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Aims and Scope

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The language of the paper must be English only.

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1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

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- compliance of the title of the paper to its content;
- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);
- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);
- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

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At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.

ANDREI ANDREEVICH SHKALIKOV

(to the 70th birthday)

Andrei Andreevich Shkalikov, corresponding member of the Russian Academy of Sciences, an outstanding mathematician with a wide range of interests, a remarkable person, professor of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University was born on November 19, 1949.

Andrei Andreevich is a leading specialist in the theory of operators and their applications, especially to problems of mechanics and mathematical physics. He is well known for his work in the theory of functions and in the theory of spaces with indefinite metrics. He is also a specialist in the theory of entire and meromorphic functions and its applications to operator theory.

Andrei Andreevich is known for solving a number of difficult problems that for many years remained unsolved. His work on the basic properties of systems of root functions of differential operators is well known worldwide. He gave a justification for Mandelstam's hypothesis about the existence of solutions satisfying the radiation principle. He solved, in general form, the Rayleigh problem on the reflection of waves from a periodic surface, obtained a solution to the Sobolev problem on the stability of the motion of a top with a cavity filled with liquid. His contribution to the construction of an abstract theory of the Orr-Sommerfeld problem is invaluable. He obtained a description of the limiting spectral portraits for a large class of functions describing the profiles of fluid flows. He is one of the founders of the modern theory of differential operators, coefficients of which are distributions, and inverse problems for such operators.

Andrei Andreevich has been a plenary speaker at many international conferences. He conducts fruitful scientific work and collaborates with many international mathematical research centers.

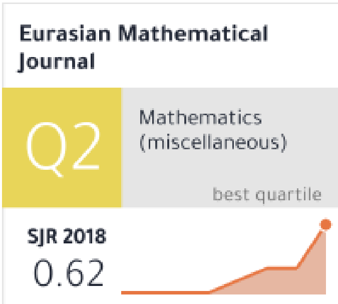
Andrei Andreevich is an author of more than 130 scientific publications. Among his pupils there are more than 20 Candidates of Sciences and 6 Doctors of Sciences. The results obtained by A.A. Shkalikov, his pupils, collaborators and followers gained worldwide recognition.

Professor Shkalikov is also an outstanding organizer. Under his supervision, many international conferences were held. In particular, conferences dedicated to the memory of I.G. Petrovsky, I.M. Gelfand, S.M. Nikol'skii, B.M. Levitan, anniversary conferences of V.A. Sadovnichy, and others.

Andrei Andreyevich is a deputy editor-in-chief of the journals Mathematical Notes, Moscow University Mathematics Bulletin, Moscow University Mechanics Bulletin, and a member of the editorial boards of the Russian Mathematical Surveys, Proceedings of the Moscow Mathematical Society and other journals, including the Eurasian Mathematical Journal.

The Editorial Board of the Eurasian Mathematical Journal cordially congratulates Andrei Andreevich on the occasion of his 70th birthday and wishes him good health, and new achievements in mathematics and mathematical education.

GOOD NEWS: EMJ IS NOW AN SJR Q2 JOURNAL



Recently the lists were published of all mathematical journals included in 2018 SCImago Journal Rank (SJR) quartiles Q1 (385 journals), Q2 (430 journals), Q3 (445 journals), and Q4 (741 journals), and Scopus CiteScore quartiles Q1 (443 journals), Q2 (375 journals), Q3 (348 journals), and Q4 (283 journals).

With great pleasure we inform our readers and authors that the Eurasian Mathematical Journal was included in the most popular scientific ranking database SJR in quartile Q2, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia. The SJR data for the Eurasian Mathematical Journal (2018) is as follows:

550. Eurasian Mathematical Journal (Kazakhstan), Q2, SJR=0.624.
(550 is the number in the list of all Q1 - Q4 journals.)

The SJR indicator is calculated by using the data of the Scopus Database of the Elsevier, the modern publishing business founded in 1880. It uses a sophisticated formula, taking into account various characteristics of journals and journals publications. This formula and related comments can be viewed on the web-page

http : //www.scimagojr.com/journalrank.php.

Some other SJR Q2 mathematical journals:
490. Studia Mathematica (Poland), SJR=0.706,
492. Comptes Rendus Mathematique (France), SJR=0.704,
522. Journal of Mathematical Physics (USA), SJR=0.667,
540. Doklady Mathematics (Russia), SJR=0.636,
570. Journal of Mathematical Sciences (Japan), SJR=0.602,
662. Journal of Applied Probability (UK), SJR=0.523,
733. Mathematical Notes (Russia), SJR=0.465,
791. Canadian Mathematical Bulletin (Canada), SJR=0.433.

Our journal ranks:

7726th place in the list of 31971 scientific journals, representing all subjects and all regions, included in this database (in the first 25% of journals of this category),

225th place in the list of 2519 scientific journals, representing all subjects, of the Asiatic region, included in this database (in the first 10% of journals of this category),

550th place in the list of 2011 mathematical journals, representing all regions, included in this database (in the first 30% of journals of this category),

19th place in the list of 165 mathematical journals of the Asiatic region, included in this database (in the first 15% of journals of this category).

On a separate page the SJR statistics for the Eurasian Mathematical Journal is attached.

Recall that the Eurasian Mathematical Journal started its work in 2010 (see [1]-[3]) and was first included in SJR indicator in 2014 (Q4, SJR=0.101, see [4], [5], [6]). So, the ambitious plan set in [6] was implemented and even essentially exceeded.

As for the Scopus CiteScore indicator, it uses another sophisticated formula, differently taking into account various characteristics journals publications. This formula and related comments can be viewed on the web-page

https : //ru.service.elsevier.com/app/answers/detail/a_id/19266/supporthub/scopus.

In this indicator the Eurasian Mathematical Journal was included in quartile Q3. The CiteScore data for the Eurasian Mathematical Journal (2018) is as follows:

333. Eurasian Mathematical Journal (Kazakhstan), Q3, CiteScore = 0,41
(333 is the number in the list of only Q3 journals.)

