all respected members of the international Editorial Board, reviewers, current authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

V.I. Burenkov, K.N. Ospanov, T.V. Tararykova, A.M. Temirkhanova.

Extract from http://www.scimagojr.com/journalrank.php
EURASIAN MATHEMATICAL JOURNAL



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MULTIDIMENSIONAL FOURIER TRANSFORMS ON AN AMALGAM TYPE SPACE

E. Liflyand

Communicated by V.I. Burenkov

Key words: amalgam space, Fourier transform, integrability, bounded variation, Young inequality, trigonometric series.

AMS Mathematics Subject Classification: 42B10, 42B35.

Abstract. Generalizing the known results on the Fourier transforms on an amalgam type space, we introduce a multidimensional analogue of such a space, a subspace of $L^1(\mathbb{R}^n_+)$. Integrability results for the Fourier transforms are obtained provided that certain derivatives of the transformed function are in that space. As an application, we obtain conditions for the integrability of multiple trigonometric series.

DOI: https://doi.org/10.32523/2077-9879-2019-10-4-63-74

1 Introduction

Among the known results on the integrability of the Fourier transform, those proved for the functions with the derivative in an amalgam type space (see [10] or [9, Chapter 3]) hold a special position among various results on the behavior of the Fourier transform of a function of bounded variation. The main reason is that, being one of the strongest conditions of its kind, it is incomparable with certain other strong conditions. The mentioned results were generalizations of the results for trigonometric series in [1] (and in [4]). The latter has a multidimensional extension in [2], while the corresponding results for the Fourier transforms exist only in dimension one. The goal of the present work is to establish such multivariate extensions. The following presentation will be more transparent if the result in [10] is given in detail.

1.1 One-dimensional precursor

We first remind the definition of the one-dimensional class of functions we are going to generalize.

We say that a function g defined on $\mathbb{R}_+=[0,\infty)$ and locally integrable on $(0,\infty)$ belongs to $A_{1,p}$, $p>1$, if

$$\|g\|_{A_{1,p}}=\sum_{m=-\infty}^{\infty}\left\{\sum_{j=1}^{\infty}\left[\int\limits_{j2^m}^{(j+1)2^m}|g(t)|\,dt\right]^p\right\}^{\frac{1}{p}}dx<\infty. \tag{1.1}$$

This space is of amalgam nature (the reader can consult on the theory of various amalgam spaces in [5], [6], [8]), since each of the summands in m is the norm in the Wiener amalgam space $W(L^1,\ell^p)$

for functions g_m equal $2^m g(2^m t)$, if $t > 2^m$, and zero otherwise. Here ℓ^p, $1 \le p < \infty$, is a space of sequences $\{d_j\}$ endowed with the norm

$$\|\{d_j\}\|_{\ell^p} = \left(\sum_{j=1}^{\infty} |d_j|^p\right)^{\frac{1}{p}}$$

and the norm of a function $g : \mathbb{R}_+ \to \mathbf{C}$ in the amalgam space $W(L^1, \ell^p)$ is taken as

$$\left[\sum_{j=0}^{\infty} \left\{\int_j^{j+1} |g(t)|\, dt\right\}^p\right] \|^{\frac{1}{p}}.$$

Amalgam spaces are usually defined on the whole $\mathbb{R}$ but there are no obstacles to define them on $\mathbb{R}_+$ as above. In other words, we can rewrite (1.1) as follows:

$$\|g\|_{A_{1,p}} = \sum_{m=-\infty}^{\infty} \|g_m(\cdot)\|_{W(L^1,\ell^p)} < \infty.$$

Theorem 1.1. *Let f be locally absolutely continuous on $(0, \infty)$, vanish at infinity, that is, $\lim_{t\to\infty} f(t) = 0$, and $f' \in A_{1,p}$, with $1 < p \le 2$. Then $\widehat{f_c}$, the cosine Fourier transform of f, is integrable, with* [2]

$$\|\widehat{f_c}\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{A_{1,p}};$$

while for $\widehat{f_s}$, the sine Fourier transform, an asymptotic formula holds: for $x > 0$,

$$\widehat{f_s}(x) = \frac{1}{x} f\left(\frac{\pi}{2x}\right) + F(x),$$

where

$$\|F\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{A_{1,p}}.$$

It is noteworthy that the spaces $A_{1,p}$ are subspaces of $L^1(\mathbb{R}_+)$. In [10] this theorem is applied to obtain the corresponding results for trigonometric series, somewhat stronger than those in [1]. Another application can be found in [11].

