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- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

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- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.

ANDREI ANDREEVICH SHKALIKOV

(to the 70th birthday)

Andrei Andreevich Shkalikov, corresponding member of the Russian Academy of Sciences, an outstanding mathematician with a wide range of interests, a remarkable person, professor of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University was born on November 19, 1949.

Andrei Andreevich is a leading specialist in the theory of operators and their applications, especially to problems of mechanics and mathematical physics. He is well known for his work in the theory of functions and in the theory of spaces with indefinite metrics. He is also a specialist in the theory of entire and meromorphic functions and its applications to operator theory.

Andrei Andreevich is known for solving a number of difficult problems that for many years remained unsolved. His work on the basic properties of systems of root functions of differential operators is well known worldwide. He gave a justification for Mandelstam's hypothesis about the existence of solutions satisfying the radiation principle. He solved, in general form, the Rayleigh problem on the reflection of waves from a periodic surface, obtained a solution to the Sobolev problem on the stability of the motion of a top with a cavity filled with liquid. His contribution to the construction of an abstract theory of the Orr-Sommerfeld problem is invaluable. He obtained a description of the limiting spectral portraits for a large class of functions describing the profiles of fluid flows. He is one of the founders of the modern theory of differential operators, coefficients of which are distributions, and inverse problems for such operators.

Andrei Andreevich has been a plenary speaker at many international conferences. He conducts fruitful scientific work and collaborates with many international mathematical research centers.

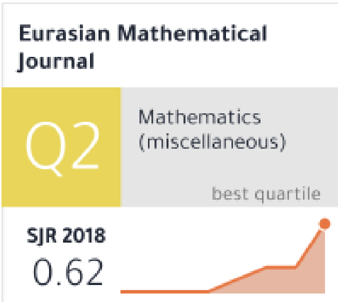
Andrei Andreevich is an author of more than 130 scientific publications. Among his pupils there are more than 20 Candidates of Sciences and 6 Doctors of Sciences. The results obtained by A.A. Shkalikov, his pupils, collaborators and followers gained worldwide recognition.

Professor Shkalikov is also an outstanding organizer. Under his supervision, many international conferences were held. In particular, conferences dedicated to the memory of I.G. Petrovsky, I.M. Gelfand, S.M. Nikol'skii, B.M. Levitan, anniversary conferences of V.A. Sadovnichy, and others.

Andrei Andreyevich is a deputy editor-in-chief of the journals Mathematical Notes, Moscow University Mathematics Bulletin, Moscow University Mechanics Bulletin, and a member of the editorial boards of the Russian Mathematical Surveys, Proceedings of the Moscow Mathematical Society and other journals, including the Eurasian Mathematical Journal.

The Editorial Board of the Eurasian Mathematical Journal cordially congratulates Andrei Andreevich on the occasion of his 70th birthday and wishes him good health, and new achievements in mathematics and mathematical education.

GOOD NEWS: EMJ IS NOW AN SJR Q2 JOURNAL



Recently the lists were published of all mathematical journals included in 2018 SCImago Journal Rank (SJR) quartiles Q1 (385 journals), Q2 (430 journals), Q3 (445 journals), and Q4 (741 journals), and Scopus CiteScore quartiles Q1 (443 journals), Q2 (375 journals), Q3 (348 journals), and Q4 (283 journals).

With great pleasure we inform our readers and authors that the Eurasian Mathematical Journal was included in the most popular scientific ranking database SJR in quartile Q2, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia. The SJR data for the Eurasian Mathematical Journal (2018) is as follows:

550. Eurasian Mathematical Journal (Kazakhstan), Q2, SJR=0.624.
(550 is the number in the list of all Q1 - Q4 journals.)

The SJR indicator is calculated by using the data of the Scopus Database of the Elsevier, the modern publishing business founded in 1880. It uses a sophisticated formula, taking into account various characteristics of journals and journals publications. This formula and related comments can be viewed on the web-page

http : //www.scimagojr.com/journalrank.php.

Some other SJR Q2 mathematical journals:
490. Studia Mathematica (Poland), SJR=0.706,
492. Comptes Rendus Mathematique (France), SJR=0.704,
522. Journal of Mathematical Physics (USA), SJR=0.667,
540. Doklady Mathematics (Russia), SJR=0.636,
570. Journal of Mathematical Sciences (Japan), SJR=0.602,
662. Journal of Applied Probability (UK), SJR=0.523,
733. Mathematical Notes (Russia), SJR=0.465,
791. Canadian Mathematical Bulletin (Canada), SJR=0.433.

Our journal ranks:

7726th place in the list of 31971 scientific journals, representing all subjects and all regions, included in this database (in the first 25% of journals of this category),

225th place in the list of 2519 scientific journals, representing all subjects, of the Asiatic region, included in this database (in the first 10% of journals of this category),

550th place in the list of 2011 mathematical journals, representing all regions, included in this database (in the first 30% of journals of this category),

19th place in the list of 165 mathematical journals of the Asiatic region, included in this database (in the first 15% of journals of this category).

On a separate page the SJR statistics for the Eurasian Mathematical Journal is attached.

Recall that the Eurasian Mathematical Journal started its work in 2010 (see [1]-[3]) and was first included in SJR indicator in 2014 (Q4, SJR=0.101, see [4], [5], [6]). So, the ambitious plan set in [6] was implemented and even essentially exceeded.

As for the Scopus CiteScore indicator, it uses another sophisticated formula, differently taking into account various characteristics journals publications. This formula and related comments can be viewed on the web-page

https : //ru.service.elsevier.com/app/answers/detail/a$_{i}$d/19266/supporthub/scopus.

In this indicator the Eurasian Mathematical Journal was included in quartile Q3. The CiteScore data for the Eurasian Mathematical Journal (2018) is as follows:

333. Eurasian Mathematical Journal (Kazakhstan), Q3, CiteScore = 0,41
(333 is the number in the list of only Q3 journals.)

Some other Scopus CiteScore Q3 mathematical journals:

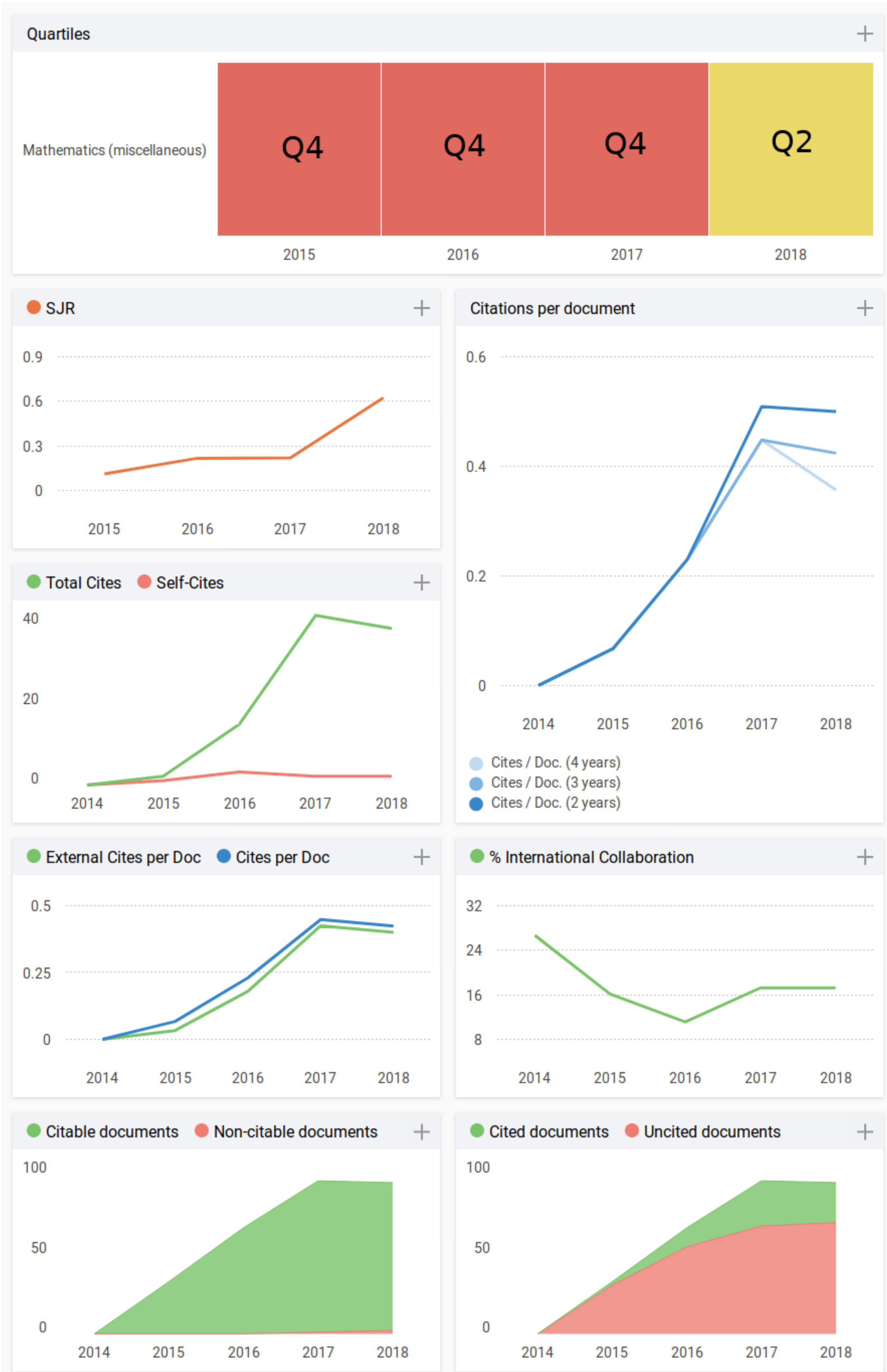
320. Czechoslovak Mathematical Journal (Czech Republic), CiteScore = 0.44,
321. Italian Journal of Pure and Applied Mathematics (Italy), CiteScore = 0.44,
323. Studia Scientiarum Mathematicarum Hungarica (Hungary), CiteScore = 0.44,
332. Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie (Romania), CiteScore = 0.41,
334. Indian Journal of Pure and Applied Mathematics (India), CiteScore = 0.41,
33. Transactions of the Moscow Mathematical Society (Russia), CiteScore = 0.41,
337. Illinois Journal of Mathematics (USA), CiteScore = 0.40,
339. Publications de l'Institut Mathematique (France), CiteScore = 0.40.