Some other Scopus CiteScore Q3 mathematical journals:

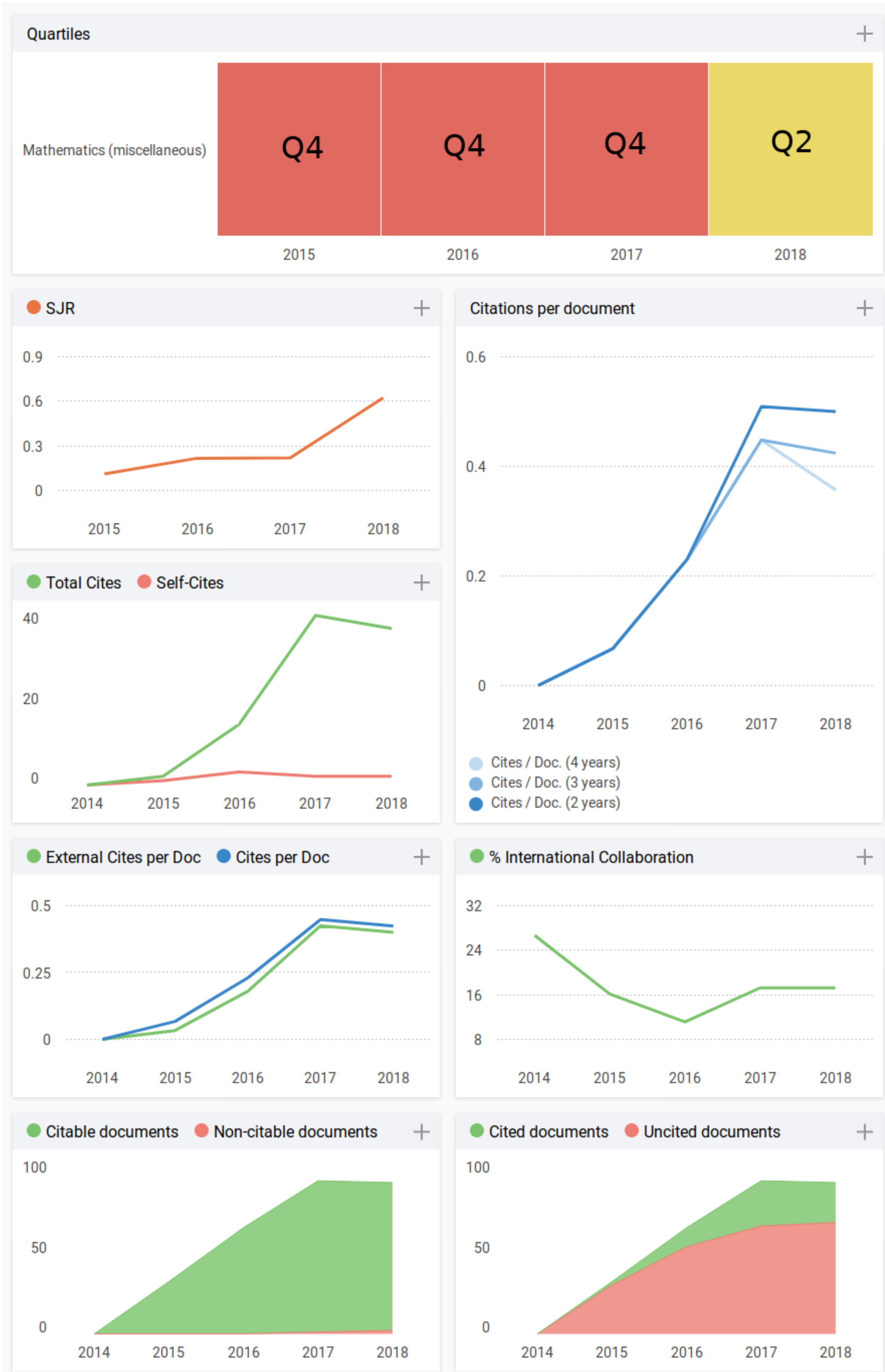
320. Czechoslovak Mathematical Journal (Czech Republic), CiteScore = 0.44,
321. Italian Journal of Pure and Applied Mathematics (Italy), CiteScore = 0.44,
323. Studia Scientiarum Mathematicarum Hungarica (Hungary), CiteScore = 0.44,
332. Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie (Romania), CiteScore = 0.41,
334. Indian Journal of Pure and Applied Mathematics (India), CiteScore = 0.41,
33. Transactions of the Moscow Mathematical Society (Russia), CiteScore = 0.41,
337. Illinois Journal of Mathematics (USA), CiteScore = 0.40,
339. Publications de l'Institut Mathematique (France), CiteScore = 0.40.

Our main current aim is to preserve the status of an SJR Q2 journal and of a Scopus CiteScore Q3 journal.

We hope that all respected members of the international Editorial Board, reviewers, current authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

V.I. Burenkov, K.N. Ospanov, T.V. Tararykova, A.M. Temirkhanova.

Extract from http://www.scimagojr.com/journalrank.php
EURASIAN MATHEMATICAL JOURNAL



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NEW GENERAL SOLUTION TO A NONLINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATION

D.S. Dzhumabaev, S.T. Mynbayeva

Communicated by K.N. Ospanov

Key words: nonlinear Fredholm integro-differential equation, special Cauchy problem, Δ_N general solution, boundary value problem.

AMS Mathematics Subject Classification: 34B15, 34G20, 45J05, 47G20.

Abstract. Partition Δ_N of the interval $[0, T]$ into N parts and introduction of additional parameters and new unknown functions on subintervals reduce a nonlinear Fredholm integro-differential equation to the special Cauchy problems for a system of nonlinear integro-differential equations with parameters. Conditions for the existence of a unique solution to the latter problem are obtained. Employing this solution we construct a Δ_N general solution to the nonlinear Fredholm integro-differential equation. Properties of the Δ_N general solution and its application to a nonlinear boundary value problem for the considered equation are discussed.

DOI: https://doi.org/10.32523/2077-9879-2019-10-4-24-33

1 Introduction

We consider the Fredholm integro-differential equation (FIDE) with nonlinear differential part

$$\frac{dx}{dt} = f(t, x) + \sum_{k=1}^{m} \varphi_k(t) \int_0^T \psi_k(\tau)x(\tau)d\tau, \quad t \in [0, T], \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous; the $n \times n$ matrices $\varphi_k(t)$, $\psi_k(\tau)$, $k = \overline{1, m}$, are continuous on $[0, T]$, $\|x\| = \max_{i=\overline{1,n}} |x_i|$.

Denote by $\mathbb{C}\big([0, T], \mathbb{R}^n\big)$ the space of all continuous functions $x : [0, T] \to \mathbb{R}^n$ with the norm $\|x\|_1 = \max_{t\in[0,T]} \|x(t)\|$. By a solution to equation (1.1) we mean a continuously differentiable on $[0, T]$ function $x(t)$ satisfying equation (1.1). Here and below in the article, we assume that the observed functions at the end-points of the intervals have one-sided derivatives.