1.2 A multidimensional amalgam type space

As mentioned above, our goal is to extend more general results in [10] to the multivariate setting. To this end, we first of all define a natural multidimensional analogue of $A_{1,p}$. Hopefully, preserving the same notation, without indicating the dependence on dimension will not result in any confusion.

We say that a function g defined on $\mathbb{R}^n_+ = \mathbb{R}_+ \times \ldots \times \mathbb{R}_+$ and locally integrable on $\mathbb{R}^n_+ \setminus \{0\}$ belongs to $A_{1,p}$, $1 < p < \infty$, if

$$\|g\|_{A_{1,p}} = \sum_{m\in\mathbb{Z}^n} \left(\sum_{\substack{1\le l_j<\infty\\ j=1,\ldots,n}} \left[\int\limits_{\substack{l_j 2^{-m_j} \le u_j \le (l_j+1)2^{-m_j}\\ j=1,\ldots,n}} |g(u)|\, du\right]^p\right)^{\frac{1}{p}} < \infty. \tag{1.2}$$

[2] Inequality $\varphi \lesssim \psi$ means that $\varphi \le C\psi$ with C being an absolute constant.

Of course, such spaces can also be defined for $p = 1$ (merely $L^1(\mathbb{R}^n_+)$) and $p = \infty$ but we are not interested in these in the present work. Moreover, we shall mostly deal with $p = 2$.

Like for trigonometric series, where the results are given in terms of belonging of the summable sequences $\{\Delta a_n\}$, $\{\Delta b_n\}$ to $a_{1,p}$ (the one-dimensional prototype of $A_{1,p}$; the multidimensional definition of $a_{1,p}$ in Section 3 completely explains the one-dimensional version as well), it is similarly expected that new conditions for the integrability of the Fourier transforms will be given in terms of belonging of certain derivative of the considered function to $A_{1,p}$. This is possible only if $A_{1,p}$ is a subspace of L^1. Indeed, this follows from

$$\|g\|_{A_{1,p}} \ge \sum_{m\in\mathbb{Z}^n} \int\limits_{[2^m,2^{m+1}]} |g(u)|\,du = \|g\|_{L^1(\mathbb{R}_+)}, \tag{1.3}$$

where $[a,b] = [a_1,b_1] \times [a_2,b_2] \times ... \times [a_n,b_n]$ denotes an n-dimensional parallelepiped.

1.3 Organization of the paper and certain notation

The paper is organized as follows. In Section 2 we formulate and prove our main results on the integrability of the Fourier transforms. Then, in Section 3, we use some of these results to readily get the above mentioned results on the integrability of trigonometric series; moreover, for the sine series, the obtained statement is stronger than that given in [2].

As is well known, the success of many of multidimensional results (more precisely, clarity of their formulations and likeness of the proof to that in dimension one) strongly depends on appropriate notation. We continue (see, e.g., [12] and in more detail [13, Chapter 5]) to use universal indicator type notation that easily allows one to distinct different phenomena on certain groups of variables and in many cases minimize the number of indices. The main tool for this are zero-one vectors η, χ and ζ. The first one will be constantly used, while the other two only in the situations where there are more than two different phenomena for one function or η is somehow involved. Thus, the vector η will always be used in the sense described below, and the vectors χ and ζ will always be of the same meaning.

Let $\eta = (\eta_1, ..., \eta_n)$ be an n-dimensional vector with the entries either 0 or 1 only. Its main task is to indicate the variable in which that or another action to be fulfilled. Correspondingly, $|\eta| = \eta_1 + ... + \eta_n$. The inequality of vectors is meant coordinate wise. If the only 1 entry is on the j-th place, while the rest are zeros, such a (basis) vector will be denoted by e_j. By x_η and dx_η we denote the $|\eta|$-tuple consisting only of x_j such that $\eta_j = 1$ and

$$dx_\eta := \prod_{j:\eta_j=1} dx_j,$$

respectively.