Our main current aim is to preserve the status of an SJR Q2 journal and of a Scopus CiteScore Q3 journal.

We hope that all respected members of the international Editorial Board, reviewers, current authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

V.I. Burenkov, K.N. Ospanov, T.V. Tararykova, A.M. Temirkhanova.

Extract from http://www.scimagojr.com/journalrank.php
EURASIAN MATHEMATICAL JOURNAL



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REMARKS ON SOBOLEV-MORREY-CAMPANATO SPACES DEFINED ON $C^{0,\gamma}$ DOMAINS

P.D. Lamberti, V. Vespri

Communicated by M.L. Goldman

Key words: Sobolev, Morrey, Campanato spaces, irregular domains.

AMS Mathematics Subject Classification: 46E35, 46E30, 42B35.

Abstract. We discuss a few old results concerning embedding theorems for Campanato and Sobolev-Morrey spaces adapting the formulations to the case of domains of class $C^{0,\gamma}$, and we present more recent results concerning the extension of functions from Sobolev-Morrey spaces defined on those domains. As a corollary of the extension theorem we obtain an embedding theorem for Sobolev-Morrey spaces on arbitrary $C^{0,\gamma}$ domains.

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1 Introduction

In the seminal papers [9, 10] Sergio Campanato introduced the spaces that nowadays are named after him, and used them to prove embedding theorems for Sobolev-Morrey spaces defined on bounded open sets Ω in $\mathbb{R}^n$. In particular, it was proved that if f is a function belonging to the Campanato space $\mathcal{L}_p^\lambda(\Omega)$ with $n<\lambda$ (and $\lambda \le n+p$) then f is Hölder continuous with exponent $(\lambda-n)/p$ that is for some $c>0$

$$|f(x)-f(y)| \le c|x-y|^{\frac{\lambda-n}{p}}, \tag{1.1}$$

for all $x,y\in\Omega$, and it was also proved that if f is a function in the Sobolev-Morrey space $W_p^{l,\lambda}(\Omega)$ with $0\le\lambda<n$, $n-\lambda<pl$ (and $pl<n-\lambda+p$) then f is Hölder continuous with exponent $l+\frac{\lambda-n}{p}$ that is for some $c>0$

$$|f(x)-f(y)| \le c|x-y|^{l+\frac{\lambda-n}{p}}, \tag{1.2}$$

for all $x,y\in\Omega$. Here, for simplicty, $l\in\mathbb{N}$ and $W_p^{l,\lambda}(\Omega)$ is the space of functions with weak derivatives up to order l in the classical Morrey space $L_p^\lambda(\Omega)$, but we note that the focus of [9, 10] was mainly on the case of fractional order of smoothness l since the case of integer exponents was already discussed in [25, 35]. See Section 2 for precise definitions.

The importance of these spaces is evident in regularity theory and harmonic analysis. The classical regularity approach was based on the singular integrals theory approach introduced by A.P. Calderón and A. Zygmund [7]. Using this approach based on the heat kernel, J. Nash [33] was able to solve the XIX Hilbert problem about the analyticity of the solutions to regular problems in the calculus of variations. One year before, E. De Giorgi [19] proved the same result with a different approach. He introduced a suitable function space, the so-called De Giorgi class, proved that any solution to regular problems in the calculus of variations belongs to this class and showed the embeddings of the De Giorgi classes in the space of Hölder continuous functions. It was a natural question to ask if this approach could recover the classical Calderón and Zygmund theory for equations with regular coefficients (i.e., continuous or Hölder continuous coefficients). This question was proposed by Ennio De Giorgi and Guido Stampacchia and solved by S. Campanato with the introduction of the Campanato spaces. The regularity in L^p spaces was proved by S. Campanato and G. Stampacchia [11] with the supplementary hypotheses of the Hölder continuity of the coefficients (for a proof of such a result, with only the assumption of the continuity of the coefficients, see [13]). These spaces were used for proving regularity of solutions to elliptic/parabolic systems/equations in variational/nonvariational form of the second (and higher) order (see, for instance [12] and [23]).

The other important field of application of these function spaces is harmonic analysis. T. Walsh [42] proved that the dual space of the Hardy space $H^p(R^N)$ is exactly the Campanato space. The theory of Hardy spaces $H^p(R^N)$ has important applications in harmonic analysis and partial differential equations (for instance, see [6, 37, 38, 14]). We recall that when $p \in (1, \infty)$, $L^p(R^N)$ and $H^p(R^N)$ are isomorphic; but when $p \in (0, 1]$, some of singular integrals (for example, Riesz transforms) are bounded on $H^p(R^N)$ but not on $L^p(R^N)$ and this fact makes the space $H^p(R^N)$ the right space where to study the theory of the boundedness of operators. In [6] C. Fefferman and E.M. Stein characterised the Hardy space $H^1(R^N)$ as the predual of the space $\mathrm{BMO}(R^N)$. The atomic and the molecular characterizations of $H^p(R^N)$ and their applications were studied by many authors; see, for example, [14, 15, 31, 32, 14, 40]. These characterizations (atomic and molecular) are necessary to extend the theory of Hardy spaces to spaces of homogeneous type in the sense of R.R. Coifman and G. Weiss [16, 17], which is, by far, one of the most general setting for singular integrals.

Going back to the initial work of S. Campanato, we note that inequality (1.1) was proved under the assumption that Ω satisfies the so-called property (A) which requires the existence of a constant $M > 0$ such that

$$|\Omega \cap B(x, r)| \geq M r^n \,, \tag{1.3}$$

for all $x \in \Omega$ and all $r > 0$ smaller than the diameter of Ω.

Inequality (1.2) was obtained under the stronger assumption that Ω is of class $C^{0,1}$, which means that, locally at the boundary, Ω can be represented as the subgraph of a Lipschitz continuous function (possibly after a rotation of coordinates).

Note that for $\lambda = 0$ we have $W^{l,\lambda}_p(\Omega) = W^l_p(\Omega)$ and inequality (1.2) is the celebrated Sobolev-Morrey inequality.