General solution plays an important role in investigating and solving problems for differential and integro-differential equations (IDEs). Integral term essentially impacts on the properties of equation (1.1). Particularly, when $f(t, x) = Ax + f_0(t)$, equation (1.1) might be unsolvable and have no solutions without any additional conditions (see [1, 3]). Consequently, a classical general solution exists not for all FIDEs. For this reason, a new concept of general solution has been introduced in [4]. Employing a regular partition Δ_N of the interval $[0, T]$ (see [2, 3]), in paper [4] a Δ_N general solution $x(\Delta_N, t, \lambda)$ is constructed to the linear FIDE. In contrast to the classical general solution, $x(\Delta_N, t, \lambda)$ exists for any linear FIDE and depends on a parameter $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^{nN}$. In [5], the Δ_N general solution to the linear loaded differential equation has been constructed. The new concept of a general solution is extended to a nonlinear ordinary differential equation in [6].

In the present paper, we introduce a new general solution to equation (1.1), establish its properties, and apply this solution to the study of nonlinear boundary value problem (BVP) for equation (1.1).

The paper is organized as follows.

In Section 2, sufficient conditions for the existence of a unique solution to the special Cauchy problem for a system of nonlinear IDEs are obtained. Employing the solution to the special Cauchy problem, we introduce a Δ_N general solution to equation (1.1) and establish its properties.

Section 3 deals with a nonlinear two-point BVP for equation (1.1). Substitution of $x(\Delta_N, x, \lambda)$, the Δ_N general solution to equation (1.1), into the boundary condition and continuity conditions gives a system of nonlinear algebraic equations in $\lambda \in \mathbb{R}^{nN}$. It is proved that the solvability of this system is equivalent to the existence of a solution to the nonlinear BVP.

2 The Δ_N general solution to equation (1.1) and its properties

Let Δ_N be a partition of the interval $[0,T)$ into N parts: $[0,T) = \bigcup_{r=1}^{N}[t_{r-1}, t_r)$ and $\overline{h} = \max_{r=\overline{1,N}}(t_r - t_{r-1})$. Given a vector $\lambda^{(0)} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_N^{(0)}\right) \in \mathbb{R}^{nN}$ and a number $\rho > 0$, we compose the set

$$G^0(\rho) = \big\{(t,x) : t \in [0,T], \big\|x - x_0(t)\big\| < \rho\big\},$$

where a piecewise constant vector–function $x_0(t)$ on $[0,T]$ is defined by the equalities $x_0(t) = \lambda_r^{(0)}$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, and $x_0(T) = \lambda_N^{(0)}$.

Condition A. *Let the following inequalities be fulfilled:*

(1) $\|f(t,x)\| \le M_0$, $(t,x) \in G^0(\rho)$, M_0 *is a constant;*

(2) $M_1\overline{h} = \big[M_0 + K_0\big(\rho + \|\lambda^{(0)}\|\big)\big]\overline{h} < \rho$,

where

$$K_0 = \sum_{k=1}^{m} \max_{t\in[0,T]} \|\varphi_k(t)\| \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \|\psi_k(\tau)\| d\tau.$$

We choose the numbers $\rho_\lambda = \rho - M_1\overline{h}$, $\rho_v = M_1\overline{h}$ and construct the following sets:

$S\big(\lambda^{(0)}, \rho_\lambda\big) = \big\{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^{nN} : \big\|\lambda_r - \lambda_r^{(0)}\big\| < \rho_\lambda, \quad r = \overline{1,N}\big\}$,

$S\big(0, \rho_v\big) = \big\{v[t] \in \widetilde{\mathbb{C}}\big([0,T], \Delta_N, \mathbb{R}^{nN}\big) : \|v[\cdot]\|_3 < \rho_v\big\}$,

$G_p^0(\rho) = \Big\{(t,x) : t \in [t_{p-1}, t_p), \|x - x_0(t)\| < \rho - M_1(t_p - t)\Big\}, \quad p = \overline{1, N-1}$,

$G_N^0(\rho) = \Big\{(t,x) : t \in [t_{N-1}, t_N], \|x - x_0(t)\| < \rho - M_1(t_N - t)\Big\}$, and

$G^0(\Delta_N, \rho) = \bigcup_{r=1}^{N} G_r^0(\rho)$.

If a function $x(t)$ satisfies equation (1.1) and $(t, x(t)) \in G^0(\Delta_N, \rho)$, then the functions $x_r(t)$, $r = \overline{1,N}$, being the restrictions of $x(t)$ to $[t_{r-1}, t_r)$, satisfy the nonlinear IDEs

$$\frac{dx_r}{dt} = f(t, x_r) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau) x_j(\tau) d\tau, \; t \in [t_{r-1}, t_r), \tag{2.1}$$

and $(t, x_r(t)) \in G_r^0(\rho)$, $r = \overline{1,N}$. Introducing the parameters $\lambda_r \hat{=} x_r(t_{r-1})$ and making the substitutions $u_r(t) = x_r(t) - \lambda_r$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, we obtain the system of nonlinear IDEs with parameters on subintervals

$$\frac{du_r}{dt} = f(t, u_r + \lambda_r) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)[u_j(\tau) + \lambda_j]d\tau, \quad t \in [t_{r-1}, t_r), \tag{2.2}$$

subject to the initial conditions

$$u_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \tag{2.3}$$

Problem (2.2), (2.3) is said to be the special Cauchy problem for the system of nonlinear IDEs with parameters on subintervals. A criterion of the unique solvability of the special Cauchy problem for the system of linear IDEs and algorithms of finding its solution are proposed in [2, 3].

Let $\mathbb{C}\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$ denote the space of all function systems $u[t] = \big(u_1(t), u_2(t), \dots, u_N(t)\big)$, where $u_r : [t_{r-1}, t_r) \to \mathbb{R}^n$ is continuous and has the finite left-sided limit $\lim\limits_{t \to t_r - 0} u_r(t)$ for any $r = \overline{1, N}$, with the norm

$$\big\|u[\cdot]\big\|_2 = \max_{r=\overline{1,N}} \sup_{t \in [t_{r-1}, t_r)} \|u_r(t)\|.$$

Solution to the special Cauchy problem (2.2), (2.3) at the fixed $\lambda = \lambda^* = \big(\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*\big) \in \mathbb{R}^{nN}$ is a function system

$$u[t, \lambda^*] = \big(u_1(t, \lambda^*), u_2(t, \lambda^*), \dots, u_N(t, \lambda^*)\big) \in \mathbb{C}\big([0,T], \Delta_N, \mathbb{R}^{nN}\big).$$

Its components $u_r(t, \lambda^*)$, $r = \overline{1, N}$, are continuously differentiable with respect to t on their domains and satisfy the system of IDEs (2.2) for $\lambda = \lambda^*$ and initial conditions (2.3).

