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# Eurasian Mathematical Journal

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## Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

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The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

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<u>Title page.</u> The title page should start with the title of the paper and authors' names (no degrees). It should contain the <u>Keywords</u> (no more than 10), the <u>Subject Classification</u> (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the <u>Abstract</u> (no more than 150 words with minimal use of mathematical symbols).

<u>Figures.</u> Figures should be prepared in a digital form which is suitable for direct reproduction.

<u>References.</u> Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

<u>Authors' data.</u> The authors' affiliations, addresses and e-mail addresses should be placed after the References.

<u>Proofs.</u> The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

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The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

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1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

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2.3. A review should cover the following topics:

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- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

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**At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.**

## ANDREI ANDREEVICH SHKALIKOV

(to the 70th birthday)

Andrei Andreevich Shkalikov, corresponding member of the Russian Academy of Sciences, an outstanding mathematician with a wide range of interests, a remarkable person, professor of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University was born on November 19, 1949.

Andrei Andreevich is a leading specialist in the theory of operators and their applications, especially to problems of mechanics and mathematical physics. He is well known for his work in the theory of functions and in the theory of spaces with indefinite metrics. He is also a specialist in the theory of entire and meromorphic functions and its applications to operator theory.

Andrei Andreevich is known for solving a number of difficult problems that for many years remained unsolved. His work on the basic properties of systems of root functions of differential operators is well known worldwide. He gave a justification for Mandelstam's hypothesis about the existence of solutions satisfying the radiation principle. He solved, in general form, the Rayleigh problem on the reflection of waves from a periodic surface, obtained a solution to the Sobolev problem on the stability of the motion of a top with a cavity filled with liquid. His contribution to the construction of an abstract theory of the Orr-Sommerfeld problem is invaluable. He obtained a description of the limiting spectral portraits for a large class of functions describing the profiles of fluid flows. He is one of the founders of the modern theory of differential operators, coefficients of which are distributions, and inverse problems for such operators.

Andrei Andreevich has been a plenary speaker at many international conferences. He conducts fruitful scientific work and collaborates with many international mathematical research centers.

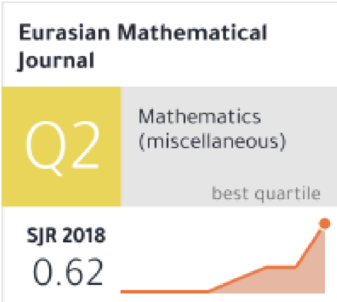
Andrei Andreevich is an author of more than 130 scientific publications. Among his pupils there are more than 20 Candidates of Sciences and 6 Doctors of Sciences. The results obtained by A.A. Shkalikov, his pupils, collaborators and followers gained worldwide recognition.

Professor Shkalikov is also an outstanding organizer. Under his supervision, many international conferences were held. In particular, conferences dedicated to the memory of I.G. Petrovsky, I.M. Gelfand, S.M. Nikol'skii, B.M. Levitan, anniversary conferences of V.A. Sadovnichy, and others.

Andrei Andreyevich is a deputy editor-in-chief of the journals Mathematical Notes, Moscow University Mathematics Bulletin, Moscow University Mechanics Bulletin, and a member of the editorial boards of the Russian Mathematical Surveys, Proceedings of the Moscow Mathematical Society and other journals, including the Eurasian Mathematical Journal.

The Editorial Board of the Eurasian Mathematical Journal cordially congratulates Andrei Andreevich on the occasion of his 70th birthday and wishes him good health, and new achievements in mathematics and mathematical education.

## GOOD NEWS: EMJ IS NOW AN SJR Q2 JOURNAL




Recently the lists were published of all mathematical journals included in 2018 SCImago Journal Rank (SJR) quartiles Q1 (385 journals), Q2 (430 journals), Q3 (445 journals), and Q4 (741 journals), and Scopus CiteScore quartiles Q1 (443 journals), Q2 (375 journals), Q3 (348 journals), and Q4 (283 journals).

With great pleasure we inform our readers and authors that the Eurasian Mathematical Journal was included in the most popular scientific ranking database SJR in quartile Q2, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia. The SJR data for the Eurasian Mathematical Journal (2018) is as follows:

550. Eurasian Mathematical Journal (Kazakhstan), Q2, SJR=0.624.
(550 is the number in the list of all Q1 - Q4 journals.)

The SJR indicator is calculated by using the data of the Scopus Database of the Elsevier, the modern publishing business founded in 1880. It uses a sophisticated formula, taking into account various characteristics of journals and journals publications. This formula and related comments can be viewed on the web-page

http : //www.scimagojr.com/journalrank.php.

Some other SJR Q2 mathematical journals:
490. Studia Mathematica (Poland), SJR=0.706,
492. Comptes Rendus Mathematique (France), SJR=0.704,
522. Journal of Mathematical Physics (USA), SJR=0.667,
540. Doklady Mathematics (Russia), SJR=0.636,
570. Journal of Mathematical Sciences (Japan), SJR=0.602,
662. Journal of Applied Probability (UK), SJR=0.523,
733. Mathematical Notes (Russia), SJR=0.465,
791. Canadian Mathematical Bulletin (Canada), SJR=0.433.

Our journal ranks:

7726th place in the list of 31971 scientific journals, representing all subjects and all regions, included in this database (in the first 25% of journals of this category),

225th place in the list of 2519 scientific journals, representing all subjects, of the Asiatic region, included in this database (in the first 10% of journals of this category),

550th place in the list of 2011 mathematical journals, representing all regions, included in this database (in the first 30% of journals of this category),

19th place in the list of 165 mathematical journals of the Asiatic region, included in this database (in the first 15% of journals of this category).

On a separate page the SJR statistics for the Eurasian Mathematical Journal is attached.

Recall that the Eurasian Mathematical Journal started its work in 2010 (see [1]-[3]) and was first included in SJR indicator in 2014 (Q4, SJR=0.101, see [4], [5], [6]). So, the ambitious plan set in [6] was implemented and even essentially exceeded.

As for the Scopus CiteScore indicator, it uses another sophisticated formula, differently taking into account various characteristics journals publications. This formula and related comments can be viewed on the web-page

https : //ru.service.elsevier.com/app/answers/detail/a$_{i}$d/19266/supporthub/scopus.

In this indicator the Eurasian Mathematical Journal was included in quartile Q3. The CiteScore data for the Eurasian Mathematical Journal (2018) is as follows:

333. Eurasian Mathematical Journal (Kazakhstan), Q3, CiteScore = 0,41
(333 is the number in the list of only Q3 journals.)