In this paper, we consider the case of open sets Ω of class $C^{0,\gamma}$ with $0 < \gamma \leq 1$ which means that the functions describing the boundary of Ω are Hölder continuous of exponent γ. It is a matter of folklore that passing from Lipschitz to Hölder continuity assumptions at the boundary of an open set is highly nontrivial (see e.g., the recent paper [29]), and it is interesting to note that also S. Campanato himself devoted his paper [8] to the study of embeddings for Sobolev spaces on open sets with power-type cusps at the boundary. We refer to the extensive monograph [34] for a recent introduction to the analysis of function spaces on irregular domains. We also refer to the classical monograph [28] for an introduction to Morrey-Campanato spaces on regular domains.

Broadly speaking, one may say that classical embedding theorems for Sobolev-Morrey-Campanato spaces hold on $C^{0,\gamma}$ domains provided one replaces (in the inequalities involved) the dimension n of the underlying space by $n_\gamma = (n-1)/\gamma + 1$, a fact which also appeared in [8]. It is important to note that $n_\gamma > n$ if $\gamma < 1$, and this typically leads to a deterioration in the estimates. For instance, if Ω is a domain with outer power-type cusps with exponent γ, property (A) above holds provided n is replaced by n_γ in the right-hand side of (1.3). On the other hand, we observe that if one wishes to control $|\Omega \cap B(x, r)|$ from above, the best one can do is to write $|\Omega \cap B(x, r)| \leq c r^n$, since it is impossible here to use n_γ. This discrepancy between the upper and lower bounds for $|\Omega \cap B(x, r)|$, indicates that the standard Euclidean metric is not suitable to deal with cusps and suggests to adapt the balls $B(x, r)$ to the type of domain under consideration. For example, if Ω is given by the cusp

$$\{(\bar{x}, x_n) \in \mathbb{R}^n : \ \bar{x} \in \mathbb{R}^{n-1}, \ x_n > |\bar{x}|^\gamma\} \tag{1.4}$$

with $\gamma < 1$, then one should replace the Euclidean ball $B(x,r)$ by the anisotropic ball

$$B_\gamma(x,r) = \left\{ y \in R^n : \ |\bar{x} - \bar{y}| < r^{\frac{1}{\gamma}}, \ |x_n - y_n| < r \right\} \tag{1.5}$$

in which case $|\Omega \cap B_\gamma(x,r)|$ is asymptotic to $r^{n\gamma}$ as $r \to 0$, and the discrepancy above disappears. Accordingly, in the right-hand side of inequalities (1.1), (1.2) one has to replace the Euclidean distance $|x - y|$ by the anisotropic one $|\bar{x} - \bar{y}|^\gamma + |x_n - y_n|$. This idea was already used by G.C. Barozzi [1] where some results of S. Campanato are extended to the case of domains with power-type cusps, and was further extended by Giuseppe Da Prato in the fundamental paper [18] where more general metrics were considered. We also note that final results for domains satisfying horn-type conditions are contained in the classical monograph [2, 3].

Although the existing literature seems to provide a complete picture of this subject, we have found it quite surprising that some results contained in the above mentioned papers, incorporate quite restrictive assumptions. In particular, in the analysis of inequality (1.2) for anisotropic metrics, [1, Theorem 3] eventually assumes for simplicity that Ω is a parallelepiped, [18, Theorem 4.1] assumes that Ω is convex and the estimate in [3, Theorem 27.4.2] is proved for all $x, y \in \Omega$ such that the segment $[x, y]$ is contained in Ω.

A different approach to the analysis of function spaces in domains of class $C^{0,\gamma}$ was suggested by Victor I. Burenkov in [4, 5] where he defined a new extension operator which, contrary to other classical extension operators, allows to deal not only with Lipschitz domains but also with $C^{0,\gamma}$ domains (as well as with anisotropic Sobolev spaces and extensions from manifolds of dimension $m < n$). Note that the flexibility of Burenkov's Extension Operator has been recently exploited in [21] where it is proved that this operator preserves general Sobolev-Morrey spaces, including the case of the classical Sobolev-Morrey spaces $W_p^{l,\lambda}(\Omega)$.

If $\gamma < 1$ then deterioration in the smoothness of the extended functions is expected and, in fact, Burenkov's Extension Operator maps the Sobolev space $W_p^l(\Omega)$ to the Sobolev space $W_p^{[\gamma l]}(\mathbb{R}^n)$ where $[\gamma l]$ is the integer part of γl. The exponent $[\gamma l]$ is sharp (in terms of Sobolev spaces). Thus, having a function extended to the whole of $\mathbb{R}^n$ allows to apply embedding theorems in $\mathbb{R}^n$ and eventually to return to Ω by mere restriction. Although the target space $W_p^{[\gamma l]}(\mathbb{R}^n)$ is sharp, it is observed already in [5] that in general the embedding theorems proved via this procedure are not sharp since the deterioration given by $[\gamma l]$ is too much for this purpose. However, this procedure has the advantage of giving at least some information even in most difficult cases.

The goal of the present paper is twofold. First, we revise the above mentioned old results by adapting their formulation to the case of elementary domains of class $C^{0,\gamma}$. In passing, we also indicate how it is possible to replace the convexity assumption in [18, Theorem 4.1] by the assumption that a Poincaré inequality for balls holds, see Theorem 2.5. Secondly, we indicate how to adapt the proofs of [21] to the case of domains of class $C^{0,\gamma}$, in order to prove that Burenkov's Extension Operator maps the Sobolev-Morrey space $W_p^{l,\lambda}(\Omega)$ to the Sobolev-Morrey space $W_p^{[\gamma l],\gamma\lambda}(\mathbb{R}^n)$, analysing also the case of Morrey norms defined by even more general weights, see Theorem 3.2. Note the extra deterioration in the Morrey exponent which passes from λ to $\gamma\lambda$. Moreover, we apply this extension result to recover an estimate of type (1.2) in domains of class $C^{0,\gamma}$, see Corollary 3.1. We observe that, although the new estimate is not sharp, it is obtained without any extra geometric assumptions on Ω or on the points $x, y \in \Omega$ as done by other authors.

It is important to observe that our extension result is obtained by using Morrey norms involving Euclidean balls both in Ω and in $\mathbb{R}^n$, even though for elementary domains of form (1.4) it would be natural to use anisotropic balls of type (1.5) in Ω. This is due to technical reasons involved in our proofs, which prevents us from controlling reflected balls in an anistropic way, see Lemma 3.3. On the other hand, since our final goal is to deal with general open sets Ω of class $C^{0,\gamma}$ (where cusps may have a different orientation depending on the part of the boundary under consideration), in principle there is no special reason why one should use the balls of type (1.5) in the whole of Ω. Thus, either one changes the definition of the Morrey spaces, adapting balls to the orientation of each local chart or uses, for uniformity, Euclidean balls in the whole of Ω. Our approach eventually leads us to choose the second option.

With reference to the problem of the extension of Sobolev-Morrey spaces, besides [21], we would also like to quote the papers [27], [30], [41].

2 Embedding theorems on elementary $C^{0,\gamma}$ domains

In this paper the elements of $\mathbb{R}^n$, $n \geq 2$, are denoted by $x = (\overline{x}, x_n)$ with $\overline{x} = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. For any $\gamma \in]0, 1]$, we consider the metric δ_γ in $\mathbb{R}^n$ defined by

$$\delta_\gamma(x, y) = \max\{|\bar{x} - \bar{y}|^\gamma, |x_n - y_n|\},$$

for all $x, y \in \mathbb{R}^N$ and we denote by $B_\gamma(x, r)$ the corresponding open balls of centre x and radius r, that is

$$\begin{aligned} B_\gamma(x, r) &= \{y \in \mathbb{R}^n : \ \delta_\gamma(x, y) < r\} \\ &= \left\{y \in R^n : \ |\bar{x} - \bar{y}| < r^{\frac{1}{\gamma}}, \ |x_n - y_n| < r\right\}. \end{aligned}$$

Note that the Lebesgue measure of $B_\gamma(x, y)$ is given by

$$|B_\gamma(x, r)| = 2\omega_{n-1} r^{n_\gamma}$$

where

$$n_\gamma = \frac{n-1}{\gamma} + 1,$$

and ω_{n-1} is the measure of the unit ball in $\mathbb{R}^{n-1}$. Note also that $n_\gamma = n + (n-1)(\frac{1}{\gamma} - 1)$, hence $n_\gamma \geq n$ and equality occurs if and only if either $n = 1$ or $\gamma = 1$.