Since we are going to deal with multidimensional variations, various will be involved. Denote by $\Delta_{u_\eta} f(x)$ the partial difference

$$\Delta_{u_\eta} f(x) = \Big(\prod_{j:\eta_j=1} \Delta_{u_j}\Big) f(x),$$

with

$$\Delta_{u_j} f(x) = f(x + u_j e_j) - f(x).$$

Here and in what follows $D^\eta f$ for $\eta = \mathbf{0} = (0,0,...,0)$ means $D^{\mathbf{0}} f := f$, that is, the function itself, while for $\eta = \mathbf{1} = (1,1,...,1)$ means the partial derivative of order n $D^{\mathbf{1}} f$ applied repeatedly in each variable, where

$$D^\eta f(x) = \left(\prod_{j:\, \eta_j=1} \frac{\partial}{\partial x_j} \right) f(x)$$

means the partial derivative of order $|\eta|$ applied to the variables indicated by η. We shall naturally denote D^{e_j} by D^j.

Certain additional notation is in order. As usual, for any vector $\alpha = (\alpha_1, ..., \alpha_n)$, we denote

$$x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}.$$

Analogously, if $\alpha \in \mathbb{Z}_+^n$, then $D^\alpha \phi$ denotes the partial derivative of order $\alpha_1 + \cdots + \alpha_n$, that is, of order α_j with respect to x_j, $j = 1 \dots, n$.

To denote the $|\eta|$-tuple consisting only of $\frac{1}{x_j}$ for j such that $\eta_j = 1$, we incorporate negative indicator vectors, that is, the described $|\eta|$-tuple will be denoted by $x_{-\eta}$. Correspondingly, the vector $(\frac{1}{x_1}, ..., \frac{1}{x_n})$ should be denoted by $x_{-\mathbf{1}}$.

Besides $A_{1,p}$ in (1.2), we shall need the cases where the corresponding amalgamation is applied to some of the variables rather than to all of them. This will be represented by the norm with respect to the η variables, that is, to those u_j for which $\eta_j = 1$:

$$\|g(\cdot, x_{\mathbf{1}-\eta}\|_{A_{1,p}^\eta} := \sum_{m_\eta \in \mathbb{Z}^{|\eta|}} \Bigg(\sum_{\substack{1 \le l_j < \infty \\ j: \eta_j = 1}} \Bigg[\int\limits_{\substack{l_j 2^{-m_j} \le u_j \le (l_j+1) 2^{-m_j} \\ j: \eta_j = 1}} |g(u_\eta, x_{\mathbf{1}-\eta})| \, du_\eta \Bigg]^p \Bigg)^{\frac{1}{p}}.$$

It is natural to use $A_{1,p}^j$ rather than $A_{1,p}^{e_j}$ in the case where this procedure is applied to the single j-th variable. Of course, $A_{1,p}^{\mathbf{1}} = A_{1,p}$, while the trivial case may be identified with L^1.

2 Main results

We study, for $\eta_j = 0$ or 1, $j = 1, 2, ..., n$, the Fourier transforms

$$\widehat{f}_\gamma(x) = \int\limits_{\mathbb{R}_+^n} f(u) \left(\prod_{j=1}^n \cos\left(x_j u_j - \frac{\pi \eta_j}{2} \right) \right) du. \tag{2.1}$$

It is clear that $\widehat{f}_\eta$ represents the cosine Fourier transforms in the variables where $\eta_j = 0$, while taking $\eta_j = 1$ gives the sine Fourier transforms.