System of IDEs (2.1) is equivalent to the special Cauchy problem with parameters on subintervals (2.2), (2.3) in the following manner. If the function system $\widetilde{x}[t] = (\widetilde{x}_1(t), \widetilde{x}_2(t), \dots, \widetilde{x}_N(t))$ is a solution to equations (2.1), then the function system $u[t, \widetilde{\lambda}] = (u_1(t, \widetilde{\lambda}), u_2(t, \widetilde{\lambda}), \dots, u_N(t, \widetilde{\lambda}))$, where $\widetilde{\lambda}_r = \widetilde{x}_r(t_{r-1})$, $u_r(t, \widetilde{\lambda}) = \widetilde{x}_r(t) - \widetilde{\lambda}_r$, $r = \overline{1, N}$, is a solution to the special Cauchy problem with parameters on subintervals (2.2), (2.3) with $\lambda = \widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_N) \in \mathbb{R}^{nN}$. And, vice versa, if the function system $u[t, \lambda^*] = (u_1(t, \lambda^*), u_2(t, \lambda^*), \dots, u_N(t, \lambda^*))$ is a solution to the special Cauchy problem with parameters on subintervals (2.2), (2.3) with $\lambda = \lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*)$, then the function system $x^*[t] = (x_1^*(t), x_2^*(t), \dots, x_N^*(t))$ with the elements $x_r^*(t) = \lambda_r^* + u_r(t, \lambda^*)$, $r = \overline{1, N}$, is a solution to equations (2.1).

Further, while constructing a new general solution to equation (1.1) and solving a BVP for this equation, we need to find the values of $\lim\limits_{t \to t_r - 0} u_r(t)$, $r = \overline{1, N}$. Therefore, it is reasonable to consider the following special Cauchy problem on the closed subintervals:

$$\frac{dv_r}{dt} = f(t, v_r + \lambda_r) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)[v_j(\tau) + \lambda_j]d\tau, \quad t \in [t_{r-1}, t_r], \tag{2.4}$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \tag{2.5}$$

Denote by $\widetilde{\mathbb{C}}\big([0,T], \Delta_N, \mathbb{R}^{nN}\big)$ the space of all function systems $v[t] = \big(v_1(t), v_2(t), \dots, v_N(t)\big)$, where $v_r : [t_{r-1}, t_r] \to \mathbb{R}^n$ is continuous for all $r = \overline{1, N}$, with the norm $\big\|v[\cdot]\big\|_3 = \max\limits_{r=\overline{1,N}} \max\limits_{t \in [t_{r-1}, t_r]} \|v_r(t)\|$.

By $\mathbb{C}\big([t_{r-1}, t_r], \mathbb{R}^n\big)$ we denote the space of all continuous functions $v : [t_{r-1}, t_r] \to \mathbb{R}^n$, with the norm $\|v\|_4 = \max\limits_{t \in [t_{r-1}, t_r]} \|v(t)\|$, $r = \overline{1, N}$.

It is obvious that if the function systems $u[t, \lambda] = \big(u_1(t, \lambda), u_2(t, \lambda), \dots, u_N(t, \lambda)\big)$ and $v[t, \lambda] = \big(v_1(t, \lambda), v_2(t, \lambda), \dots, v_N(t, \lambda)\big)$ are solutions to problems (2.2), (2.3) and (2.4), (2.5), respectively, then

$$u_r(t, \lambda) = v_r(t, \lambda), \quad t \in [t_{r-1}, t_r),$$

$$\lim_{t\to t_r-0} u_r(t,\lambda) = v_r(t_r,\lambda), \quad r=\overline{1,N}.$$

For a fixed parameter $\widehat{\lambda} \in S\big(\lambda^{(0)},\rho_\lambda\big)$, we get

$$\frac{dv_r}{dt} = f(t, v_r + \widehat{\lambda}_r) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)\big[v_j(\tau)+\widehat{\lambda}_j\big]d\tau,\ t\in[t_{r-1},t_r], \tag{2.6}$$

$$v_r(t_{r-1}) = 0, \quad r=\overline{1,N}. \tag{2.7}$$

Let us introduce the following notation:

$G(\Delta_N) = \big(G_{p,k}(\Delta_N)\big)$ is the $nm\times nm$ matrix consisting of the $n\times n$ matrices

$$G_{p,k}(\Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \psi_p(\tau) \int_{t_{r-1}}^{\tau} \varphi_k(s)ds d\tau, \quad p,k=\overline{1,m}.$$

If the matrix $[I-G(\Delta_N)]$ is invertible, then we can represent the inverse matrix in the form $[I-G(\Delta_N)]^{-1} = \big(R_{k,p}(\Delta_N)\big)$, $k,p=\overline{1,m}$, where I is the identity matrix of dimension nm, $R_{k,p}(\Delta_N)$ are the square matrices of dimension n.

Theorem 2.1. *Let Condition A be fulfilled, the matrix $I-G(\Delta_N)$ be invertible and the following inequalities be valid:*

(i) $\big\|f(t,x')-f(t,x'')\big\| \le L_0\|x'-x''\|$, L_0 *is a constant,* $(t,x'),(t,x'')\in G^0(\rho)$;
(ii) $\Big(L_0+K_0\Big)\overline{h} < 1$;
(iii) $\chi\cdot\Big(M_0+K_0\cdot\big(\rho_\lambda+\|\lambda^{(0)}\|\big)\Big)\overline{h} < \rho_v$, *where*

$$\chi = 1+\overline{h}\sum_{k=1}^{m}\max_{t\in[0,T]}\Big\|\varphi_k(t)\Big\|\sum_{p=1}^{m}\Big\|R_{k,p}(\Delta_N)\Big\|\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\|\psi_p(s)\|ds.$$

Then, for any $\widehat{\lambda}\in S(\lambda^{(0)},\rho_\lambda)$, there exists a unique function system $v[t,\widehat{\lambda}] = (v_1(t,\widehat{\lambda}), v_2(t,\widehat{\lambda}),\dots,v_N(t,\widehat{\lambda}))$, the solution to the special Cauchy problem (2.6), (2.7) *in* $S(0,\rho_v)$.