Some other Scopus CiteScore Q3 mathematical journals:

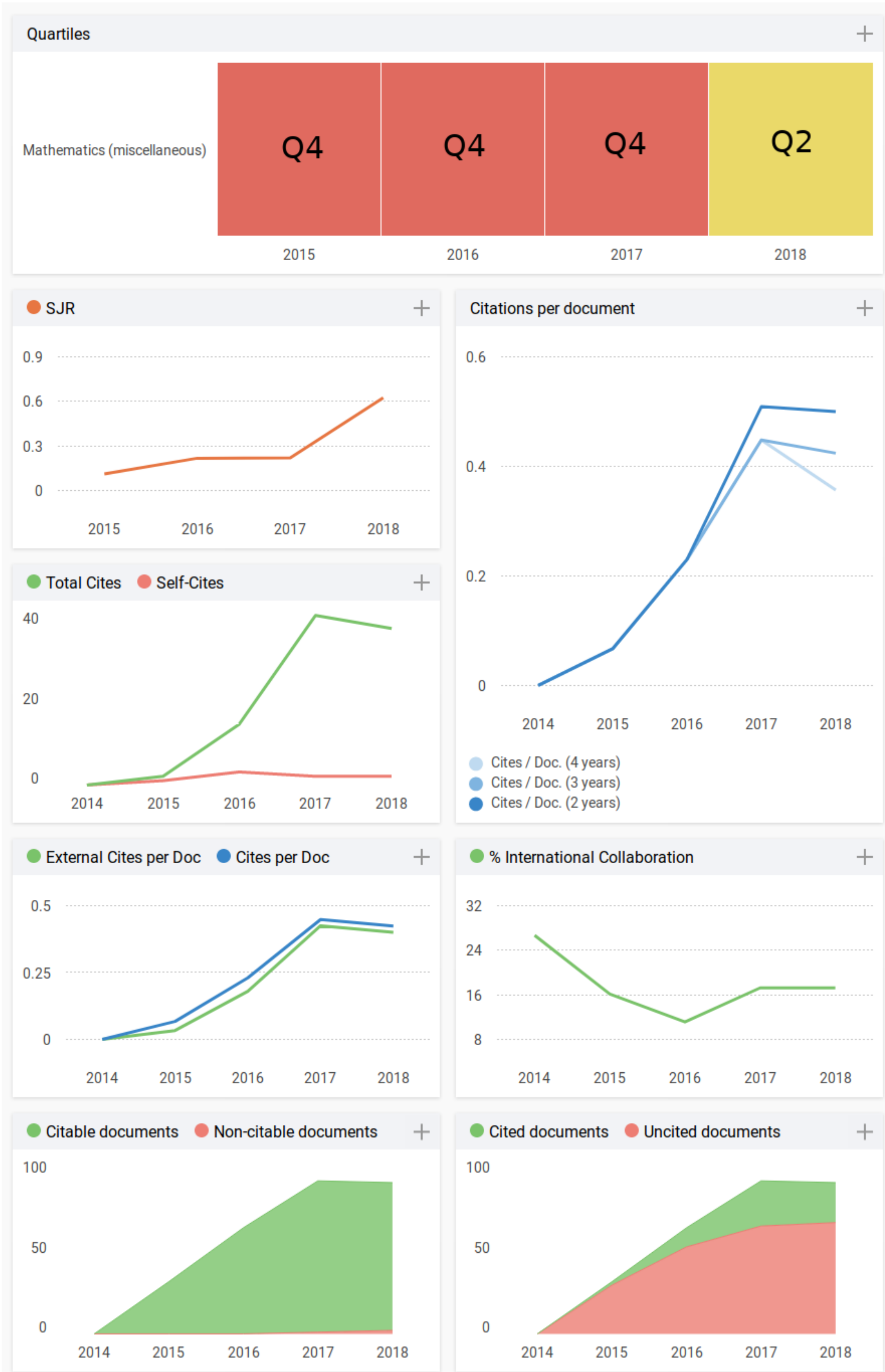
320. Czechoslovak Mathematical Journal (Czech Republic), CiteScore = 0.44,
321. Italian Journal of Pure and Applied Mathematics (Italy), CiteScore = 0.44,
323. Studia Scientiarum Mathematicarum Hungarica (Hungary), CiteScore = 0.44,
332. Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie (Romania), CiteScore = 0.41,
334. Indian Journal of Pure and Applied Mathematics (India), CiteScore = 0.41,
33. Transactions of the Moscow Mathematical Society (Russia), CiteScore = 0.41,
337. Illinois Journal of Mathematics (USA), CiteScore = 0.40,
339. Publications de l'Institut Mathematique (France), CiteScore = 0.40.

Our main current aim is to preserve the status of an SJR Q2 journal and of a Scopus CiteScore Q3 journal.

We hope that all respected members of the international Editorial Board, reviewers, current authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

V.I. Burenkov, K.N. Ospanov, T.V. Tararykova, A.M. Temirkhanova.

Extract from http://www.scimagojr.com/journalrank.php
EURASIAN MATHEMATICAL JOURNAL



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# ON A KUDRYAVTSEV TYPE FUNCTION SPACE

**A.A. Kalybay, Zh.A. Keulimzhayeva, R. Oinarov**

Communicated by M.L. Goldman

**Key words:** function space, weight function, multiweighted derivative, norm, boundary conditions, stabilization to a polynomial.

**AMS Mathematics Subject Classification:** 46E35, 46E30.

**Abstract.** In the paper we introduce a Kudryavtsev type space, the norm of which contains a differential operator called the multiweighted derivative. This type of spaces has been studied in details for power weights. Here we consider general weights. The main result of the paper is the proof of the existence of a generalized polynomial, to which functions of this space stabilize at a singular point, so that the coefficients of this polynomial can be considered as the characteristics of the behaviour of a function nearby this singularity.

**DOI: https://doi.org/10.32523/2077-9879-2019-10-4-34-46**

## 1 Introduction

Let $I = (0, \infty)$, $n$ be a natural number, $\bar{\rho} = (\rho_1, \rho_2, ..., \rho_n)$, $\rho_i : I \to R$, $i = 1, 2, ..., n$, be positive functions locally summable on $I$ such that

$$\rho_i^{-1} \equiv \frac{1}{\rho_i} \in L_1^{loc}(I), \quad i = 1, 2, ..., n-1, \quad \rho_n^{-1} \in L_{p'}^{loc}(I), \quad 1 \leq p' \leq \infty. \tag{1.1}$$

For a function $f : I \to R$ we assume that $D_{\bar{\rho}}^0 f(t) \equiv f(t)$ and $D_{\bar{\rho}}^k f(t) = \rho_k(t) \frac{d}{dt} D_{\bar{\rho}}^{k-1} f(t)$, $t \in I$, $k = 1, 2, ..., n$, where all functions $D_{\bar{\rho}}^k f$, $k = 0, 1, ..., n-1$, are locally absolutely continuous on $I$. We call this operation $D_{\rho}^k f$, $k = 1, 2, ..., n$, when it has sense, the $k$ th order $\bar{\rho}$ – multiweighted derivative of the function $f$ on $I$.

Let $L_{p,\bar{\rho}}^n(I)$ be the class of all functions $f : I \to R$ that have the $n$ th order $\bar{\rho}$ – multiweighted derivatives on $I$ and finite value $\|D_{\bar{\rho}}^n f\|_{p,I}$, where $\|\cdot\|_{p,I}$ is the usual norm of the space $L_p(I)$, $1 \leq p \leq \infty$. Let $\frac{1}{p} + \frac{1}{p'} = 1$.

For any function $f \in L_{p,\bar{\rho}}^n(I)$ and any $x_0 \in I$ there exists $D_{\bar{\rho}}^k f(x_0)$, $k = 0, 1, ..., n-1$, in particular, $D_{\bar{\rho}}^k f(1)$, $k = 0, 1, ..., n-1$. Therefore, on the class $L_{p,\bar{\rho}}^n(I)$ we can define the functional

$$\|f\|_{W_{p,\bar{\rho}}^n} = \|D_{\bar{\rho}}^n f\|_{p,I} + \sum_{i=0}^{n-1} |D_{\bar{\rho}}^i f(1)|, \tag{1.2}$$

that turns the class $L_{p,\bar{\rho}}^n(I)$ into a normed space. We denote by $W_{p,\bar{\rho}}^n(I)$ the class $L_{p,\bar{\rho}}^n(I)$ with norm (1.2). By using the standard method, we can prove that $W_{p,\bar{\rho}}^n(I)$ is a Banach space.