We should discuss a multidimensional notion of absolute continuity: $f \in AC(\mathbb{R}_+^n)$; see, e.g., [7]. It suffices to define such functions f as those representable as

$$f(x) = \int_0^{x_1} ... \int_0^{x_n} h(u)\, du + \sum_{\zeta \ne \mathbf{1}} f_\zeta(x_\zeta), \tag{2.2}$$

where $h \in L^1(\mathbb{R}_+^n)$ and marginal functions f_ζ depending on a smaller number of variables than n, i.e., $|\zeta| < n$ (since $|\zeta| = n$ if and only if $\zeta = \mathbf{1}$) are absolutely continuous on $\mathbb{R}_+^{|\zeta|}$. This inductive definition is correct since reduces to the usual absolute continuity on $\mathbb{R}_+$ for marginal functions of one variable. It is also plain that the partial derivatives $D^\zeta f$ exist almost everywhere. In particular, $D^{\mathbf{1}} f = h$ almost everywhere.

Locally absolute continuity on $\mathbb{R}^n_+ \setminus \{0\}$ means absolute continuity on every finite rectangle $[a,b] = [a_1,b_1] \times ... \times [a_n,b_n] \subset \mathbb{R}^n_+ \setminus \{0\}$. In this case, a_1,...,a_n, respectively, should replace $-\infty$ in (2.2).

Similarly to above, we introduce the notion and notation of partial (local) absolute continuity AC_ζ or LAC_ζ, which means the (local) absolute continuity with respect to the variables x_ζ only. How (2.2) should be modified in this case is completely obvious.

Theorem 2.1. *Let $f : \mathbb{R}^n_+ \to \mathbb{C}$ be locally absolutely continuous on $\mathbb{R}^n_+ \setminus \{0\}$ and let f vanish at infinity along with all $D^\eta f$ except $\eta = \mathbf{1}$. Let $1 < p \le 2$. For $\eta = \mathbf{0}$, we have*

$$\int_{\mathbb{R}^n_+} |\widehat{f}_{\mathbf{0}}(x)|\,dx \lesssim \sum_\eta \int_{\mathbb{R}^{\mathbf{1}-\eta}_+} \|D^{\mathbf{1}} f(\cdot, x_{\mathbf{1}-\eta})\|_{A^\eta_{1,p}} dx_{\mathbf{1}-\eta}; \tag{2.3}$$

for $\eta \ne \mathbf{1}, \mathbf{0}$,

$$\int_{\mathbb{R}^n_+} |\widehat{f}_\eta(x)|\,dx \lesssim \sum_{\mathbf{0}\le\chi\le\eta} \sum_{\substack{\zeta:\zeta_i=0\\ if\ \chi_i=1}} \int_{\mathbb{R}^{|\zeta+\chi|}_+} \Big(\prod_{j:\chi_j=1} \frac{1}{x_j}\Big) \|D^{\mathbf{1}-\chi} f(\frac{\pi}{2} x_{-\chi}, x_\zeta, \cdot)\|_{A^{\mathbf{1}-\chi-\zeta}_{1,p}} dx_{\zeta+\chi}; \tag{2.4}$$

and for $\eta = \mathbf{1}$,

$$\widehat{f}_{\mathbf{0}}(x) = \Big(\prod_{j=1}^n \frac{1}{x_j}\Big) f(\frac{\pi}{2} x_{\mathbf{1}}) + F(x), \tag{2.5}$$

with

$$\int_{\mathbb{R}^n_+} |F(x)|\,dx \lesssim \sum_{\mathbf{0}\le\chi\le\mathbf{1}} \sum_{\substack{\zeta:\zeta_i=0\\ if\ \chi_i=1}} \int_{\mathbb{R}^{|\zeta+\chi|}_+} \Big(\prod_{j:\chi_j=1} \frac{1}{x_j}\Big) \|D^{\mathbf{1}-\chi} f(\frac{\pi}{2} x_{-\chi}, x_\zeta, \cdot)\|_{A^{\mathbf{1}-\chi-\zeta}_{1,p}} dx_{\zeta+\chi}. \tag{2.6}$$

We note that most of the following arguments and calculations will be provided for $p = 2$. The estimates for $1 < p < 2$ then immediately follow by a standard inequality.