Given $p \in [1, \infty[$, a function $\phi :]0, \infty[\to]0, \infty[$ and an open set Ω in $\mathbb{R}^n$, for all $f \in L^p(\Omega)$ we set

$$\|f\|_{L^\phi_{p,\gamma}(\Omega)} := \sup_{x \in \Omega} \sup_{r > 0} \left(\frac{1}{\phi(r)} \int_{B_\gamma(x,r) \cap \Omega} |f(y)|^p dy \right)^{\frac{1}{p}}$$

and

$$|f|_{\mathcal{L}^\phi_{p,\gamma}(\Omega)} := \sup_{x \in \Omega} \sup_{r > 0} \left(\frac{1}{\phi(r)} \int_{B_\gamma(x,r) \cap \Omega} |f(y) - \fint_{B_\gamma(x,r) \cap \Omega} f(z) dz|^p dy \right)^{\frac{1}{p}}.$$

The generalised Morrey spaces are defined by

$$L^\phi_{p,\gamma}(\Omega) = \{f \in L^p(\Omega) : \ \|f\|_{L^\phi_{p,\gamma}(\Omega)} < \infty\},$$

and the generalised Campanato spaces are defined by

$$\mathcal{L}^\phi_{p,\gamma}(\Omega) = \{f \in L^p(\Omega) : \ |f|_{\mathcal{L}^\phi_{p,\gamma}(\Omega)} < \infty\}.$$

For any $l \in \mathbb{N}$, we consider also the Sobolev-Morrey spaces

$$W^{l,\phi}_{p,\gamma}(\Omega) = \{f \in L^p(\Omega) : \ D^\alpha f \in L^\phi_{p,\gamma}(\Omega), \ \forall |\alpha| \leq l\}$$

endowed with the norm

$$\|f\|_{W^{l,\phi}_{p,\gamma}(\Omega)} = \sum_{|\alpha|\le l} \|D^\alpha f\|_{L^\phi_{p,\gamma}(\Omega)} .$$

If $\lambda \ge 0$ and $\phi(r) = \min\{r^\lambda, 1\}$ for all $r > 0$ then the corresponding spaces will be denoted by $L^\lambda_{p,\gamma}(\Omega)$, $\mathcal{L}^\lambda_{p,\gamma}(\Omega)$, $W^{l,\lambda}_{p,\gamma}(\Omega)$. Since $|\cdot|_{\mathcal{L}^\lambda_{p,\gamma}(\Omega)}$ is a semi-norm, it is customary to endow the Campanato space $\mathcal{L}^\lambda_{p,\gamma}(\Omega)$ with the norm defined by

$$\|f\|_{\mathcal{L}^\lambda_{p,\gamma}(\Omega)} := \|f\|_{L^p(\Omega)} + |f|_{\mathcal{L}^\lambda_{p,\gamma}(\Omega)} ,$$

for all $f \in \mathcal{L}^\lambda_{p,\gamma}(\Omega)$.

Note that $L^\lambda_{p,1}(\Omega)$, $\mathcal{L}^\lambda_{p,1}(\Omega)$ are the classical Morrey and Campanato spaces respectively (recall that $L^\lambda_{p,1}(\Omega)$ contains only the zero function for $\lambda > n$ and it coincides with $L^\infty(\Omega)$ for $\lambda = n$ by the Lebesgue differentiation theorem, see [28] for more details concerning the limiting cases).

We consider elementary Hölder continuous domains Ω in $\mathbb{R}^n$ with exponent $\gamma \in]0,1]$ of the form

$$\Omega = \{x = (\bar{x}, x_n) \in \mathbb{R}^n : \ \bar{x} \in W, \ a < x_n < \varphi(\bar{x})\} , \tag{2.1}$$

where $-\infty \le a < \infty$, W is a smooth or convex open set in $\mathbb{R}^{n-1}$, and $\varphi : W \to \mathbb{R}$ is a Hölder continuous function with exponent γ satisfying the condition $\varphi(\bar{x}) > a + \delta$ for some $\delta > 0$. In particular, there exists a positive constant M such that

$$|\varphi(\bar{x}) - \varphi(\bar{y})| \le M|\bar{x} - \bar{y}|^\gamma , \ \forall\, \bar{x}, \bar{y} \in \mathbb{R}^{n-1} . \tag{2.2}$$

The best constant M in inequality (2.2) is denoted by $\mathrm{Lip}_\gamma \varphi$. For $\gamma = 1$ we obtain Lipschitz continuous domains. It is well known that Lipschitz continuous domains satisfy the usual cone condition. Similarly, Hölder continuous domains satisfy a generalisation of that condition which we call the cusp condition. Namely, for any $x \in \mathbb{R}^n$ and $h > 0$, we set

$$C_\gamma(x, h, M) = \{y \in \mathbb{R}^N : \ x_n - h < y_n < x_n - M|\bar{y} - \bar{x}|^\gamma\} \tag{2.3}$$

and we call it a cusp with exponent γ, vertex x, height h and opening M. Then we can prove the following simple lemma which, by the way, is essential in order to apply the general results of [1, 3, 18].

Lemma 2.1. *Let $\gamma \in]0,1]$ and Ω be an elementary Hölder continuous domain in $\mathbb{R}^n$ as in (2.1) with $W = \mathbb{R}^{n-1}$ and $a = -\infty$. Then for all $x \in \bar{\Omega}$ and $h > 0$, we have*

$$C_\gamma(x, h, \mathrm{Lip}_\gamma \varphi) \subset \Omega. \tag{2.4}$$

Moreover, there exists $c > 0$ depending only on n, γ and $\mathrm{Lip}_\gamma \varphi$ such that

$$|B_\gamma(x, r) \cap \Omega| \ge c r^{n_\gamma}, \tag{2.5}$$

for all $x \in \bar{\Omega}$ and $r > 0$.

Proof. Given a cusp $C_\gamma(x, h, \mathrm{Lip}_\gamma \varphi)$ as in the statement, for any point $y \in C_\gamma(x, h, \mathrm{Lip}_\gamma \varphi)$ we have

$$y_n < x_n - \mathrm{Lip}_\gamma \varphi\, |\bar{x} - \bar{y}|^\gamma \le \varphi(\bar{x}) - \mathrm{Lip}_\gamma \varphi\, |\bar{x} - \bar{y}|^\gamma \le \varphi(\bar{y}) ,$$

where the third inequality follows from the Hölder continuity of φ. Thus, $C_\gamma(x, h, \mathrm{Lip}_\gamma \varphi) \subset \Omega$. Inequality (2.5), easily follows from (2.4), the inclusion $C_\gamma(x, r, 1) \subset B_\gamma(x, r)$ and the fact that $|C_\gamma(x, h, M)| = c h^{n_\gamma}$ where c is a positive constant depending only on n, γ, M. $\square$

Given two function spaces $X(\Omega)$ $Y(\Omega)$, we write $X(\Omega) \simeq Y(\Omega)$ to indicate that any function $f \in X(\Omega)$ equals almost everywhere in Ω a function $g \in Y(\Omega)$ and viceversa, and that the two norms $\|\cdot\|_{X(\Omega)}$, $\|\cdot\|_{Y(\Omega)}$ are equivalent. Note that, for the sake of simplicity, two functions f, g as above will be denoted by the same symbol (being aware of this identification is particularly important when stating Hölder continuity estimates).

The following theorem can be deduced by the general result [18, Theorem 3.1] combined with inequality (2.5) which guarantees that Ω is of type (A) as required in [18, Theorem 3.1]. Here, $C^{0,\alpha}(\bar{\Omega}, \delta_\gamma)$ denotes the space of Hölder continuous functions with exponent α with respect to the metric δ_γ.

Theorem 2.1 (Campanato-Da Prato). *Let Ω be a bounded elementary Hölder continuous domain with exponent $\gamma \in]0,1]$, $\lambda > 0$. The following statements hold:*

(i) If $\lambda < n_\gamma$ then $\mathcal{L}^\lambda_{p,\gamma}(\Omega) \simeq L^\lambda_{p,\gamma}(\Omega)$.