Proof. We choose $v^{(0)}[t] = (0,0,\dots,0)$ and compose a sequence of function systems $v^{(\nu)}\big[t,\widehat{\lambda}\big] = \big(v_1^{(\nu)}(t,\widehat{\lambda}), v_2^{(\nu)}(t,\widehat{\lambda}),\dots,v_N^{(\nu)}(t,\widehat{\lambda})\big)$, $\nu=1,2,\dots,$ by solving the special Cauchy problems for the system of linear IDEs

$$\frac{dv_r}{dt} = F_r(t,\nu,\widehat{\lambda}) + \sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)v_j(\tau)d\tau, t\in[t_{r-1},t_r], \tag{2.8}$$

$$v_r(t_{r-1}) = 0, \quad r=\overline{1,N}, \tag{2.9}$$

where $F_r(t,\nu,\widehat{\lambda}) = f(t, v_r^{(\nu-1)}(t)+\widehat{\lambda}_r) + \sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)d\tau\widehat{\lambda}_j.$

Since the matrix $I-G(\Delta_N)$ is invertible, in accordance with [3, pp. 345-346], the linear special Cauchy problem (2.8), (2.9) has a unique solution $v^{(\nu)}[t,\widehat{\lambda}]$ with the elements

$$v_r^{(\nu)}(t,\widehat{\lambda}) = \int_{t_{r-1}}^{t}\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}R_{k,p}(\Delta_N)g_p(\Delta_N,F,\nu)d\tau +$$

$$+\int_{t_{r-1}}^{t}F_r(\tau,\nu,\widehat{\lambda})d\tau, \quad t\in[t_{r-1},t_r], \quad r=\overline{1,N}, \quad \nu=1,2,\dots, \tag{2.10}$$

where $g_p(\Delta_N,F,\nu)=\sum\limits_{r=1}^{N}\int\limits_{t_{r-1}}^{t_r}\psi_p(\tau)\int\limits_{t_{r-1}}^{\tau}F_r(s,\nu,\widehat{\lambda})dsd\tau$, $p=\overline{1,m}$, are the n vectors.

It is easily seen that the functions $v_r^{(\nu)}(t,\widehat{\lambda})$ belong to $\mathbb{C}\big([t_{r-1},t_r],\mathbb{R}^n\big)$, $r=\overline{1,N}$, $\nu=1,2,\ldots$. Denote by V_r the set of functions $v_r^{(\nu)}(t,\widehat{\lambda})$. Since

$$\Big\|v_r^{(\nu)}(t,\widehat{\lambda})\Big\|=\bigg\|\int\limits_{t_{r-1}}^{t}\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}R_{k,p}(\Delta_N)g_p(\Delta_N,F,\nu)d\tau+\int\limits_{t_{r-1}}^{t}F_r(\tau,\nu,\widehat{\lambda})d\tau\bigg\|\le$$
$$\le\bigg[1+\overline{h}\sum_{k=1}^{m}\max_{t\in[t_{r-1},t_r]}\Big\|\varphi_k(t)\Big\|\sum_{p=1}^{m}\Big\|R_{k,p}(\Delta_N)\Big\|\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\|\psi_p(s)\|ds\bigg]\times$$
$$\times\Big[M_0+K_0\cdot\big(\rho_\lambda+\big\|\lambda^{(0)}\big\|\big)\Big]\overline{h},\quad(2.11)$$

the set V_r is uniformly bounded on $[t_{r-1},t_r]$, $r=\overline{1,N}$.

Now, by virtue of (2.10) and (2.11), for the points $t_r',t_r''\in[t_{r-1},t_r]$, $r=\overline{1,N}$, we get the inequality

$$\Big\|v_r^{(\nu)}(t_r'',\widehat{\lambda})-v_r^{(\nu)}(t_r',\widehat{\lambda})\Big\|=$$
$$=\bigg\|\int\limits_{t_r'}^{t_r''}\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}R_{k,p}(\Delta_N)g_p(\Delta_N,F,\nu)d\tau+\int\limits_{t_r'}^{t_r''}F_r(\tau,\nu,\widehat{\lambda})d\tau\bigg\|\le$$
$$\le\bigg[1+\overline{h}\sum_{k=1}^{m}\max_{t\in[t_{r-1},t_r]}\Big\|\varphi_k(t)\Big\|\sum_{p=1}^{m}\Big\|R_{k,p}(\Delta_N)\Big\|\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\|\psi_p(s)\|ds\bigg]\times$$
$$\times\Big[M_0+K_0\cdot\big(\rho_\lambda+\big\|\lambda^{(0)}\big\|\big)\Big]|t_r''-t_r'|.$$

Therefore, the functions $v_r^{(\nu)}(t,\widehat{\lambda})$ are equicontinuous, and, by Arzela's theorem [7, p. 207], each set V_r, $r=\overline{1,N}$, is compact. Then we can select a subsequence $v_r^{(\nu_l)}(t,\widehat{\lambda})$, which uniformly converges to $v_r(t,\widehat{\lambda})$ as $l\to\infty$ on $[t_{r-1},t_r]$ for all $r=\overline{1,N}$. Let us compose the function system $v[t,\widehat{\lambda}]=(v_1(t,\widehat{\lambda}),v_2(t,\widehat{\lambda}),\ldots,v_N(t,\widehat{\lambda}))$. Since for the functions $v_r^{(\nu_l)}(t,\widehat{\lambda})$, the equalities

$$v_r^{(\nu_l)}(t,\widehat{\lambda})=\int\limits_{t_{r-1}}^{t}\Big[f(\tau,v_r^{(\nu_l-1)}(\tau,\widehat{\lambda})+\widehat{\lambda}_r)+\sum_{k=1}^{m}\varphi_k(\tau)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\psi_k(s)ds\Big]d\tau+$$
$$+\int\limits_{t_{r-1}}^{t}\sum_{k=1}^{m}\varphi_k(\tau)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\psi_k(s)v_j^{(\nu_l)}(s,\widehat{\lambda})dsd\tau,\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N}\quad(2.12)$$

are true, passing in (2.12) to the limit as $l\to\infty$, we get

$$v_r(t,\widehat{\lambda})=\int\limits_{t_{r-1}}^{t}\Big[f(\tau,v_r(\tau,\widehat{\lambda})+\widehat{\lambda}_r)+\sum_{k=1}^{m}\varphi_k(\tau)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\psi_k(s)ds\Big]d\tau+$$
$$+\int\limits_{t_{r-1}}^{t}\sum_{k=1}^{m}\varphi_k(\tau)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}\psi_k(s)v_j(s,\widehat{\lambda})dsd\tau,\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N}.$$

It is easily seen that the function system $v[t,\widehat{\lambda}]=(v_1(t,\widehat{\lambda}),v_2(t,\widehat{\lambda}),\ldots,v_N(t,\widehat{\lambda}))$ is a solution to the special Cauchy problem (2.6), (2.7) in $S(0,\rho_v)$.