We will see that in order to control the L^1 norm of the Fourier transform of a function from the considered class, the crucial role belongs to the bounds of a special sequence of integrals over combinations of the dyadic intervals $[2^{m_j}, 2^{m_j+1}]$. Given a function $g(u)$, $u \in \mathbb{R}^k_+$, we define the sequence of functions of $y \in \mathbb{R}^k_+$

$$\widehat{G_m}(y) = \int_{[2^{-m},\infty)} g(u) e^{-i\langle x,u\rangle}\,du.$$

Obviously, this function is the Fourier transform of the function $G_m(u)$ which is $g(u)$ for $u \in [2^{-m}, \infty)$ and zero otherwise.

The mentioned above integrals are estimated in the next lemma the statement and the proof of which are inspired by Lemma 1 in [10].

Lemma 2.1. *Let g be an integrable function on $\mathbb{R}^k_+$. Then for $m : m_j = 0, \pm1, \pm2, ...,\ j = 1, 2, ..., k,$*

$$\int\limits_{[2^m,2^{m+1}]} \frac{|\widehat{G_m}(y)|}{y_1...y_k}\,dy \lesssim \left(\sum_{\substack{1\le l_j<\infty\\ j=1,\dots,k}}\left[\int\limits_{\substack{l_j2^{-m_j}\le u_j\le(l_j+1)2^{-m_j}\\ j=1,\dots,k}} |g(u)|\,du\right]^2\right)^{\frac12}.$$

Proof. We start with the following inequality:

$$\int\limits_{[2^m,2^{m+1}]} \frac{|\widehat{G_m}(y)|}{y_1...y_k}\,dy \lesssim \int\limits_{[2^m,2^{m+1}]} |\widehat{S_{2^{-m}}}(y)\,\widehat{G_m}(y)|\,dy, \tag{2.7}$$

where

$$\widehat{S_a}(y) = \prod_{j=1}^k \frac{\sin a_j y_j}{y_j}, \qquad a_j > 0.$$

The latter can be considered as the Fourier transform, up to a constant multiple, of the indicator function of the parallelepiped $[0, a]$. This follows from the formula (see (5) in [3, Chapter I, §4]; it is mentioned in Remark 12 in the cited literature of [3] that the formula goes back to Fourier)

$$\int_0^\infty \frac{\sin bt}{t}\cos zt\,dt = \begin{cases} \frac{\pi}{2}, & t < b;\\ \frac{\pi}{4}, & t = b;\\ 0, & t > b.\end{cases}$$

By the Schwarz-Cauchy-Bunyakovskii inequality, the right-hand side of (2.7) does not exceed

$$2^{\frac{|m|}{2}}\left(\int\limits_{[2^m,2^{m+1}]} |\widehat{S_{2^{-m}}}(y)\,\widehat{G_m}(y)|^2\,dy\right)^{\frac12}.$$

In fact, we no more need the integral over $[2^m, 2^{m+1}]$ (the factor $2^{\frac{|m|}{2}}$ is contributed by it) and have to estimate

$$2^{\frac{|m|}{2}}\left(\int\limits_{\mathbb{R}^k} |\widehat{S_{2^{-m}}}(y)\,\widehat{G_m}(y)|^2\,dy\right)^{\frac12}.$$

Since G_m is integrable and S_m is square integrable, $\widehat{S_m}(y)\,\widehat{G_m}(y)$ is the Fourier transform of their convolution, and both are square integrable. By Parseval's identity, we estimate

$$2^{\frac{|m|}{2}}\left(\int\limits_{\mathbb{R}^k} |(S_m * G_m)(y)|^2\,dy\right)^{\frac12}. \tag{2.8}$$

Further,

$$\widehat{G_m}(y) = \sum_{\substack{1\le l_j<\infty\\ j=1,\dots,k}} \int\limits_{\substack{l_j 2^{-m_j}\le u_j\le (l_j+1)2^{-m_j}\\ j=1,\dots,k}} g(u)e^{-i\langle x,u\rangle}\,du = \sum_{\substack{1\le l_j<\infty\\ j=1,\dots,k}} \widehat{G_{m,l}}(y),$$

where

$$\widehat{G_{m,l}}(y) = \int\limits_{\substack{l_j 2^{-m_j}\le u_j\le (l_j+1)2^{-m_j}\\ j=1,\dots,k}} g(u)e^{-i\langle x,u\rangle}\,du.$$