From conditions (1.1) it follows that the functions $\rho_i$, $i = 1, 2, ..., n$, can have singularities at the endpoints of the interval $I$. Therefore, it is very important to determine the behaviour of a function in $W^n_{p,\bar\rho}(I)$ in a neighbourhood of the endpoints of $I$ with respect to the degree of singularity of the weight functions $\rho_i$, $i = 1, 2, ..., n$.

For $\rho_i \equiv 1$, $i = 1, 2, ..., n-1$, and $\rho_n \equiv \varphi$ we have $D^k_{\bar\rho} f(t) = f^{(k)}(t)$, $k = 0, 1, ..., n-1$, $D^n_{\bar\rho} f(t) = \varphi(t) f^{(n)}(t)$, $\|D^n_{\bar\rho} f\|_{p,I} = \|\varphi f^{(n)}\|_{p,I}$ and $D^i_{\bar\rho} f(1) = f^{(i)}(1)$, $i = 0, 1, ..., n-1$, i.e., we get the Kudryavtsev's spaces $L^n_{p,\varphi}(I)$ and $W^n_{p,\varphi}(I)$, which were studied in the works [4] and [5] in connection with the problem of the setting of boundary conditions for singular elliptic equations.

Let us note that when $\rho_i$ are power functions in the form $\rho_i(t) = t^{\alpha_i}$, $i = 1, 2, ..., n$, $\alpha_i \in R$, the space $W^n_{p,\bar\rho}(I)$ has been studied in details (see [1], [2], [3] and references given there).

The main aim of this work is to determine the boundary behavior of functions in $W^n_{p,\bar\rho}(I)$.

## 2 Auxiliary facts and statements

For $i, j = 0, 1, ..., n-1$ we define the functions $K_{j,i+1}$ and $\bar K_{j,i+1}$:

$$K_{j,i+1}(x,s) = (-1)^{j-i} \int_s^x \rho_j^{-1}(t_j) \int_s^{t_j} \rho_{j-1}^{-1}(t_{j-1}) ... \int_s^{t_{i+2}} \rho_{i+1}^{-1}(t_{i+1}) dt_{i+1} dt_{i+2} ... dt_j$$

and

$$\bar K_{j,i+1}(x,s) = \int_s^x \rho_j^{-1}(t_j) \int_{t_j}^x \rho_{j-1}^{-1}(t_{j-1}) ... \int_{t_{i+2}}^x \rho_{i+1}^{-1}(t_{i+1}) dt_{i+1} dt_{i+2} ... dt_j$$

for $j > i$, $K_{i,i+1}(x,s) \equiv \bar K_{i,i+1}(x,s) \equiv 1$ and $K_{j,i+1}(x,s) \equiv \bar K_{j,i+1}(x,s) \equiv 0$ for $j < i$. Assuming for any $s$ and $x$ that $\int_s^x = -\int_x^s$, we have $K_{j,i+1}(x,s) = \bar K_{j,i+1}(s,x)$ and $\bar K_{j,i+1}(x,s) = K_{j,i+1}(s,x)$.

The functions $K_{j,i+1}$ and $\bar K_{j,i+1}$ have the properties given in Lemma 2.1.

**Lemma 2.1.** *Let* $0 < x, t, s < \infty$ *and* $0 \le i \le j \le n-1$. *Then*

$$K_{j,i+1}(x,s) = \sum_{k=i}^{j} K_{j,k+1}(x,t) K_{k,i+1}(t,s), \tag{2.1}$$

$$\bar K_{j,i+1}(x,s) = \sum_{k=i}^{j} \bar K_{j,k+1}(t,s) \bar K_{k,i+1}(x,t). \tag{2.2}$$

*Proof.* For $j = i$ we have $K_{i,i+1}(x,s) = \bar K_{i,i+1}(x,s) \equiv 1$; moreover, (2.1) and (2.2) hold. Let $j = i+1$. Then

$$K_{i+1,i+1}(x,s) = \int_s^x \rho_{i+1}^{-1}(\tau) d\tau = \int_t^x \rho_{i+1}^{-1}(\tau) d\tau + \int_s^t \rho_{i+1}^{-1}(\tau) d\tau$$

$$= K_{i+1,i+1}(x,t) + K_{i+1,i+1}(t,s) = \sum_{k=i}^{i+1} K_{i+1,k+1}(x,t) K_{k,i+1}(t,s),$$

i.e., (2.1) holds for $j = i+1$. Let (2.1) hold for $j = i+1, i+2, ..., m$, where $m < n-1$ and $0 \le i < j$. Let us prove that (2.1) also holds for $j = m+1$.

We have that

$$K_{m+1,i+1}(x,s) = \int_s^x \rho_{m+1}^{-1}(\tau) K_{m,i+1}(\tau,s) d\tau$$

$$= \int_s^t \rho_{m+1}^{-1}(\tau) K_{m,i+1}(\tau,s) d\tau + \int_t^x \rho_{m+1}^{-1}(\tau) K_{m,i+1}(\tau,s) d\tau$$

$$= K_{m+1,i+1}(t,s) + \sum_{k=i}^{m} \int_t^x \rho_{m+1}^{-1}(\tau) K_{m,k+1}(\tau,t) K_{k,i+1}(t,s) d\tau$$

$$= K_{m+1,i+1}(t,s) K_{m,m+1}(x,t) + \sum_{k=i}^{m} K_{m+1,k+1}(x,t) K_{k,i+1}(t,s) d\tau$$

$$= \sum_{k=i}^{m+1} K_{m+1,k+1}(x,t) K_{k,i+1}(t,s),$$

i.e., (2.1) holds for $j = m+1$. Therefore, this equality also holds for $0 \le i \le j \le n-1$. The validity of (2.2) can be proved similarly. □

Let $I_0 = (0,1]$ and $I_\infty = [1,\infty)$. Let us introduce the spaces $W^n_{p,\bar\rho}(I_0)$ and $W^n_{p,\bar\rho}(I_\infty)$ with norms

$$\|f\|_{W^n_{p,\bar\rho}(I_0)} = \|D^n_{\bar\rho} f\|_{p,I_0} + \sum_{i=0}^{n-1} |D^n_{\bar\rho} f(1)|, \tag{2.3}$$

$$\|f\|_{W^n_{p,\bar\rho}(I_\infty)} = \|D^n_{\bar\rho} f\|_{p,I_\infty} + \sum_{i=0}^{n-1} |D^n_{\bar\rho} f(1)|, \tag{2.4}$$

as restrictions of the function $f \in W^n_{p,\bar\rho}(I)$ to $I_0$ and $I_\infty$, respectively. If $f \in W^n_{p,\bar\rho}(I)$, then the restrictions $f_{|I_0} = f_1$ and $f_{|I_\infty} = f_2$, respectively, belong to $W^n_{p,\bar\rho}(I_0)$ and $W^n_{p,\bar\rho}(I_\infty)$. Moreover,

$$D^i_{\bar\rho} f_1(1) = D^i_{\bar\rho} f_2(1), \ i = 0,1,...,n-1. \tag{2.5}$$

Conversely, if $f_1 \in W^n_{p,\bar\rho}(I_0)$ and $f_2 \in W^n_{p,\bar\rho}(I_\infty)$ with condition (2.5), then for the functions $f = f_1$ on $I_0$ and $f = f_2$ on $I_\infty$ the functions $D^i_{\bar\rho} f$, $i = 0,1,...,n-1$, are locally absolute continuous on $I$. It means that there exists $D^n_{\bar\rho} f$ almost everywhere on $I$ and, in addition, $\|D^n_{\bar\rho} f\|_{p,I} < \infty$ since

$$\|D^n_{\rho} f\|_{p,I} \le \|D^n_{\bar\rho} f_1\|_{p,I_0} + \|D^n_{\bar\rho} f_2\|_{p,I_\infty}. \tag{2.6}$$