(ii) If $\lambda > n_\gamma$ then $\mathcal{L}^\lambda_{p,\gamma}(\Omega) \simeq C^{0,\alpha}(\bar{\Omega}, \delta_\gamma)$ where

$$\alpha = \frac{\lambda - n_\gamma}{p};$$

in particular, there exists $c > 0$ such that for all $f \in \mathcal{L}^\lambda_{p,\gamma}(\Omega)$ and for all $x, y \in \Omega$ we have

$$|f(x) - f(y)| \le c|f|_{\mathcal{L}^\lambda_{p,\gamma}(\Omega)}(|\bar{x} - \bar{y}|^\gamma + |x_n - y_n|)^{\frac{\lambda - n_\gamma}{p}}. \tag{2.6}$$

The following result is direct application of a general result in [3, Theorem 27.4.2, Remark 27.4.3] combined with inclusion (2.4) which guarantees that Ω satisfies the γ-horn condition described in [2, p. 153]. As customary, we denote by $[x, y]$ the segment connecting two points x and y in $\mathbb{R}^n$.

Theorem 2.2 (Sobolev-Morrey Embedding for elementary $C^{0,\gamma}$ domains). *Let Ω be an elementary Hölder continuous domain with exponent $\gamma \in]0,1]$. Let $l \in \mathbb{N}$, $\lambda > 0$, $p \in [1, \infty[$ be such that*

$$pl > n_\gamma - \lambda$$

and[1] $\gamma(l + \frac{\lambda - n_\gamma}{p}) < 1$. Then there exists $c > 0$ such that for all $f \in W^{l,\lambda}_{p,\gamma}(\Omega)$ and for all $x, y \in \Omega$ such that $[x, y] \subset \Omega$ we have

$$|f(x) - f(y)| \le c\|f\|_{W^{l,\lambda}_{p,\gamma}(\Omega)}|x - y|^{\gamma\left(l + \frac{\lambda - n_\gamma}{p}\right)}. \tag{2.7}$$

Note that by setting formally $l = 0$ in (2.7), one essentially obtains estimate (2.6). It is interesting to observe that the previous result (with minor modifications) was proved in [1] in the case of a parallelepiped.

Theorem 2.3 (Barozzi). *Let Ω be a parallelepiped in $\mathbb{R}^n$ of the form $\Omega = \Pi_{i=1}^n]a_i, b_i[$ with $-\infty < a_i < b_i < \infty$ for all $i = 1, \dots, n$. Let $\gamma \in]0,1]$, $l \in \mathbb{N}$, $\lambda > 0$, $p \in [1, \infty[$ be such that*

$$pl > n_\gamma - \lambda$$

and such that $l + \frac{\lambda - n_\gamma}{p} \le 1$. Then for any $\epsilon > 0$ there exists $c > 0$ such that for all $f \in W^{l,\lambda}_{p,\gamma}(\Omega)$ and for all $x, y \in \Omega$ we have

$$|f(x) - f(y)| \le c\|f\|_{W^{l,\lambda}_{p,\gamma}(\Omega)}(|\bar{x} - \bar{y}|^\gamma + |x_n - y_n|)^{l + \frac{\lambda - n_\gamma}{p} - \epsilon}.$$

[1]If viceversa $\gamma(l + \frac{\lambda - n_\gamma}{p}) > 1$ then one has Lipschitz continuity; in the case $\gamma(l + \frac{\lambda - n_\gamma}{p}) = 1$ one gets Hölder continuity with any exponent less than 1.

Moreover, the following theorem can be deduced by a more general result obtained by G. Da Prato in [18, Theorem 4.1] for $l = 1$ in the case of a convex set.

Theorem 2.4. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. Let $\gamma \in]0,1]$, and $\eta = \frac{n_\gamma}{n} + n - n_\gamma$. Let $\lambda > 0$, $p \in [1, \infty[$ be such that*

$$p\eta > n_\gamma - \lambda\,.$$

Then there exists $c > 0$ such that for all $f \in W^{1,\lambda}_{p,\gamma}(\Omega)$ and for all $x, y \in \Omega$ we have

$$|f(x) - f(y)| \le c\|f\|_{W^{1,\lambda}_{p,\gamma}(\Omega)}(|\bar{x} - \bar{y}|^\gamma + |x_n - y_n|)^{\eta + \frac{\lambda - n_\gamma}{p}}\,.$$

Remark 1. *We note that the constant η in Theorem 2.4 replaces the constant $l = 1$ in the previous theorems. Since $\eta < 1$ for $\gamma < 1$, we have a deterioration in the estimates. This seems to be due to the fact that the result in [18, Theorem 4.1] is quite general and is stated in order to embrace more general types of metrics.*

We now explain where the exponent η in Theorem 2.4 comes from. The main ingredient is a quantitative Poincaré-Wirtinger inequality for bounded convex domains B in $\mathbb{R}^n$, namely the inequality

$$\|f - f_B\|_{L^p(B)} \le \left(\frac{\omega_n}{|B|}\right)^{1-\frac{1}{n}} d^n \|\nabla f\|_{L^p(B)}, \quad \forall f \in W^1_p(B), \tag{2.8}$$

where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$, d denotes the Euclidean diameter of B and $f_B = ⨍_B f(x)dx$ (see, e.g., [24, p.164]).

It follows from (2.8) and Hölder's inequality that if Ω is a convex domain in $\mathbb{R}^n$ and $f \in W^1_p(\Omega)$ then for all $x \in \Omega$ and $r > 0$ we have

$$\begin{aligned}\|f - f_{\Omega\cap B_\gamma(x,r)}\|&_{L^1(\Omega\cap B_\gamma(x,r))}\\ &\le \omega_n^{1-\frac{1}{n}} |\Omega \cap B_\gamma(x,r)|^{\frac{1}{n}-\frac{1}{p}} d_r^n \|\nabla f\|_{L^p(\Omega\cap B_\gamma(x,r))},\end{aligned} \tag{2.9}$$

where d_r denotes the Euclidean diameter of $\Omega \cap B_\gamma(x,r)$. If in addition we have that $\nabla f \in L^\lambda_{p,\gamma}(\Omega)$, we obtain

$$\|f - f_{\Omega\cap B_\gamma(x,r)}\|_{L^1(\Omega\cap B_\gamma(x,r))} \le c|\Omega \cap B_\gamma(x,r)|^{\frac{1}{n}-\frac{1}{p}} d_r^n r^{\frac{\lambda}{p}}$$

hence

$$\|f - f_{\Omega\cap B_\gamma(x,r)}\|_{L^1(\Omega\cap B_\gamma(x,r))} \le c r^{n_\gamma(\frac{1}{n}-\frac{1}{p})} r^{\frac{\lambda}{p}} d_r^n \tag{2.10}$$

since the measure of $|\Omega \cap B_\gamma(x,r)|$ is controlled from above by a multiple of r^{n_γ}.

In the general framework of [18], it is then assumed that $d_r \le c r^\beta$ for some constant $\beta \ge 1$ which in our case is $\beta = 1$ and cannot be better. Keeping track of β, we obtain from (2.10) that

$$\|f - f_{\Omega\cap B_\gamma(x,r)}\|_{L^1(\Omega\cap B_\gamma(x,r))} \le c r^{n_\gamma(\frac{1}{n}-\frac{1}{p})} r^{\frac{\lambda}{p}} r^{n\beta} \tag{2.11}$$

which means that $f \in \mathcal{L}^\theta_{1,\gamma}(\Omega)$ with

$$\theta = \frac{n_\gamma}{n} + n\beta + \frac{\lambda - n_\gamma}{p}\,.$$

If $\theta > n_\gamma$, that is $(n_\gamma/n + n\beta - n_\gamma)p > n_\gamma - \lambda$, we deduce by Theorem 2.1 (ii) that $u \in C^{0,\alpha}(\bar{\Omega}, \delta_\gamma)$ with

$$\alpha = \frac{n_\gamma}{n} + n\beta + \frac{\lambda - n_\gamma}{p} - n_\gamma,$$

which for $\beta = 1$ yields

$$\alpha = \eta + \frac{\lambda - n_\gamma}{p}.$$

This explains the appearance of η in Theorem 2.4.