Correspondingly,

$$G_m(u) = \sum_{\substack{1\le l_j<\infty\\ j=1,\dots,k}} g_{m,l}(u),$$

with $g_{m,l}(u)=g(u)$ if $l_j 2^{-m_j}\le u_j\le (l_j+1)2^{-m_j}$, $j=1,\dots,k$, and zero otherwise. Representing (2.8) as

$$2^{\frac{|m|}{2}}\left(\int\limits_{\mathbb{R}^k}\left|\sum_{\substack{1\le l_j<\infty\\ j=1,\dots,k}} S_m * g_{m,j}(u)\right|^2 du\right)^{\frac12},$$

let us analyze what the support of each summand

$$S_m * g_{m,j}(u) = \int\limits_{\substack{l_j 2^{-m_j}\le v_j\le (l_j+1)2^{-m_j}\\ j=1,\dots,k}} S_m(u-v)g(v)\,dv$$

is. Since we have $0<u_j-v_j<2^{-m_j}$, such a summand is supported within the parallelepiped

$$\prod_{j=1}^{k}[l_j j 2^{-m_j},(l_j+2)2^{-m_j}].$$

Only 2^k neighbouring parallelepipeds may have an intersection of positive measure. Therefore, the value in (2.8) is dominated by 2^k summands of type

$$2^{\frac{|m|}{2}}\left(\sum_{\substack{1\le l_j<\infty\\ j=1,\dots,k}}\int\limits_{\mathbb{R}^k}\left|S_m * g_{m,j}(u)\right|^2 du\right)^{\frac12},$$

where each l_j is only even or only odd.

The bound for each of the 2^k values is the same and can be obtained by means of Young's inequality for convolution (see, e.g., [14, Chapter V, §1]): If $\varphi\in L^r(\mathbb{R}^k)$ and $\psi\in L^q(\mathbb{R}^k)$, then for $\frac1r+\frac1q=\frac1p+1$, $1\le p,q,r\le\infty$,

$$\|\varphi * \psi\|_p \le \|\varphi\|_r \|\psi\|_q.$$

Before this point the proof of the lemma is of superposition type, that is, we, in fact, repeat the proof of Lemma 1 in [10] in each variable. However, we cannot make the proof purely inductive and continue to deal with each variable separately or, technically, reduce it to induction in dimension k. Here we are obliged to apply the same but general multidimensional version of Young's inequality.

Taking $\varphi = S_m$ and $\psi = g_{m,j}$, $q = 1$ and $p = r = 2$, we obtain in each of the cases

$$2^{\frac{|m|}{2}} \left(\sum_{\substack{1 \le l_j < \infty \\ j=1,\dots,k}} \|S_m\|_2^2 \, \|g_{m,j}\|_1^2 \right)^{\frac{1}{2}}.$$

Since

$$\|S_m\|_2^2 = \int\limits_{[0,2^{-m}]} du = 2^{-|m|},$$

we get the required bound

$$\left(\sum_{\substack{1 \le l_j < \infty \\ j=1,\dots,k}} \left[\int\limits_{\substack{l_j 2^{-m_j} \le u_j \le (l_j+1)2^{-m_j} \\ j=1,\dots,k}} |g(u)| \, du \right]^2 \right)^{\frac{1}{2}}.$$

This completes the proof of the lemma. $\square$

Now, the proof of the theorem runs as follows.

Proof of Theorem 2.1. It follows from the routine calculations for the one-dimensional case in [10] that

$$\int\limits_0^\infty f(u) \cos\left(x_j u_j - \frac{\pi \eta_j}{2}\right) du_j = \frac{1}{x_j} f\left(u_{\mathbf{1}-\mathbf{e_j}}, \frac{\pi}{2x_j}\right) \sin \frac{\pi \eta_j}{2}$$
$$- \frac{1}{x_j} \int\limits_0^{\frac{\pi}{2x_j}} D^j f(u) \left[\sin\left(x_j u_j - \frac{\pi \eta_j}{2}\right) + \sin \frac{\pi \eta_j}{2} \right] du_j$$
$$- \frac{1}{x_j} \int\limits_{\frac{\pi}{2x_j}}^\infty D^j f(u) \sin\left(x_j u_j - \frac{\pi \eta_j}{2}\right) du_j.$$