On the other hand

$$\|D^n_{\rho} f\|_{p,I_0} + \|D^n_{\bar\rho} f\|_{p,I_\infty} \le 2\|D^n_{\bar\rho} f\|_{p,I}, \ \forall f \in W^n_{p,\bar\rho}(I), \tag{2.7}$$

therefore, from (2.3), (2.4), (2.5), (2.6) and (2.7) we have

$$\frac{1}{2}\left(\|f\|_{W^n_{p,\bar\rho}(I_0)} + \|f\|_{W^n_{p,\bar\rho}(I_\infty)}\right) \le \|f\|_{W^n_{p,\bar\rho}(I)} \le 2\left(\|f\|_{W^n_{p,\bar\rho}(I_0)} + \|f\|_{W^n_{p,\bar\rho}(I_\infty)}\right) \tag{2.8}$$

for all $f \in W^n_{p,\bar\rho}(I)$. On the basis of (2.8) the investigation of the space $W^n_{p,\bar\rho}(I)$ can be reduced to investigation of the spaces $W^n_{p,\bar\rho}(I_0)$ and $W^n_{p,\bar\rho}(I_\infty)$.

## 3 Space $W^n_{p,\bar\rho}(I_0)$

In this Section, for brevity, we denote $W^n_{p,\bar\rho}(I_0) \equiv W^n_{p,\bar\rho}$, $\|\cdot\|_p \equiv \|\cdot\|_{p,I_0}$ and $L_p \equiv L_p(I_0)$. Let us note that the system of functions $\{K_{i,1}(1,t), t \in I_0\}_{i=0}^{n-1}$ is a system of linearly independent solutions of the homogeneous equation

$$D^n_{\bar\rho} f(t) = 0, \quad t \in I_0. \tag{3.1}$$

In [6] L.D. Kudryavtsev introduced the concept of stabilization of a function belonging to $W^n_{p,\varphi}(I)$ to an algebraic polynomial of the $(n-1)$ th order, and coefficients of this polynomial can be considered as "boundary values" at infinity of this function. Let us consider the polynomial with respect to the system of functions $\{K_{i,1}(1,t)\}_{i=0}^{n-1}$ such that

$$P_n(t) \equiv P_n(\bar\rho, t) = \sum_{i=0}^{n-1} a_i K_{i,1}(1,t),$$

where $a_i$, $i = 0, 1, ..., n-1$, are real numbers. Since for the space $W^n_{p,\bar\rho} = W^n_{p,\bar\rho}(I_0)$ zero is a singular point, then as in [6] by L.D. Kudryavtsev we introduce Definition 1.

**Definition 1.** We say that a function $f \in W^n_{p,\bar\rho}$ stabilizes at zero to the polynomial $P_n(t) \equiv P_n(t, f)$, if

$$\lim_{t\to 0^+} D^i_{\bar\rho}[f(t) - P_n(t,f)] = 0, \quad i = 0, 1, ..., n-1. \tag{3.2}$$

From (3.2) it follows that if $f \in W^n_{p,\bar\rho}$ stabilizes at $t = 0$ to the polynomial $P_n(t, f)$, then the coefficients of the polynomial $P_n(t, f)$ can be consequently defined from the relations

$$a_i(f) = \lim_{t\to 0^+} [D^i_{\bar\rho} f(t) - \sum_{j=i+1}^{n-1} a_j(f) K_{j,i+1}(1,t)], \quad i = n-1, n-2, ..., 0. \tag{3.3}$$

The values $a_i(f)$, $i = 0, 1, ..., n-1$, can be interpreted as "boundary values" of the function $f \in W^n_{p,\bar\rho}$ at zero. Let us now consider the conditions of stabilization of the function $f \in W^n_{p,\bar\rho}$ at zero to the polynomial $P_n(t, f)$.

**Theorem 3.1.** *Let $1 < p < \infty$. Then each function $f \in W^n_{p,\bar\rho}$ stabilizes at zero to the polynomial $P_n(t, f)$, if and only if for all $t \in (0, 1]$ and $i = 0, 1, ..., n-1$, the following integrals*

$$\int_0^t \bar K^{p'}_{n-1,i+1}(t,s) \rho_n^{-p'}(s) ds \tag{3.4}$$

*converge. Moreover, the following relations*

$$D^i_{\bar\rho} f(t) = \sum_{j=i}^{n-1} a_j(f) K_{j,i+1}(1,t) + \int_0^t \bar K_{n-1,i+1}(t,s) \rho_n^{-1}(s) D^n_{\bar\rho} f(s) ds \tag{3.5}$$

*hold for $i = 0, 1, ..., n-1$.*

First we prove the following statement.

**Lemma 3.1.** *If for a function $f : I_0 \to R$ the operation $D^n_{\bar\rho} f$ has sense on the interval $I_0$ and the function $f$ stabilizes at zero to the polynomial $P_n(t, f)$, then for all $t \in (0, 1]$ and $i = 0, 1, ..., n-1$, the integrals*

$$I_{i+1,n}(t,f) = \int\limits_0^t \rho_{i+1}^{-1}(t_{i+1}) \int\limits_0^{t_{i+1}} \rho_{i+2}^{-1}(t_{i+2}) ... \int\limits_0^{t_{n-1}} \rho_n^{-1}(t_n) D^n_{\bar\rho} f(t_n) dt_n dt_{n-1} ... dt_{i+1} \tag{3.6}$$

*converge. Moreover, the following relations*

$$D^i_{\bar\rho} f(t) = \sum_{j=i}^{n-1} a_j K_{j,i+1}(1,t) + I_{i+1,n}(t,f) \tag{3.7}$$

*hold for $i = 0, 1, ..., n-1$ (each integral (3.6), in general, is understood in improper sense).*

*Proof.* Let the function $f : I_0 \to R$, for which the operation $D^n_{\bar\rho} f$ has sense, stabilize at zero to the polynomial $P_n(t, f)$. Then from (3.3) for $i = n-1$ we have

$$\lim_{\varepsilon \to 0} D^{n-1}_{\bar\rho} f(\varepsilon) = a_{n-1}$$

and for $0 < \varepsilon < t \le 1$

$$\lim_{\varepsilon\to 0} \int\limits_\varepsilon^t \rho_n^{-1}(s) D^n_{\bar\rho} f(s) ds = \lim_{\varepsilon\to 0} \int\limits_\varepsilon^t \frac{d}{ds} D^{n-1}_{\bar\rho} f(s) ds =$$

$$= D^{n-1}_{\bar\rho} f(t) - \lim_{\varepsilon\to 0} D^{n-1}_{\bar\rho} f(\varepsilon) = D^{n-1}_{\bar\rho} f(t) - a_{n-1} < \infty,$$

i.e., the integral $I_{n,n}(t,f) = \int\limits_0^t \rho_n^{-1}(s) D^n_{\bar\rho} f(s) ds$ converges and the relation (3.7) holds for $i = n-1$. Let these conditions hold for $i = n-1, n-2, ..., k$, $k > 0$. Now we need to show that the integral (3.6) converges for $i = k-1$ and the relation (3.7) holds for $i = k-1$. The relation (3.7) for $i = k$ we rewrite in the form