We now reformulate the statement of [18, Theorem 4.1] in order to relax a bit the convexity assumption on Ω. Namely, assume that Ω is a bounded domain in $\mathbb{R}^n$ such that condition (2.5) is satisfied and such that the following p-Poincaré inequality holds

$$\fint_{\Omega\cap B_\gamma(x,r)} |f - f_{\Omega\cap B_\gamma(x,r)}|dx \leq c_p r^{\tilde{\eta}} \left(\fint_{\Omega\cap B_\gamma(x,\tau r)} |\nabla f|^p dx \right)^{\frac{1}{p}}, \tag{2.12}$$

for all $f \in W^1_p(\Omega)$ and $r > 0$, where $\tau \geq 1$ and $\tilde{\eta} > 0$ are a fixed constants. In particular, if $f \in W^{1,\lambda}_{p,\gamma}(\Omega)$ we have

$$\begin{aligned} &\|f - f_{\Omega\cap B_\gamma(x,r)}\|_{L^1(\Omega\cap B_\gamma(x,r))} \\ &\qquad \leq c_p r^{\tilde{\eta}} |\Omega \cap B_\gamma(x,\tau r)|^{1-\frac{1}{p}} \|\nabla f\|_{L^p(\Omega\cap B_\gamma(x,r))} \\ &\qquad \leq c r^{\tilde{\eta}} (\tau r)^{n_\gamma(1-\frac{1}{p})} \|\nabla f\|_{L^p(\Omega\cap B_\gamma(x,r))} \leq c r^{n_\gamma(1-\frac{1}{p})+\frac{\lambda}{p}+\tilde{\eta}}, \end{aligned}$$

for some $c > 0$ independent of r.

This implies that $f \in \mathcal{L}^\theta_{1,\gamma}(\Omega)$ with

$$\theta = n_\gamma + \tilde{\eta} + \frac{\lambda - n_\gamma}{p}.$$

If $\theta > n_\gamma$, that is $p\tilde{\eta} > n_\gamma - \lambda$, by the original result of [18, Theorem 3.I] we deduce that $u \in C^{0,\alpha}(\bar{\Omega}, \delta_\gamma)$ with

$$\alpha = \tilde{\eta} + \frac{\lambda - n_\gamma}{p}.$$

Note that for applying [18, Theorem 3.I] we need only condition (2.5).

In conclusion, the following variant of Theorem 2.4 holds.

Theorem 2.5. *Let Ω be a bounded domain in $\mathbb{R}^n$ such that condition* (2.5) *holds, and let $p \in [1, \infty[$. Assume that the p-Poincaré inequality* (2.12) *holds. Let $\gamma \in]0, 1]$ and $\lambda > 0$ be such that*

$$p\tilde{\eta} > n_\gamma - \lambda.$$

Then there exists $c > 0$ such that for all $f \in W^{1,\lambda}_{p,\gamma}(\Omega)$ and for all $x, y \in \Omega$ we have

$$|f(x) - f(y)| \leq c\|f\|_{W^{1,\lambda}_{p,\gamma}(\Omega)} (|\bar{x} - \bar{y}|^\gamma + |x_n - y_n|)^{\tilde{\eta}+\frac{\lambda-n_\gamma}{p}}.$$

We observe that inequality (2.9) implies the validity of inequality (2.12) with $\tilde{\eta} = \frac{n_\gamma}{n} + n - n_\gamma$, which is the constant η used in Theorem 2.4. We also note that assuming the validity of p-Poincaré inequalities of type (2.12) is nowadays standard in Analysis on Metric Spaces. For instance, we refer to the celebrated paper [26] where general p-Poincaré inequalities of the form

$$\fint_B |f - f_B|d\mu \leq c_p r \left(\fint_{\tau B} g^p d\mu \right)^{\frac{1}{p}} \tag{2.13}$$

are considered. Here g is the upper gradient of f, B is an arbitrary ball of radius r in a metric space space X, τB is the concentric ball of radius τr for a fixed $\tau \geq 1$ and μ is a suitable measure in X. Sufficient conditions ensuring the validity of (2.13) are known in the literature and are discussed e.g., in [26, § 10]. See also [20] for a more recent work on this subject. We note that the study of inequalities of the type (2.12) in domains with cusps or domains of class $C^{0,\gamma}$ is very delicate and in general one does not expect their validity, in particular for outer cusps. Conditions for the validity of global (p,p)-Poincaré inequalities (which means that the power p appears also in the left-hand side of (2.13)) in domains with inner cusps and more generally John domains or L^p-averaging domains are given in [36] where, besides an interesting counterexample, a class of domains admitting moderately sharp outer 'spires' is also analyzed.

3 Extension of Sobolev-Morrey spaces on $C^{0,\gamma}$ domains

3.1 The case of elementary domains of class $C^{0,\gamma}$

Let Ω be an elementary Hölder continuous domain in $\mathbb{R}^n$ with exponent $\gamma \in]0,1]$ as in (2.1), with $W = \mathbb{R}^{n-1}$ and $a = -\infty$. Following [5, 6], we set $G = \mathbb{R}^n \setminus \overline{\Omega}$ and

$$G_k = \{x \in G : 2^{-k-1} < \rho_n(x) \leq 2^{-k}\}$$

for all $k \in \mathbb{Z}$, where $\rho_n(x) = x_n - \varphi(\overline{x})$ is the signed distance from $x \in \mathbb{R}^n$ to ∂G in the x_n direction and we consider a partition of unity associated with the covering $\{G_k\}_{k\in\mathbb{Z}}$ of G satisfying a number of properties. Namely, it is proved in [5] that for every $k \in \mathbb{Z}$ there exists $\psi_k \in C^\infty(\mathbb{R}^n)$ such that

(i) $\displaystyle\sum_{k=-\infty}^{\infty} \psi_k = \begin{cases} 1, & \text{if } x \in G, \\ 0, & \text{if } x \notin G; \end{cases}$

(ii) $G = \cup_{k=-\infty}^{\infty} \mathrm{supp}\psi_k$ and the covering $\{\mathrm{supp}\psi_k\}_{k\in\mathbb{Z}}$ has multiplicity equal to 2;

(iii) $G_k \subset \mathrm{supp}\psi_k \subset G_{k-1} \cup G_k \cup G_{k+1}$, for all $k \in \mathbb{Z}$;

(iv) $|D^\alpha \psi_k(x)| \leq c(\alpha) 2^{k(\frac{|\overline{\alpha}|}{\gamma}+\alpha_n)}$, for all $x \in \mathbb{R}^n, k \in \mathbb{Z}, c(\alpha) > 0$ is independent of x and k.

Note the appearance of γ in the exponent in item (iv) above.

Burenkov's Extension Operator was defined in [5] as follows. Let $l \in \mathbb{N}$ and $1 \leq p \leq \infty$. For every $f \in W^{l,p}(\Omega)$, we set

$$(Tf)(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ \displaystyle\sum_{k=-\infty}^{\infty} \psi_k(x) f_k(x), & \text{if } x \in G, \end{cases} \tag{3.1}$$

where

$$f_k(x) = \int_{\mathbb{R}^n} f(\overline{x} - 2^{-\frac{k}{\gamma}}\overline{z}, x_n - A2^{-k}z_n)\omega(z)dz =$$
$$= A^{-1}2^{\frac{k}{\gamma}(n-1)+k} \int_{\mathbb{R}^n} \omega(2^{\frac{k}{\gamma}}(\overline{x} - \overline{y}), A^{-1}2^k(x_n - y_n))f(y)dy\,,$$

A is a sufficiently large constant depending only on n and M in (2.2) (in [5] it is chosen for example $A = 200(1 + Mn)$) and $\omega \in C_c^\infty(\mathbb{R}^n)$ is a kernel of mollification defined by

$$\omega(x) = \omega_1(x_1)\cdots\omega_n(x_n),\ \omega_i \in C_c^\infty(1/2, 1),\ \int_{-\infty}^{+\infty} \omega(x_i)dx_i = 1,\ \int_{-\infty}^{+\infty} \omega_i(x_i)x_i^k dx_i = 0$$

for all $i = 1, \ldots, n$, $k = 1, \ldots, l$.

Among other results (in particular, concerning anisotropic Sobolev spaces), it is proved in [5] that the operator T is a linear continuous operator from $W_p^l(\Omega)$ to $W_p^{[\gamma l]}(\mathbb{R}^n)$ where $[\gamma l]$ is the integer part of γl.

The following theorem is a generalisation of the extension theorem proved in [21] in the case of Lispchitz domains, that is for $\gamma = 1$. Considering a number of technical issues appearing in the case $\gamma < 1$, we assume for simplicity that the function ϕ defining the Morrey norm satisfies the condition $\phi(r) = 1$ for all $r > 1$.