Applying this to each variable, we get the terms of the form

$$\frac{1}{x_1...x_n}\int\limits_{\frac{\pi}{2x_1}}^{\infty}...\int\limits_{\frac{\pi}{2x_k}}^{\infty}\left\{\prod_{j=1}^{k}\sin\left(x_ju_j-\frac{\pi\eta_j}{2}\right)\,du_j\right\}$$
$$\int\limits_{0}^{\frac{\pi}{2x_{k+1}}}...\int\limits_{0}^{\frac{\pi}{2x_{k+i}}}\left\{\prod_{j=k+1}^{k+i}\left[\sin\left(x_ju_j-\frac{\pi\eta_j}{2}\right)+\sin\frac{\pi\eta_j}{2}\right]\,du_j\right\}$$
$$D^{\mathbf{1}}f\left(u_1,...,u_{k+i},\frac{\pi}{2x_{k+i+1}},...,\frac{\pi}{2x_n}\right)\,du_{k+i+1}...du_n.$$

Of course, the three groups of variables can be in many other orders, some of them empty, but these will show the needed calculations without loss of generality. Moreover, since the third group will remain untouched, it suffices to apply the needed calculations and order for the first two groups, with each containing only one variable.

Since

$$\int\limits_0^\infty\int\limits_0^{\frac{\pi}{2x_j}} s_j|D^jf(s_1,...,s_{j-1},s_j,s_{j+1},...,s_n)|\,ds_j\,dx_j$$
$$=\frac{\pi}{2}\int\limits_0^\infty|D^jf(s_1,...,s_{j-1},s_j,s_{j+1},...,s_n)|\,ds_j,$$

it follows from (1.3) and the remark thereafter that to prove the theorem it remains to estimate

$$\frac{1}{x_1...x_k}\int\limits_{\frac{\pi}{2x_1}}^{\infty}...\int\limits_{\frac{\pi}{2x_k}}^{\infty}. \tag{2.9}$$

Integrating them in $x_1,...,x_k$ over k-repeated integral $\int\limits_0^\infty...\int\limits_0^\infty$, we can represent each of these integrals as

$$\sum_{m_j=-\infty}^{\infty}\int\limits_{2^{m_j}}^{2^{m_j+1}},$$

$j=1,...,k$. As in dimension one in [10] (see also [9]), we can take the lower limits in (2.9) to be 2^{-m_j} rather than $\frac{\pi}{2x_j}$. Applying now the proven Lemma 2.1, we complete the proof of the theorem, since m_j runs from $-\infty$ to ∞ and we can write 2^{m_j} instead of 2^{-m_j}. Recall that the results for $1<p<2$ immediately follow from the proven theorem for $p=2$. $\square$

3 Applications

In [12] (see also [13]), interrelation between the multidimensional Fourier integral of a function of bounded Hardy's variation and the trigonometric series with the coefficients being the values of that function at the lattice points is described in detail. Such an interrelation is called there a "stable bridge". With this stable bridge in hand, let us cross it towards results on trigonometric series. Since $D^{\mathbf{1}}f \in A_{1,p}$ and f is locally absolutely continuous, f is of bounded Hardy variation (for definitions and details, see, e.g., [12] and [13]). This allows us to apply the approach from [12].

First, we have to define a discrete version of $A_{1,p}$, for the sequences of numbers $a = \{a_k\}$, $k \in \mathbb{Z}_+^n$. *We say that a sequence a defined on $\mathbb{Z}_+^n$ belongs to $a_{1,p}$, $1 < p < \infty$, if*

$$\|a\|_{a_{1,p}} = \sum_{m\in\mathbb{Z}^n} \Bigg(\sum_{\substack{1\le l_j<\infty \\ j=1,\dots,n}} \Bigg[\sum_{\substack{l_j 2^{-m_j} \le k_j \le (l_j+1)2^{-m_j} \\ j=1,\dots,n}} |a_k| \Bigg]^p \Bigg)^{\frac{1}{p}} < \infty. \tag{3.1}$$

Of course, $a_{1,p}^\eta$ is understood exactly in the same manner as $A_{1,p}^\eta$ above.