$$D^k_{\bar\rho}[f(t) - \sum_{j=1}^{n-1} a_j K_{j,1}(1,t)] = I_{k+1,n}(t,f).$$

Multiplying both sides of this equality by $\rho_k^{-1}(t)$, we find integrals of both sides from $\varepsilon > 0$ to $t \in I$, where $0 < \varepsilon < t < 1$:

$$\int\limits_\varepsilon^t \rho_k^{-1}(s) D^k_{\bar\rho} \left[ f(s) - \sum_{j=1}^{n-1} a_j K_{j,1}(1,s) \right] ds = \int\limits_\varepsilon^t \rho_k^{-1}(x) I_{k+1,n}(x,f) dx,$$

$$\int\limits_\varepsilon^t \frac{d}{ds} D^{k-1}_{\bar\rho} \left[ f(s) - \sum_{j=1}^{n-1} a_j K_{j,1}(1,s) \right] ds = \int\limits_\varepsilon^t \rho_k^{-1}(x) I_{k+1,n}(x,f) dx,$$

$$D^{k-1}_{\bar\rho} \left[ f(t) - \sum_{j=1}^{n-1} a_j K_{j,1}(1,t) \right] - \lim_{\varepsilon\to 0} D^{k-1}_{\bar\rho} \left[ f(t) - \sum_{j=1}^{n-1} a_j K_{j,1}(1,t) \right]$$

$$= \lim_{\varepsilon\to 0} \int\limits_\varepsilon^t \rho_k^{-1}(x) I_{k+1,n}(x,f) dx,$$

$$D_{\bar{\rho}}^{k-1}\left[f(t)-\sum_{j=1}^{n-1} a_j K_{j,1}(1,t)\right]=\lim_{\varepsilon\to 0}\int_{\varepsilon}^{t}\rho_k^{-1}(x) I_{k+1,n}(x,f)dx.$$

The last gives that the integral $I_{k,n}(t,f)$ converges and the relation (3.7) holds for $i=k-1$. □

*Proof of Theorem 3.1.* Let each function $f\in W_{p,\bar{\rho}}^n$ stabilize at zero to the polynomial $P_n(t,f)$. Then by Lemma 3.1 for $f\in W_{p,\bar{\rho}}^n$ the conditions (3.6) and (3.7) hold. For any $F\in L_p$ the function

$$f(t)=\sum_{i=0}^{n-1}\alpha_i K_{i,1}(1,t)+(-1)^n\int_t^1 \rho_n^{-1}(s)K_{n-1,1}(s,t)F(s)ds,\ t\in I,$$

belongs to $W_{p,\bar{\rho}}^n$ and $D_{\bar{\rho}}^n f(t)=F(t)$. Then for $f\in W_{p,\bar{\rho}}^n$ and $D_{\bar{\rho}}^n f(t)=F(t)\ge 0$ the integrals $I_{i+1,n}(t,f)$, $i=0,1,...,n-1$, absolutely converge and hence, they are summable in Lebesgue sense. Therefore, by Fubini's theorem, we can change the order of integration and get

$$I_{i+1,n}(t,f)=\int_0^t K_{n-1,i+1}(t,s)\rho_n^{-1}(s)D_{\bar{\rho}}^n f(s)ds<\infty, \forall t\in I_0, \tag{3.8}$$

for all $f\in W_{p,\bar{\rho}}^n$ and $i=0,1,...,n-1$. If $f$ runs over entire $W_{p,\bar{\rho}}^n$, then $D_{\bar{\rho}}^n f$ runs over entire $L_p$. Hence, from (3.8) it follows (3.4); moreover, the relation (3.5) is a corollary of (3.7) and (3.8).

Inversely, let (3.4) hold. Then we show that for any $f\in W_{p,\bar{\rho}}^n$ there exists a polynomial $P_n(t,f)$ such that (3.2) holds. From (3.4) for $i=n-1$ we get that $\rho_n^{-1}\in L_{p'}$. Then for any $f\in W_{p,\bar{\rho}}^n$ and $t\in I_0$ we have

$$\int_0^t \rho_n^{-1}(s)D_{\bar{\rho}}^n f(s)ds<\infty.$$

Consequently,

$$D_{\bar{\rho}}^{n-1}f(t)-\lim_{\varepsilon\to 0}D_{\bar{\rho}}^{n-1}f(\varepsilon)=\int_0^t \rho_n^{-1}(s)D_{\bar{\rho}}^n f(s)ds.$$

Assuming $\lim_{\varepsilon\to 0}D_{\bar{\rho}}^{n-1}f(\varepsilon)=a_{n-1}$, from the last we have

$$D_{\bar{\rho}}^{n-1}f(t)=a_{n-1}+\int_0^t \rho_n^{-1}(s)D_{\bar{\rho}}^n f(s)ds$$

or

$$D_{\bar{\rho}}^{n-1}[f(t)-a_{n-1}K_{n-1,1}(1,t)]=\int_0^t \rho_n^{-1}(s)D_{\bar{\rho}}^n f(s)ds. \tag{3.9}$$

Then

$$\lim_{t\to 0}D_{\bar{\rho}}^{n-1}[f(t)-a_{n-1}K_{n-1,1}(1,t)]=0.$$

Multiplying both sides of (3.9) by $\rho_{n-1}^{-1}(\cdot)$ and finding integrals from 0 to $t$, we have

$$\int_0^t \frac{d}{d\tau}D_{\bar{\rho}}^{n-2}[f(\tau)-a_{n-1}K_{n-1,1}(1,\tau)]d\tau=\int_0^t \rho_{n-1}^{-1}(x)\int_0^x \rho_n^{-1}(s)D_{\bar{\rho}}^n f(s)ds$$

$$= \int_0^t \rho_n^{-1}(s) K_{n-1,n-1}(t,s) D_{\bar\rho}^n f(s) ds < \infty.$$

Denoting $\lim\limits_{\varepsilon\to 0} D_{\bar\rho}^{n-2}[f(\varepsilon) - a_{n-1}K_{n-1,1}(1,\varepsilon)] = a_{n-2}$, from the last we get

$$D_{\bar\rho}^{n-2}[f(t) - \sum_{i=n-2}^{n-1} a_i K_{i,1}(1,t)] = \int_0^t \rho_{n-1}^{-1}(s) K_{n-1,n-1}(t,s) D_{\bar\rho}^n f(s) ds.$$

Suppose that for $i = n-1, n-2, ..., k$, $k > 0$, we can find numbers $a_i$, $i = n-1, n-2, ..., k$, such that

$$D_{\rho}^i[f(t) - \sum_{j=i}^{n-1} a_j K_{j,1}(1,t)] = \int_0^t \rho_{n-1}^{-1}(s) K_{n-1,i+1}(t,s) D_{\bar\rho}^n f(s) ds. \tag{3.10}$$