Theorem 3.1. *Let Ω be an elementary Hölder continuous domain in $\mathbb{R}^n$ with exponent $\gamma \in]0, 1]$, with $W = \mathbb{R}^{n-1}$ and $a = -\infty$. Let $l \in \mathbb{N}$, $p \in [1, \infty[$, and $\phi :]0, \infty[\to]0, \infty[$ satisfy the condition $\phi(r) = 1$ for all $r > 1$. Then the operator T maps $W_{p,1}^{l,\phi}(\Omega)$ continuously to $W_{p,1}^{[\gamma l],\phi_\gamma}(\mathbb{R}^n)$, where ϕ_γ is defined by $\phi_\gamma(r) = \phi(r^\gamma)$ for all $r \geq 0$. In particular, T maps the space $W_{p,1}^{l,\lambda}(\Omega)$ to the space $W_{p,1}^{[\gamma l],\gamma\lambda}(\mathbb{R}^n)$, for any $\lambda \geq 0$.*

The proof of Theorem 3.1 can be carried out by adapting the corresponding proof of [21] in a suitable way. Since the adaptation is quite technical and touches a number of delicate points, we indicate here the main steps starting from the first but crucial lemmas which we combine in the following statement. Here $\tilde{G}_k = G_{k-1} \cup G_k \cup G_{k+1} = \{x \in G : 2^{-k-2} < \rho_n(x) \leq 2^{-k+1}\}$ for all $k \in \mathbb{Z}$ and $\operatorname{diam} C$ denotes the Euclidean diameter of a set C.

Lemma 3.1. *Assume that $B_1(x, r) \cap G \neq \emptyset$ for some $x \in \mathbb{R}^n$ and $r > 0$. Let $h \in \mathbb{Z}$ be the minimal integer such that $B_1(x, r) \cap G_h \neq \emptyset$. Let $k \in \mathbb{Z}$ be such that $k \geq h + 3$ and $B_1(x, r) \cap \tilde{G}_k \neq \emptyset$. Then*

$$|2^{-(h+3)} - 2^{-k}| \leq c(r + r^\gamma)\,, \tag{3.2}$$

where c depends only on γ and $\mathrm{Lip}_\gamma\varphi$.

Moreover, given $E > 0$ there exists $S > 0$ depending only on γ, $\mathrm{Lip}_\gamma\varphi$, E, and a lower bound for h such that for every $\eta \in \mathbb{R}^n$, with $|\eta| < E$,

$$\operatorname{diam}\left(\bigcup_{k=h+3}^{\infty}\left(B_1(x, r) \cap \tilde{G}_k - (2^{-\frac{k}{\gamma}}\bar{\eta}, 2^{-k}\eta_n)\right)\right) \leq S(r + r^\gamma)\,. \tag{3.3}$$

Proof. By our assumptions we deduce that $\{x \in B_1(x, r) : \rho_n(x) = 2^{-h-2}\}, \{x \in B_1(x, r) : \rho_n(x) = 2^{-k+1}\} \neq \emptyset$ hence there exist $y, w \in B_1(x, r)$ with $y_n - \varphi(\bar{y}) = 2^{-h-2}$ and $w_n - \varphi(\bar{w}) = 2^{-k+1}$. Since $|y_n - x_n|, |\bar{y} - \bar{w}| < 2r$, by the Hölder continuity of φ we get

$$\begin{aligned}|2^{-(h+3)} - 2^{-k}| = \frac{1}{2}|2^{-h-2} - 2^{-k+1}| &= \frac{1}{2}|y_n - \varphi(\bar{y}) - w_n + \varphi(\bar{w})| \\ &\leq \frac{1}{2}(|y_n - w_n| + \mathrm{Lip}_\gamma\varphi|\bar{y} - \bar{w}|^\gamma) \leq \frac{1}{2}\left(2r + \mathrm{Lip}_\gamma\varphi(2r)^\gamma\right)\end{aligned}$$

and (3.2) follows.

We now prove (3.3). Let $k \geq h+3$ be such that $B_1(x,r)\cap \tilde{G}_k \neq \emptyset$. Let $a \in B_1(x,r)\cap \tilde{G}_{h+3}$ and $b \in B_1(x,r)\cap \tilde{G}_k$. By (3.2), for all $\eta \in \mathbb{R}^n$, with $|\eta| < E$, we have

$$|b_n - 2^{-k}\eta_n - (a_n - 2^{-(h+3)}\eta_n)| \leq |b_n - a_n| + |2^{-k} - 2^{-(h+3)}||\eta_n| \leq 2r + cE(r + r^\gamma)$$

and

$$|\bar{b} - 2^{-\frac{k}{\gamma}}\bar{\eta} - (\bar{a} - 2^{-\frac{h+3}{\gamma}}\bar{\eta})| \leq |\bar{b} - \bar{a}| + |2^{-\frac{k}{\gamma}} - 2^{-\frac{h+3}{\gamma}}||\bar{\eta}| \leq c\max\{1, 2^{-h(1-\gamma)/\gamma}\}(r + r^\gamma)$$

which proves (3.3). □

Another crucial step in the proof is [21, Lemma 2.4, (ii)] which has to be modified as follows. As in [6, Chapter 6], for every $k \in \mathbb{Z}$ we set

$$\tilde{\Omega}_k = \{x \in \Omega : 2^{-k-2} < |\rho_n(x)| \leq b2^{-k+1}\},$$

where $b = 10A$.

Lemma 3.2. *Assume that $B_1(x,r)\cap G \neq \emptyset$ for some $x \in \mathbb{R}^n$ and $r > 0$. Let $f \in W^{l,p}(\Omega)$ and $\mathcal{U} \subset \mathbb{R}^n$ be a fixed measurable set with $d := \sup\{\rho_n(x) : \ x \in B_1(x,r)\cap \mathcal{U}\} < \infty$. Then there exists $c > 0$ and $m \in \mathbb{N}$ depending only on n, l, p, M, ω, d, and for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq l$ there exists a function g_α independent of $r,\mathcal{U}$, such that for every $z \in \mathbb{R}^n$ with $|z| \leq c$ there exist m balls $B_1(x_z^{(i)}, r^\gamma)$, $i = 1,\dots,m$, such that*

$$\|D^\alpha f_k - g_\alpha\|^p_{L^p(B_1(x,r)\cap\mathcal{U}\cap\tilde{G}_k)} \leq c2^{pk(\frac{|\bar{\alpha}|}{\gamma}+\alpha_n-l)}\int_{|z|\leq c}\sum_{|\beta|=l}\|D^\beta f\|^p_{L^p(\cup_{i=1}^m B_1(x_z^{(i)},r^\gamma)\cap\tilde{\Omega}_k)}dz, \tag{3.4}$$

for all $k \in \mathbb{N}$.

The proof of the previous lemma follows the lines of [21, Lemma 2.4, (ii)]. We omit the lengthy details but we explain how this lemma is used and how the modified exponent $pk(\frac{|\bar{\alpha}|}{\gamma}+\alpha_n-l)$ affects the final result. Namely, in order to prove Theorem 3.1, one has to estimate the derivatives $D^\alpha Tf$ of the extension Tf of a function f. By applying the Leibnitz rule one ends up with estimating $D^{\alpha-\beta}\psi_k D^\beta f_k$ for all $\beta \leq \alpha$. The difficult part of the work concerns the case $\beta < \alpha$ and $k > 0$. One observes that $\sum_{k\in\mathbb{Z}} D^{\alpha-\beta}\psi_k D^\beta f_k = \sum_{k\in\mathbb{Z}} D^{\alpha-\beta}\psi_k(D^\beta f_k - g_\beta)$ for $\beta < \alpha$ since g_β does not depend on k. Thus, one has to estimate $D^\beta f_k - g_\beta$. By combining the previous lemma with property (iv) of the partition of unity, we have

$$\begin{aligned}
\|D^{\alpha-\beta}&\psi_k(D^\beta f_k - g_\beta)\|^p_{L^p(B_1(x,r)\cap\mathcal{U}\cap\tilde{G}_k)} \\
&\leq c2^{pk(\frac{|\bar{\alpha}-\bar{\beta}|}{\gamma}+\alpha_n-\beta_n)}\|D^\beta f_k - g_\beta\|^p_{L^p(B_1(x,r)\cap\mathcal{U}\cap\tilde{G}_k)} \\
&\leq c2^{pk(\frac{|\bar{\alpha}-\bar{\beta}|}{\gamma}+\alpha_n-\beta_n)}2^{pk(\frac{|\bar{\beta}|}{\gamma}+\beta_n-l)}\int_{|z|\leq c}\sum_{|\beta|=l}\|D^\beta f\|^p_{L^p(\cup_{i=1}^m B_1(x_z^{(i)},r^\gamma)\cap\tilde{\Omega}_k)}dz\,.
\end{aligned} \tag{3.5}$$