Let us prove the uniqueness of a solution. Assume that there exists another solution $\widetilde{v}[t,\widehat{\lambda}] \in S(0,\rho_v)$ to problem (2.6), (2.7) for $\widehat{\lambda} \in S(\lambda^{(0)},\rho_\lambda)$, i.e.

$$\widetilde{v}_r(t,\widehat{\lambda}) = \int_{t_{r-1}}^t f(\tau,\widehat{\lambda}_r + \widetilde{v}_r(\tau,\widehat{\lambda}))d\tau +$$
$$+ \int_{t_{r-1}}^t \sum_{k=1}^m \varphi_k(\tau) \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \psi_k(s)\big[\widehat{\lambda}_j + \widetilde{v}_j(s,\widehat{\lambda})\big]ds d\tau, \quad t \in [t_{r-1},t_r], \quad r = \overline{1,N}.$$

Then

$$\Big\|v_r(t,\widehat{\lambda}) - \widetilde{v}_r(t,\widehat{\lambda})\Big\| \le \int_{t_{r-1}}^t L_0\Big\|v_r(\tau,\widehat{\lambda}) - \widetilde{v}_r(\tau,\widehat{\lambda})\Big\|d\tau +$$
$$+ \int_{t_{r-1}}^t \sum_{k=1}^m \|\varphi_k(\tau)\| \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|\psi_k(s)\| \cdot \|v_j(s,\widehat{\lambda}) - \widetilde{v}_j(s,\widehat{\lambda})\|ds d\tau, \; t \in [t_{r-1},t_r], \; r = \overline{1,N},$$

and by virtue of Condition $\mathcal{A}$

$$\Big\|v[\cdot,\widehat{\lambda}] - \widetilde{v}[\cdot,\widehat{\lambda}]\Big\|_3 \le (L_0 + K_0)\overline{h}\Big\|v[\cdot,\widehat{\lambda}] - \widetilde{v}[\cdot,\widehat{\lambda}]\Big\|_3.$$

Condition (ii) of the theorem provides that $v_r(t,\widehat{\lambda}) = \widetilde{v}_r(t,\widehat{\lambda})$ for all $t \in [t_{r-1},t_r]$, $r = \overline{1,N}$. □

Definition 1. Let the conditions of Theorem 2.1 be fulfilled and the function system $v[t,\lambda] = \big(v_1(t,\lambda), v_2(t,\lambda), \ldots, v_N(t,\lambda)\big) \in S(0,\rho_v)$ be a solution to the special Cauchy problem (2.4), (2.5) with the parameter $\lambda = (\lambda_1,\lambda_2,...,\lambda_N)$, where $\lambda_r \in S\big(\lambda_r^{(0)},\rho_\lambda\big)$, $r = \overline{1,N}$. Then the function $x(\Delta_N,t,\lambda)$, given by the equalities $x(\Delta_N,t,\lambda) = \lambda_r + v_r(t,\lambda)$ for $t \in [t_{r-1},t_r)$, $r = \overline{1,N}$, and $x(\Delta_N,T,\lambda) = \lambda_N + v_N(T,\lambda)$, is called a Δ_N general solution to equation (1.1) in $G^0(\Delta_N,\rho)$.

The conditions of Theorem 2.1 ensure the existence and uniqueness of a Δ_N general solution to equation (1.1) in $G^0(\Delta_N,\rho)$. For any $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_N) \in \mathbb{R}^{nN}$ with $\lambda_r \in S\big(\lambda_r^{(0)},\rho_\lambda\big)$, $r = \overline{1,N}$, the function $x(\Delta_N,t,\lambda)$ satisfies equation (1.1) for all $t \in (0,T)\backslash\{t_p, p = \overline{1,N-1}\}$, and the pair $(t, x(\Delta_N,t,\lambda))$ belongs to $G^0(\Delta_N,\rho)$.

We assume that the conditions of Theorem 2.1 are fulfilled and $x(\Delta_N,t,\lambda)$ is a Δ_N general solution to equation (1.1) in $G^0(\Delta_N,\rho)$.

Theorem 2.2. *Let a piecewise continuous on $[0,T]$ function $\widetilde{x}(t)$ with the possible discontinuity points $t = t_p$, $p = \overline{1,N-1}$, be given, and $(t,\widetilde{x}(t)) \in G^0(\Delta_N,\rho)$. Assume that the function $\widetilde{x}(t)$ has a continuous derivative and satisfies equation (1.1) for all $t \in (0,T)\backslash\{t_p, p = \overline{1,N-1}\}$. Then there exists a unique $\widetilde{\lambda} = (\widetilde{\lambda}_1,\widetilde{\lambda}_2,\ldots,\widetilde{\lambda}_N) \in \mathbb{R}^{nN}$ with $\widetilde{\lambda}_r \in S\big(\lambda_r^{(0)},\rho_\lambda\big)$, $r = \overline{1,N}$, such that the equality $x(\Delta_N,t,\widetilde{\lambda}) = \widetilde{x}(t)$ holds for all $t \in [0,T]$.*