Now we need to show that there exists $a_{k-1}$ and (3.10) holds also for $i = k-1$. To show this, in (3.10) we assume $i = k$, multiply both sides by $\rho_k^{-1}(\cdot)$ and integrate both sides from 0 to $t$, $0 < t \le 1$. Then, using (3.4), we get

$$\int_0^t \frac{d}{d\tau} D_\rho^{k-1}[f(\tau) - \sum_{j=k}^{n-1} a_j K_{j,1}(1,\tau)] d\tau = \int_0^t \rho_k^{-1}(x) \int_0^x \rho_n^{-1}(s) K_{n-1,k+1}(t,s) D_{\bar\rho}^n f(s) ds$$

$$= \int_0^t \rho_n^{-1}(s) K_{n-1,k}(t,s) D^n \bar\rho f(s) ds < \infty.$$

Therefore, assuming $\lim\limits_{\varepsilon\to 0} D_{\bar\rho}^{k-1}[f(t) - \sum\limits_{j=k}^{n-1} a_j K_{j,1}(1,t)] = a_{k-1}$, we have

$$D_{\bar\rho}^{k-1}[f(t) - \sum_{j=k}^{n-1} a_j K_{j,1}(1,t)] - a_{k-1} = \int_0^t \rho_n^{-1}(s) K_{n-1,k}(t,s) D_{\bar\rho}^n f(s) ds,$$

i.e., (3.10) holds for $i = k-1$. Then (3.10) holds for all $i = n-1, n-2, ..., 1$. Moreover, for $i = 1$ we have that there exist numbers $a_i = a_i(f)$, $i = 1, 2, ..., n-1$, such that

$$f(t) = \sum_{i=1}^{n-1} a_i K_{i,1}(1,t) + \int_0^t \rho_n^{-1}(s) K_{n-1,1}(t,s) D_{\bar\rho}^n f(s) ds.$$

The last gives that

$$D_{\bar\rho}^i[f(t) - \sum_{j=1}^{n-1} a_j K_{j,1}(1,t)] = \int_0^t \rho_{n-1}^{-1}(s) K_{n-1,i+1}(t,s) D_{\bar\rho}^n f(s) ds < \infty \tag{3.11}$$

and

$$\lim_{t\to 0} D_{\bar\rho}^i[f(t) - \sum_{j=1}^{n-1} a_j K_{j,1}(1,t)] = 0, \;\; i = 0, 1, ..., n-1,$$

i.e., the function $f \in W_{p,\bar\rho}^n$ stabilizes at zero to the polynomial $P_n(t,f) = \sum\limits_{i=1}^{n-1} a_i(f) K_{i,1}(1,t)$. The relation (3.5) follows from (3.11). □

For the integral operator

$$K_i f(t) = \int_0^t \rho_n^{-1}(s)\bar{K}_{n-1,i+1}(t,s)f(s)ds, \;\; 0 \le i \le n-1, \tag{3.12}$$

we consider the inequality

$$\|u_i K_i f\|_{p,I_0} \le C\|f\|_{p,I_0}, \forall f \in L_p(I_0), \tag{3.13}$$

where $u_i$ is a nonnegative function locally summable on $I_0$, where $I_0 = (0,1]$.

**Lemma 3.2.** *Let $1 < p < \infty$. Then there exists a constant $C > 0$ and (3.13) holds if and only if*

$$\sup_{0<x<1} \int_x^1 u_i^p(t) \left( \int_0^x \rho_n^{-p'}(s)\bar{K}_{n-1,i+1}^{p'}(t,s)ds \right)^{p-1} dt < \infty. \tag{3.14}$$

*Proof.* By Lemma 2.1 the kernel $\bar{K}_{n-1,i+1}(t,s)$, $i = 0,1,...,n-1$, has the form (2.2), where $j = n-1$. Now we rewrite this form in the following way

$$\bar{K}_{n-1,i+1}(t,s) = \sum_{j=0}^{n-i-1} \bar{K}_{n-1,i+j+1}(x,s)\bar{K}_{i+j,i+1}(t,x). \tag{3.15}$$

Denote $\bar{K}_{n-1,i+1}(t,s) \equiv \hat{K}_{n-i-1}(t,s)$, $\bar{K}_{i+j,i+1}(t,x) \equiv \hat{K}_j(t,x)$ and $\bar{K}_{n-1,i+j+1}(x,s) = \hat{K}_{n-i-1-j}(x,s)$. Then, assuming $n-i-1 = m$, from (3.15) we have

$$\hat{K}_m(t,s) = \sum_{j=0}^{m} \hat{K}_{m-j}(x,s)\hat{K}_j(t,x). \tag{3.16}$$

In the work [7] the classes $P_n$, $n \ge 0$, of the kernels in the form (3.12) are introduced. Moreover, there it is proved a criterion for the validity of (3.13) in the case when the kernel of the operator in the form (3.12) belongs to $P_n$ for certain $n \ge 0$. From the relation (3.16) it follows that the kernel $\bar{K}_{n-1,i+1}(t,s) = \hat{K}_m(t,s)$ of the integral operator (3.12) belongs to $P_m$, $m = n-i-1 \ge 0$, $i = 0,1,...,n-1$. Then by Theorem 1 of [7] it follows that the inequality (3.13) holds if and only if (3.14) holds. □

We assume that

$$r_{n,i}(t) = \|\rho_n^{-1}(\cdot)\bar{K}_{n-1,i+1}(t,\cdot)\|_{p',(0,t)}^{-1}.$$

**Theorem 3.2.** *Let $1 < p < \infty$ and (3.4) hold. Then for any $f \in W_{p,\bar{\rho}}^n$ there exists a unique polynomial $P_n(t,f)$ such that*

$$(i) \; \sup_{0<t<x} |r_{n,i}(t)D_{\bar{\rho}}^i[f(t) - P_n(t,f)| \le \|D_{\bar{\rho}}^n f\|_{p,(0,x)}, \;\; i = 0,1,...,n-1; \tag{3.17}$$

*(ii) if for $i : 0 \le i \le n-1$ there exists a locally summable function $u_i(\cdot) \ge 0$ such that*

$$\sup_{0<x<1} \int_x^1 u_i^p(t) \left( \int_0^x \rho_n^{-p'}(s)K_{n-1,i+1}^{p'}(x,s)ds \right)^{p-1} dx < \infty, \tag{3.18}$$

*then the estimate*

$$\|u_i D_{\bar{\rho}}^i[f - P_n(f)]\|_p \le C\|D_{\bar{\rho}}^n f\|_p, \;\; \forall f \in W_{p,\bar{\rho}}^n, \tag{3.19}$$

*holds and in particular, if* (3.18) *holds for $u_i = \rho_i^{-1}$, then*

$$\|\frac{d}{dt}D_{\bar{\rho}}^{i-1}[f - P_n(f)]\|_p \le C\|D_{\bar{\rho}}^n f\|_p, \;\; \forall f \in W_{p,\bar{\rho}}^n. \tag{3.20}$$

*Proof of Theorem 3.2.* Let (3.4) hold. Then by Theorem 3.1 for any $f \in W^n_{p,\bar{\rho}}$ there exists a unique polynomial $P_n(t) \equiv P_n(t,f)$ and (3.5) holds. Then for $i : 0 \leq i \leq n-1$ we have

$$\int_0^t \bar{K}_{n-1,i+1}(t,s)\rho_n^{-1}(s)D^n_{\bar{\rho}}f(s)ds = D^i_{\bar{\rho}}f(t) - \sum_{j=i}^{n-1} a_j(f)K_{j,i+1}(1,t) = D^i_{\bar{\rho}}[f(t) - P_n(t,f)]. \tag{3.21}$$

On the basis of Hölder's inequality, the last gives

$$|D^i_{\bar{\rho}}[f(t) - P_n(t,f)]| \leq \left(\int_0^t \rho^{-p'}(s)K^{p'}_{n-1,i+1}(t,s)ds\right)^{\frac{1}{p'}} \left(\int_0^t |D^n_{\bar{\rho}}f(s)|^p ds\right)^{\frac{1}{p}}.$$

Therefore, (3.17) holds.