We note that the exponent of the power of 2 in the right-hand side of (3.5) equals

$$pk\left(\frac{|\bar{\alpha}-\bar{\beta}|}{\gamma}+\alpha_n-\beta_n\right) + pk\left(\frac{|\bar{\beta}|}{\gamma}+\beta_n-l\right) = pk\left(\frac{|\bar{\alpha}|}{\gamma}+\alpha_n-l\right)$$

hence one can control the right-hand side of (3.5), provided that exponent is non-positive, that is

$$|\bar{\alpha}| + \gamma\alpha_n \leq \gamma l\,. \tag{3.6}$$

Inequality (3.6) explains why one gets $[\gamma l]$ as index of smoothness in the target Sobolev space $W^{[\gamma l],\phi}_{\lambda,\gamma}(\mathbb{R}^n)$ in Theorem 3.1.

Moreover, in estimate (3.4) we have the quantity

$$\|D^\beta f\|^p_{L^p(\cup_{i=1}^m B_1(x_z^{(i)},r^\gamma)\cap\tilde{\Omega}_k)}$$

and, since the balls have radius r^γ, one eventually controls that quantity via

$$\phi(r^\gamma)\|D^\beta f\|^p_{L^\phi_{p,1}(\Omega)}$$

which explains the appearance of the new weight ϕ_λ in Theorem 3.1. For further details, we refer to the proof of [21, Theorem 2.5].

3.2 The case of general domains of class $C^{0,\gamma}$

We recall the definition of open sets with $C^{0,\gamma}$ boundary. Here and in the sequel, given a set C in $\mathbb{R}^n$ and $d>0$ we denote by C_d the set $\{x\in C : \mathrm{dist}(x,\partial C)>d\}$.

Definition 1. *Let $\gamma\in]0,1]$, $d>0$, $M\geq 0$, $s\in\mathbb{N}\cup\{\infty\}$. Let $\{V_j\}_{j=1}^s$ be a family of cuboids, i.e. for every $j=\overline{1,s}$ there exists an isometry λ_j in $\mathbb{R}^n$ such that*

$$\lambda_j(V_j)=\Pi_{i=1}^n]a_{i,j},b_{i,j}[$$

where $0<a_{i,j}<a_{i,j}+d<b_{i,j}$. Assume that $D:=\sup_{j=\overline{1,s}}\mathrm{diam}V_j<\infty$, $(V_j)_d\neq\emptyset$ for all $j=\overline{1,s}$, and that the multiplicity of the covering $\{V_j\}_{j=1}^s$ is finite. We then say that $\mathcal{A}=(s,d,\{V_j\}_{j=1}^s,\{\lambda_j\}_{j=1}^s)$ is an atlas.

Let $M\geq 0$. We say that an open set Ω in $\mathbb{R}^n$ is of class $C^{0,\gamma}_M(\mathcal{A})$ if the following conditions are satisfied:

(i) For every $j=\overline{1,s}$, we have $\Omega\cap(V_j)_d\neq\emptyset$.

(ii) $\Omega\subset\cup_{j=1}^s(V_j)_d$.

(iii) For every $j=\overline{1,s}$, the set $\mathcal{H}_j:=\lambda_j(\Omega\cap V_j)$ satisfies the following condition: either $\mathcal{H}_j=\Pi_{i=1}^n]a_{i,j},b_{i,j}[$ (in which case $V_j\subset\Omega$), or $\mathcal{H}_j$ is a bounded elementary Hölder continuous domain of the form

$$\mathcal{H}_j=\{x\in\mathbb{R}^n:\ \bar{x}\in W_j,\ a_{n,j}<x_n<\varphi_j(\bar{x})\}$$

where φ_j is a real-valued Hölder continuous function with exponent γ, defined on $W_j=\Pi_{i=1}^{n-1}]a_{i,j},b_{i,j}[$ such that

$$a_{n,j}+d<\varphi_j \quad\text{and}\quad \mathrm{Lip}_\gamma\varphi_j\leq M$$

(in which case $V_j\cap\partial\Omega\neq\emptyset$).

Finally, we say that an open set Ω in $\mathbb{R}^n$ is of class $C^{0,\gamma}$ if it is of class $C^{0,\gamma}_M(\mathcal{A})$ for some M and $\mathcal{A}$.

The definition of Burenkov's Extension Operator for a general domain of class $C^{0,\gamma}$ is given by pasting together the extension operators defined on each chart of the atlas as follows. Following [6, p. 265], given an open set Ω of class $C^{0,\gamma}_M(\mathcal{A})$, we consider a family of functions $\{\psi\}_{j=1}^s$ such that $\psi_j \in C_c^\infty(\mathbb{R}^n)$, $\mathrm{supp}\psi_j \subset (V_j)_d$, $0 \le \psi_j \le 1$, $\sum_{j=1}^s \psi_j^2(x) = 1$ for all $x \in \Omega$ and such that $\|D^\alpha \psi_j\|_{L^\infty(\mathbb{R}^n)} \le M$ for all $j = \overline{1,s}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le l$, where M depends only on n, l, d.

Burenkov's Extension Operator T is defined from $W_p^l(\Omega)$ to $W_p^{[\gamma l]}(\mathbb{R}^n)$ by

$$Tf = \sum_{j=1}^s \psi_j T_j(f\psi_j), \tag{3.7}$$

for all $f \in W^{l,p}(\Omega)$, where T_j are the extension operators defined on each domain $\Omega \cap V_j$. See [21] for details.

Then, we have he following. Recall that ϕ_γ is defined by $\phi_\gamma(r) = \phi(r^\gamma)$ for all $r \ge 0$.

Theorem 3.2. *Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,\gamma}$ with $\gamma \in]0,1]$. Let $l \in \mathbb{N}$, $p \in [1,\infty[$, and $\phi :]0,\infty[\to]0,\infty[$ satisfying the condition $\phi(r) = 1$ for all $r > 1$. Then the operator T maps $W_{p,1}^{l,\phi}(\Omega)$ continuously to $W_{p,1}^{[\gamma l],\phi_\gamma}(\mathbb{R}^n)$. In particular, T maps the space $W_{p,1}^{l,\lambda}(\Omega)$ to the space $W_{p,1}^{[\gamma l],\gamma\lambda}(\mathbb{R}^n)$, for any $\lambda \ge 0$.*

The proof of Theorem 3.2 can be carried out by pasting together local extensions operators provided by Theorem 3.1 in each cuboid of the covering of Ω. This argument is described in detail in the proof of [21, Theorem 3.3]. Finally, we can deduce the following

Corollary 3.1. *Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,\gamma}$ with $\gamma \in]0,1]$. Let $l \in \mathbb{N}$, $p \in [1,\infty[$, and $\lambda > 0$. If*

$$p[\gamma l] > n - \gamma\lambda$$

and $[\gamma l] + \frac{\gamma\lambda - n}{p} < 1$ then there exists $c > 0$ such that for all $f \in W_{p,1}^{l,\lambda}(\Omega)$ and for all $x, y \in \Omega$ we have

$$|f(x) - f(y)| \le c\|f\|_{W_{p,1}^{l,\lambda}(\Omega)} |x - y|^{[\gamma l] + \frac{\gamma\lambda - n}{p}} . \tag{3.8}$$

The proof of the previous corollary follows immediately by Theorem 3.2 and estimate (2.7) applied with $\gamma = 1$ and l replaced by $[\gamma l]$. Indeed, by Theorem 3.2, any functions $f \in W_{p,1}^{l,\lambda}(\Omega)$ is extended to the whole of $\mathbb{R}^n$ as a function of $W_{p,1}^{[\gamma l],\gamma\lambda}(\mathbb{R}^n)$ to which the classical Sobolev-Morrey Theorem applies.

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