Proof. Let $\widetilde{x}_r(t)$ be the restriction of $\widetilde{x}(t)$ to $[t_{r-1},t_r)$, $r = \overline{1,N}$, and $\widetilde{x}[t] = (\widetilde{x}_1(t),\widetilde{x}_2(t),...,\widetilde{x}_N(t))$. The assumptions of the theorem imply that the functions $\widetilde{x}_r(t)$, $r = \overline{1,N}$, satisfy equation (2.1) and $(t,\widetilde{x}_r(t)) \in G_r^0(\rho)$ for all $r = \overline{1,N}$. For the function $\widetilde{x}(t)$, we assign the parameter $\widetilde{\lambda} = (\widetilde{\lambda}_1,\widetilde{\lambda}_2,\ldots,\widetilde{\lambda}_N) \in \mathbb{R}^{nN}$ with $\widetilde{\lambda}_r = \widetilde{x}(t_{r-1})$. It is clear that $\widetilde{\lambda}_r \in S\big(\lambda_r^{(0)},\rho_\lambda\big)$ for any $r = \overline{1,N}$. By Theorem 2.1, there exists $v[t,\widetilde{\lambda}] = \big(v_1(t,\widetilde{\lambda}), v_2(t,\widetilde{\lambda}),\ldots,v_N(t,\widetilde{\lambda})\big)$, a unique solution to the special Cauchy problem (2.4), (2.5) with $\lambda = \widetilde{\lambda}$, in $S(0,\rho_v)$. Since

$$u_r(t,\widetilde{\lambda}) = \int_{t_{r-1}}^t f(\tau,u_r(\tau,\widetilde{\lambda}) + \widetilde{\lambda}_r)d\tau + \int_{t_{r-1}}^t \sum_{k=1}^m \varphi_k(\tau)\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \psi_k(s)[u_j(s,\widetilde{\lambda}) + \widetilde{\lambda}_j]ds d\tau,$$

and

$$\left\|\widetilde{\lambda}_r + u_r(t,\widetilde{\lambda}) - \lambda_r^{(0)}\right\| \leq M_1(t-t_{r-1}) + \rho - M_1\overline{h} \leq M_1(t-t_{r-1}) + \rho - M_1(t_r - t_{r-1}) =$$

$$= \rho - M_1(t_r - t), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N},$$

the pair $(t, \widetilde{\lambda}_r + u_r(t,\widetilde{\lambda}))$ belongs to $G_r^0(\rho)$, $r = \overline{1,N}$.

Definition 1 and the interrelation mentioned above between solutions to equations (2.1) and special Cauchy problem (2.2), (2.3) lead to the following equalities $\widetilde{x}(t) = \widetilde{\lambda}_r + v_r(t,\widetilde{\lambda}) = x(\Delta_N, t, \widetilde{\lambda})$ for $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, and $\widetilde{x}(T) = \widetilde{\lambda}_N + v_N(T,\widetilde{\lambda}) = x(\Delta_N, T, \widetilde{\lambda})$.

Let us show the uniqueness of $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_N) \in \mathbb{R}^{nN}$, $\widetilde{\lambda}_r \in S(\lambda_r^{(0)}, \rho_\lambda)$, $r = \overline{1,N}$. Suppose, contrary to our claim, there exists another $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in \mathbb{R}^{nN}$, with $\lambda_r^* \in S(\lambda_r^{(0)}, \rho_\lambda)$, $r = \overline{1,N}$, such that $\widetilde{x}(t) = x(\Delta_N, t, \lambda^*)$ for all $t \in [0,T]$. Then, according to Definition 1, we have the equalities $\widetilde{x}(t) = \lambda_r^* + v_r(t,\lambda^*)$ for $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, and $\widetilde{x}(T) = \lambda_N^* + v_N(T,\lambda^*)$, where the function system $v[t,\lambda^*] = \big(v_1(t,\lambda^*), v_2(t,\lambda^*), \ldots, v_N(t,\lambda^*)\big) \in S(0,\rho_v)$ is a solution to the special Cauchy problem (2.4), (2.5) with $\lambda = \lambda^*$. Now, using initial conditions (2.5), we obtain $\widetilde{\lambda}_r = \widetilde{x}(t_{r-1}) = \lambda_r^* + v_r(t_{r-1},\lambda^*) = \lambda_r^*$ for $r = \overline{1,N}$. □

Corollary 2.1. *Let $x^*(t)$ be a solution to equation* (1.1)*, and $(t, x^*(t)) \in G^0(\Delta_N, \rho)$. Then there exists a unique $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in \mathbb{R}^{nN}$ with $\lambda_r^* \in S\big(\lambda_r^{(0)}, \rho_\lambda\big)$, $r = \overline{1,N}$, such that the equality $x(\Delta_N, t, \lambda^*) = x^*(t)$ holds for all $t \in [0,T]$.*

3 Solvability of nonlinear BVP for equation (1.1)

In this Section, we consider equation (1.1) with the boundary condition

$$g[x(0), x(T)] = 0, \tag{3.1}$$

where $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

If $x(t)$ is a solution to equation (1.1), and $x[t] = (x_1(t), x_2(t), ..., x_N(t))$ is a function system of its restrictions to the subintervals $[t_{r-1}, t_r)$, $r = \overline{1,N}$, then the equations

$$\lim_{t \to t_p - 0} x_p(t) = x_{p+1}(t_p), \qquad p = \overline{1, N-1}, \tag{3.2}$$

hold. Equations (3.2) are the continuity conditions for solutions to equation (1.1) at the interior points of partition Δ_N.

Theorem 3.1. *Let a function system $x[t] = (x_1(t), x_2(t), \ldots, x_N(t))$ belong to $\mathbb{C}([0,T], \Delta_N, \mathbb{R}^{nN})$ and the pair $(t, x_r(t))$ belong to $G_r^0(\rho)$ for all $r = \overline{1,N}$. Assume that the functions $x_r(t)$, $r = \overline{1,N}$, satisfy equations* (2.1) *and continuity conditions* (3.2)*. Then the function $x^*(t)$, given by the equalities*

$$x^*(t) = x_r(t), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}, \quad \text{and} \quad x^*(T) = \lim_{t \to T-0} x_N(t),$$

is continuous on $[0,T]$, continuously differentiable on $(0,T)$, and satisfies equation (1.1)*. Moreover, $(t, x^*(t)) \in G^0(\Delta_N, \rho)$.*

Proof. By assumption, $x[t] = (x_1(t), x_2(t), \ldots, x_N(t)) \in \mathbb{C}([0,T], \Delta_N, \mathbb{R}^{nN})$. Therefore, the equality $x^*(T) = \lim\limits_{t \to T-0} x_N(t)$ and equations (3.2) provide the continuity of the function $x^*(t)$ on $[0,T]$. In view of $(t, x_r(t)) \in G_r^0(\rho)$, $r = \overline{1,N}$, the pair $(t, x^*(t))$ belongs to $G^0(\Delta_N, \rho)$. Since the functions

$x_r(t)$, $r=\overline{1,N}$, satisfy equations (2.1), it follows that the function $x^*(t)$ has a continuous derivative and satisfies equation (1.1) for all $t\in(0,T)\backslash\{t_p, p=\overline{1,N-1}\}$.