Let for $i : 0 \leq i \leq n-1$ there exists a non-negative locally summable function $u_i$ and (3.18) holds. Then, multiplying the both sides of (3.21) by $u_i$ and taking $L_p$ norm in the both sides, we have

$$\|u_i D^i_{\bar{\rho}}[f - P_n(f)]\|_p \leq \left(\int_0^1 u_i^p(t)\left|\int_0^t \bar{K}_{n-1,i+1}(t,s)\rho_n^{-1}(s)D^n_{\bar{\rho}}f(s)ds\right|^p dt\right)^{\frac{1}{p}} = \|u_i K_i D^n_{\bar{\rho}}f\|_p, \tag{3.22}$$

where $K_i$ is the operator (3.12).

By Lemma 3.2 on the basis of (3.18) the following estimate

$$\|u_i K_i D^n_{\bar{\rho}}f\|_p \leq C\|D^n_{\bar{\rho}}f\|_p, \ \forall f \in W^n_{p,\bar{\rho}},$$

holds. This estimate and (3.22) give (3.19), and the relation (3.20) follows from $\rho_i^{-1}D^i_{\bar{\rho}}[f - P_n(f)] = \frac{d}{dt}D^{i-1}_{\bar{\rho}}[f - P_n(f)]$. □

**Corollary 3.1.** *Let $1 < p < \infty$ and (3.4) hold. Then for $i = 0, 1, ..., n-1$, we have*

$$\lim_{t \to 0} r_{n,i}(t)D^i_{\bar{\rho}}[f(t) - P_n(t,f)] = 0 \tag{3.23}$$

*and*

$$|D^i_{\bar{\rho}}f(1) - a_i(f)| \leq C\|D^n_{\bar{\rho}}f\|_p. \tag{3.24}$$

*Proof.* The relation (3.23) follows from (3.17). From (3.21) for $t = 1$ we have

$$D^i_{\bar{\rho}}f(1) - a_i(f) = \int_0^1 \rho_n^{-1}(s)\bar{K}_{n-1,i+1}(t,s)D^n_{\bar{\rho}}f(s)ds.$$

This equality and (3.4) give (3.24). □

**Corollary 3.2.** *If (3.4) holds, then the condition $\lim_{t \to 0} D^i_{\bar{\rho}}f(t) = 0, i = 0, 1, ..., n-1$, is equivalent to the condition $a_i(f) = 0$, $i = 0, 1, ..., n-1$. Moreover, for all $i = 0, 1, ..., n-1$, we have*

$$\lim_{t \to 0} r_{n,i}(t)D^i_{\bar{\rho}}f(t) = 0,$$

$$\sup_{0<t<x} |r_{n,i}(t)D^i_{\bar{\rho}}f(t)| \leq C\|D^n_{\bar{\rho}}f\|_{p,(0,x)}, \ 0 < x \leq 1,$$

*and if the condition (3.18) holds, then the estimate*

$$\|u_i D^i_{\bar{\rho}}f\|_p \leq C\|D^n_{\bar{\rho}}f\|_p$$

*is valid.*

*Proof.* The proof of Corollary 3.2 directly follows from Theorems 3.1 and 3.2. □

Let $N_1$ and $N_2$ be subsets of the set $N = \{0, 1, ..., n-1\}$ such that $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = N$. Let $N_1 = \{i_1, i_2, ..., i_k\}$ and $N_2 = \{j_1, j_2, ..., j_m\}$.

**Theorem 3.3.** *Let $1 < p < \infty$ and (3.4) hold. If $N_2 = N$, then the functional*

$$\|f\|^1_{W^n_{p,\bar\rho}} = \|D^n_{\bar\rho} f\|_p + \sum_{j=0}^{n-1} |a_j(f)|, \tag{3.25}$$

*and if $N_1$ and $N_2$ are not empty, then the functional*

$$\|f\|^2_{W^n_{p,\bar\rho}} = \|D^n_{\bar\rho} f\|_p + \sum_{\mu=1}^{k} |D^{i_\mu}_{\bar\rho} f(1)| + \sum_{\lambda=1}^{m} |a_{j_\lambda}(f)|, \tag{3.26}$$

*are equivalent to the norm $\|f\|_{W^n_{p,\bar\rho}}$ of the space $W^n_{p,\bar\rho}$.*

*Proof of Theorem 3.3.* The norm $\|f\|_{W^n_{p,\bar\rho}}$ of the space $W^n_{p,\bar\rho}$ has the form

$$\|f\|_{W^n_{p,\bar\rho}} = \|D^n_{\bar\rho} f\|_p + \sum_{i=0}^{n-1} |D^i_{\bar\rho} f(1)|. \tag{3.27}$$

Due to the condition (3.4) each function $f \in W^n_{p,\bar\rho}$ stabilizes at zero to the polynomial $P_n(t, f)$ with the coefficients $a_i(f)$, $i = 0, 1, ..., n-1$, i.e., for $f \in W^n_{p,\bar\rho}$ there exists a constant $a_i(f)$, $i = 0, 1, ..., n-1$. Therefore, for (3.24) for $i : 0 \le i \le n-1$ we have that

$$|D^i_{\bar\rho} f(1)| \le |D^i_{\bar\rho} f(1) - a_i(f)| + |a_i(f)| \le C_i \|D^n_{\bar\rho} f\|_p + |a_i(f)|, \tag{3.28}$$

$$|a_i(f)| \le |D^i_{\bar\rho} f(1) - a_i(f)| + |D^i_{\bar\rho} f(1)| \le C_i \|D^n_{\bar\rho} f\|_p + |D^i_{\bar\rho} f(1)|. \tag{3.29}$$

Substituting (3.28) into (3.27) and (3.29) into (3.25), we get that there exists a constant $C > 0$ such that the inequality

$$\frac{1}{C} \|f\|^1_{W^n_{p,\bar\rho}} \le \|f\|_{W^n_{p,\bar\rho}} \le C \|f\|^1_{W^n_{p,\bar\rho}}, \quad \forall f \in W^n_{p,\bar\rho},$$

holds, i.e., the norms $\|f\|^1_{W^n_{p,\bar\rho}}$ and $\|f\|_{W^n_{p,\bar\rho}}$ are equivalent.

Similarly, substituting (3.28) into (3.27) for $i = j_1, j_2, ..., j_m$ and (3.29) into (3.25) for $i = i_1, i_2, ..., i_k$, we get that $\|f\|^2_{W^n_{p,\bar\rho}}$ and $\|f\|_{W^n_{p,\bar\rho}}$ are equivalent. □

## 4 Space $W^n_{p,\bar\rho}(I_\infty)$

Let us consider the space $W^n_{p,\bar\rho}(I_\infty)$ of functions $f : I_\infty \to R$, for which the following functional is finite

$$\|f\|_{W^n_{p,\bar\rho}} = \|D^n_{\bar\rho} f\|_p + \sum_{i=0}^{n-1} |D^i_{\bar\rho} f(1)|.$$

By changing of the variables $t = \frac{1}{x}$ the function $f \in W^n_{p,\bar\rho}(I_\infty)$ can be reduced to the function $\tilde f(x) = f(\frac{1}{x}) : (0, 1] \to R$. Since $\frac{df(t)}{dt} = -x^2 \frac{d\tilde f(x)}{dx}$, then assuming $x^2 \rho_i(\frac{1}{x}) \equiv \tilde\rho_i(x)$, $i = 0, 1, ..., n-1$, and $t^{\frac{2}{p'}} \rho_n(\frac{1}{t}) = \tilde\rho_n(t)$, we have $D^i_{\bar\rho} f(t) = (-1)^i D^i_{\bar{\tilde\rho}} \tilde f(x)$, $i = 0, 1, ..., n-1$, and

$$\|D^n_{\bar\rho} f\|_{p,I_\infty} = \|D^n \bar{\tilde\rho} \tilde f\|_{p,I_0}.$$

Therefore, the investigation of the space $W^n_{p,\bar\rho}(I_\infty)$ is equivalent to the investigation of the space $W^n_{p,\bar{\tilde\rho}}(I_0)$. Hence, from the results obtained for the space $W^n_{p,\bar{\tilde\rho}}(I_0)$ we can easily obtain the corresponding results for the space $W^n_{p,\bar\rho}(I_\infty)$.