The result for the trigonometric series

$$\sum_{k\in\mathbb{Z}_+^n} a_k \prod_{i:\eta_i=1} \cos x_i k_i \prod_{i:\eta_i=0} \sin x_i k_i,$$

with the null sequence of coefficients a_k and with, for $k \in [k_1, k_1+1) \times \dots \times [k_n, k_n+1)$,

$$A(x) = \sum_{\mathbf{0}\le\eta\le\mathbf{1}} \prod_{j:\eta_j=1} (k_j - x_j)\, \Delta_\eta a_k, \tag{3.2}$$

where

$$\Delta_\eta a_k = \prod_{j:\eta_j=1} (a_{k_1,\dots,k_{j-1},k_j,k_{j+1},\dots,k_n} - a_{k_1,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_n}),$$

reads as follows.

Theorem 3.1. *Let a be a sequence on $\mathbb{Z}_+^n$ such that a vanishes at infinity along with all $\Delta^\eta a$ except $\eta = \mathbf{1}$. Let $1 < p \le 2$.*

For $\eta = \mathbf{0}$, we have

$$\begin{aligned} &\int_{\mathbb{T}_+^n} \Bigg| \sum_{k\in\mathbb{Z}_+^n} a_k \prod_{i:\eta_i=1} \cos x_i k_i \prod_{i:\eta_i=0} \sin x_i k_i \Bigg| dx \\ &\lesssim \sum_{\eta} \sum_{k\in\mathbb{Z}_+^{\mathbf{1}-\eta}} \|\Delta^{\mathbf{1}} a_{\cdot,k_{\mathbf{1}-\eta}}\|_{a_{1,p}^\eta}; \end{aligned} \tag{3.3}$$

for $\eta \ne \mathbf{1}, \mathbf{0}$,

$$\begin{aligned} &\int_{\mathbb{T}_+^n} \Bigg| \sum_{k\in\mathbb{Z}_+^n} a_k \prod_{i:\eta_i=1} \cos x_i k_i \prod_{i:\eta_i=0} \sin x_i k_i \Bigg| dx \\ &\lesssim \sum_{\mathbf{0}\le\chi\le\eta} \sum_{\zeta:\zeta_i=0 \quad if \quad \chi_i=1} \sum_{\mathbb{Z}_+^{|\zeta+\chi|}} \Bigg(\prod_{j:\chi_j=1} \frac{1}{x_j} \Bigg) \|\Delta^{\mathbf{1}-\chi} A(\frac{\pi}{2} x_{-\chi}, k_\zeta, \cdot)\|_{a_{1,p}^{\mathbf{1}-\chi-\zeta}}; \end{aligned} \tag{3.4}$$

and for $\eta = \mathbf{1}$,

$$\sum_{k\in\mathbb{Z}_+^n} a_k \prod_{i=1}^{n} \sin x_i k_i = A(\frac{\pi}{2x_1},...,\frac{\pi}{2x_n}) \prod_{j=1}^{n} \frac{1}{x_j} + F(x), \tag{3.5}$$

with

$$\int_{\mathbb{T}_+^n} |F(x)|\,dx \lesssim \sum_{\mathbf{0}\le\chi\le\mathbf{1}} \sum_{\substack{\zeta:\zeta_i=0\\ if\ \ \chi_i=1}} \sum_{\mathbb{Z}_+^{|\zeta+\chi|}} \left(\prod_{j:\chi_j=1} \frac{1}{x_j}\right) \|\Delta^{\mathbf{1}-\chi} A(\frac{\pi}{2}x_{-\chi}, k_\zeta, \cdot)\|_{a_{1,p}^{\mathbf{1}-\chi-\zeta}}. \tag{3.6}$$

As in many other cases, this result is somewhat stronger than the earlier prototype in [2].

Proof. The proof goes along the same lines as the proof of Theorem 5.4 in [12], just the estimates are taken from Theorem 2.1. □

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