**Definition 2.** We say that the function $f \in W^n_{p,\bar\rho}(I_\infty)$ stabilizes at infinity to the polynomial

$$\bar P_n(t,f)=\sum_{i=0}^{n-1}\bar a_i\bar K_{i,1}(t,1)$$

if for all $j=0,1,...,n-1$ we have that $\lim\limits_{t\to\infty} D^j_{\bar\rho}[f(t)-\bar P_n(t,f)]=0$. Moreover, the coefficients $\bar a_i=\bar a_i(f)$, $i=0,1,...,n-1$, of the polynomial $\bar P_n(t,f)$ can be defined from the following relations

$$\bar a_j(f)=\lim_{t\to\infty}(D^j_{\bar\rho}f(t)-\sum_{i=j}^{n-1}\bar a_j(f)\bar K_{i,j+1}(t,1)),\ j=n-1,n-2,...,0.$$

**Theorem 4.1.** *Let $1<p<\infty$. Then each function $f\in W^n_{p,\bar\rho}(I_\infty)$ stabilizes at infinity to the polynomial $\bar P_n(t,f)$ if and only if for all $t\in I_\infty$ and $i=0,1,...,n-1$, the following integrals*

$$\int_t^\infty \rho_n^{-p'}(s)K^{p'}_{n-1,i+1}(s,t)ds \tag{4.1}$$

*converge. Moreover, the following relations*

$$D^i_{\bar\rho}f(t)=\sum_{j=i}^{n-1}\bar a_j(f)\bar K_{j,i+1}(t,1)+\int_t^\infty K_{n-1,i+1}(s,t)\rho_n^{-1}(s)D^n_{\bar\rho}f(s)ds$$

*hold for $i=0,1,...,n-1$.*

Let

$$\bar r_{n,i}(t)=\|\rho_n^{-1}(\cdot)K_{n-1,i+1}(\cdot,t)\|^{-1}_{p',(t,\infty)}.$$

**Theorem 4.2.** *Let $1<p<\infty$ and (4.1) hold. Then for any $f\in W^n_{p,\bar\rho}(I_\infty)$ there exists a unique polynomial $\bar P_n(t,f)$ such that*

$$(i)\ \sup_{x<t<\infty}|\bar r_{n,i}(t)D^i_{\bar\rho}[f(t)-\bar P_n(t,f)]|\le\|D^n_{\bar\rho}f\|_{p,(x,\infty)},\ \ i=0,1,...,n-1;$$

*(ii) if for $i: 0\le i\le n-1$ there exists a locally summable function $\bar u_i(\cdot)\ge0$ such that*

$$\sup_{x>1}\int_1^x\bar u_i^p(t)\left(\int_x^\infty\rho^{-p'}(s)K^{p'}_{n-1,i+1}(s,t)ds\right)^{p-1}dt<\infty, \tag{4.2}$$

*then the estimate*

$$\|\bar u_iD^i_{\bar\rho}[f-\bar P_n(f)]\|_p\le C\|D^n_{\bar\rho}f\|_p,\ \ \forall f\in W^n_{p,\bar\rho},$$

*holds and in particular, if (4.2) holds for $\bar u_i=\rho_i^{-1}$, then*

$$\|\frac{d}{dt}D^{i-1}_{\bar\rho}[f-\bar P_n(f)]\|_p\le C\|D^n_{\bar\rho}f\|_p,\ \ \forall f\in W^n_{p,\bar\rho}.$$

**Corollary 4.1.** *Let $1<p<\infty$ and (4.1) hold. Then for $i=0,1,...,n-1$, we have*

$$\lim_{t\to\infty}\bar r_{n,i}(t)D^i_{\bar\rho}[f(t)-\bar P_n(t,f)]=0$$

*and*

$$|D^i_{\bar\rho}f(1)-\bar a_i(f)|\le C\|D^n_{\bar\rho}f\|_p.$$

**Corollary 4.2.** *If* (4.1) *holds, then the condition* $\lim_{t\to\infty} D^i_{\bar\rho} f(t) = 0$, $i = 0, 1, ..., n-1$, *is equivalent to the condition* $\bar a_i(f) = 0$, $i = 0, 1, ..., n-1$. *Moreover, for all* $i = 0, 1, ..., n-1$, *we have*

$$\lim_{t\to\infty} \bar r_{n,i}(t) D^i_{\bar\rho} f(t) = 0,$$

$$\sup_{x<t} |\bar r_{n,i} D^i_{\bar\rho} f(t)| \le C \|D^i_{\bar\rho} f\|_{p,(x,\infty)}, \quad x \ge 1,$$

*and if the condition* (4.2) *holds, then the estimate*

$$\|\bar u_i D^i_{\bar\rho} f\|_p \le C \|D^n_{\bar\rho} f\|_p$$

*is valid.*

**Theorem 4.3.** *Let* $1 < p < \infty$ *and* (4.1) *hold. If* $N_2 = N$, *then the functional*

$$\|f\|^1_{W^n_{p,\bar\rho}} = \|D^n_{\bar\rho} f\|_p + \sum_{j=0}^{n-1} |\bar a_j(f)|,$$

*and if* $N_1$ *and* $N_2$ *are not empty, then the functional*

$$\|f\|^2_{W^n_{p,\bar\rho}} = \|D^n_{\bar\rho} f\|_p + \sum_{\mu=1}^{k} |D^{i_\mu}_{\bar\rho} f(1)| + \sum_{\lambda=1}^{m} |\bar a_{j_\lambda}(f)|,$$

*are equivalent to the norm* $\|f\|_{W^n_{p,\bar\rho}}$ *of the space* $W^n_{p,\bar\rho}$.

Using (2.8), on the basis of Theorems 3.3 and 4.3 we can write Theorem 4.4 for the entire interval $I = (0, \infty)$.

**Theorem 4.4.** *Let* $1 < p < \infty$. *Let* (3.4) *and* (4.1) *hold. If* $N_2 = N$, *then the functionals*

$$\|f\|^1_{W^n_{p,\bar\rho}(I)} = \|D^n_{\bar\rho} f\|_p + \sum_{j=0}^{n-1} |a_j(f)|,$$

$$\|f\|^2_{W^n_{p,\bar\rho}(I)} = \|D^n_{\bar\rho} f\|_p + \sum_{j=0}^{n-1} |\bar a_j(f)|,$$

*and if* $N_1$ *and* $N_2$ *are not empty, then the functional*

$$\|f\|^3_{W^n_{p,\bar\rho}(I)} = \|D^n_{\bar\rho} f\|_p + \sum_{\mu=1}^{k} |a_{j_\mu}(f)| + \sum_{\lambda=1}^{m} |\bar a_{j_\lambda}(f)|,$$

*are equivalent to the norm* $\|f\|_{W^n_{p,\bar\rho}(I)}$ *of the space* $W^n_{p,\bar\rho}(I)$.

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