The existence and continuity of $\dot{x}^*(t_p)$, $p=\overline{1,N-1}$, follow from the equalities

$$\lim_{t\to t_p-0}\dot{x}^*(t)=\lim_{t\to t_p-0}\left[f(t,x^*(t))+\sum_{k=1}^m\varphi_k(t)\int_0^T\psi_k(\tau)x^*(\tau)d\tau\right]=$$

$$=f(t_p,x^*(t_p))+\sum_{k=1}^m\varphi_k(t_p)\int_0^T\psi_k(\tau)x^*(\tau)d\tau=\lim_{t\to t_p+0}\dot{x}^*(t),\qquad p=\overline{1,N-1}.$$

These relations show that the function $x^*(t)$ satisfies equation (1.1) at the interior points of partition Δ_N as well. □

The Δ_N general solution to equation (1.1) allows us to reduce the solvability of BVP (1.1), (3.1) to the solvability of the system of nonlinear algebraic equations in parameters $\lambda_r\in\mathbb{R}^n$, $r=\overline{1,N}$. To this end, we write the continuity conditions (3.2) as follows:

$$\lim_{t\to t_p-0}x(\Delta_N,t,\lambda)-x(\Delta_N,t_p,\lambda)=0,\qquad p=\overline{1,N-1},$$

where $x(\Delta_N,t,\lambda)$ is a Δ_N general solution to equation (1.1) in $G^0(\Delta_N,\rho)$. Substituting the corresponding expressions of the Δ_N general solution into boundary condition (3.1) and continuity conditions (3.2), we obtain the system of nonlinear algebraic equations

$$g[\lambda_1,\lambda_N+v_N(T,\lambda)]=0,\tag{3.3}$$

$$\lambda_p+v_p(t_p,\lambda)-\lambda_{p+1}=0,\qquad p=\overline{1,N-1}.\tag{3.4}$$

Rewrite system (3.3), (3.4) in the form:

$$Q_*(\Delta_N;\lambda)=0,\qquad \lambda\in\mathbb{R}^{nN}.\tag{3.5}$$

Theorem 3.2. *Let the conditions of Theorem 2.1 be fulfilled and $x(\Delta_N,t,\lambda)$ be a Δ_N general solution to equation (1.1) in $G^0(\Delta_N,\rho)$. Assume that a function $x^*(t)$ be a solution to problem (1.1), (3.1) and $(t,x^*(t))\in G^0(\Delta_N,\rho)$. Then the vector $\lambda^*=(\lambda_1^*,\lambda_2^*,\ldots,\lambda_N^*)$ with $\lambda_r^*=x^*(t_{r-1})$, $r=\overline{1,N}$, is a solution to equation (3.5), and $\lambda_r^*\in S\big(\lambda_r^{(0)},\rho_\lambda\big)$, $r=\overline{1,N}$. Vice versa, if $\widetilde{\lambda}=(\widetilde{\lambda}_1,\widetilde{\lambda}_2,\ldots,\widetilde{\lambda}_N)$ with $\widetilde{\lambda}_r\in S\big(\lambda_r^{(0)},\rho_\lambda\big)$, $r=\overline{1,N}$, is a solution to equation (3.5), then the function $\widetilde{x}(t)=x(\Delta_N,t,\widetilde{\lambda})$ is a solution to problem (1.1), (3.1), and $(t,\widetilde{x}(t))\in G^0(\Delta_N,\rho)$.*

Proof. If a function $x^*(t)$ is a solution to problem (1.1), (3.1), then the equalities

$$g[x^*(0),x^*(T)]=0,$$

$$\lim_{t\to t_p-0}x^*(t)-x^*(t_{p+1})=0,\quad p=\overline{1,N-1}$$

are true. Take $\lambda^*=(\lambda_1^*,\lambda_2^*,\ldots,\lambda_N^*)$ with $\lambda_r^*=x^*(t_{r-1})$, $r=\overline{1,N}$. Belonging of $(t,x^*(t))$ to $G^0(\Delta_N,\rho)$ provides $\lambda_r^*\in S\big(\lambda_r^{(0)},\rho_\lambda\big)$ for any $r=\overline{1,N}$. By Theorem 2.1, the special Cauchy problem (2.4), (2.5) has a unique solution $v[t,\lambda^*]\in S(0,\rho_v)$. Since the function $x^*(t)$ satisfies equation (1.1) as well, by Theorem 2.2, the equality $x^*(t)=x(\Delta_N,t,\lambda^*)$ holds for all $t\in[0,T]$. Substituting the corresponding expression of $x(\Delta_N,t,\lambda^*)$ into (3.3) and (3.4), we get:

$$g[\lambda_1^*,\lambda_N^*+v_N(T,\lambda^*)]=0,$$

$$\lambda_p^* + v_p(t_p, \lambda^*) - \lambda_{p+1}^* = 0, \quad p = \overline{1, N-1},$$

i.e. $\lambda^* \in \mathbb{R}^{nN}$ is a solution to equation (3.5).

Assume now that the vector $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_N)$ with $\widetilde{\lambda}_r \in S\big(\lambda_r^{(0)}, \rho_\lambda\big)$, $r = \overline{1, N}$, is a solution to equation (3.5). Then the special Cauchy problem (2.4), (2.5) has a unique solution $v[t, \widetilde{\lambda}] \in S(0, \rho_v)$. Substituting $\widetilde{\lambda}$ into the Δ_N general solution, we obtain the function $\widetilde{x}(t) = x(\Delta_N, t, \widetilde{\lambda})$. By using Condition $\mathcal{A}$, it is easily proved that $(t, \widetilde{x}(t)) \in G^0(\Delta_N, \rho)$. Since $Q_*(\Delta_N; \widetilde{\lambda}) = 0$ implies

$$g[\widetilde{\lambda}_1, \widetilde{\lambda}_N + v_N(T, \widetilde{\lambda})] = 0,$$

$$\widetilde{\lambda}_p + v_p(t_p, \widetilde{\lambda}) - \widetilde{\lambda}_{p+1} = 0, \quad p = \overline{1, N-1},$$

and, by Definition 1,

$$\widetilde{x}(t) = x(\Delta_N, t, \widetilde{\lambda}) = \widetilde{\lambda}_r + v_r(t, \widetilde{\lambda}), \qquad t \in [t_{r-1}, t_r), \qquad r = \overline{1, N-1};$$

$$\widetilde{x}(t) = x(\Delta_N, t, \widetilde{\lambda}) = \widetilde{\lambda}_N + v_N(t, \widetilde{\lambda}), \qquad t \in [t_{N-1}, t_N]$$

the function $\widetilde{x}(t)$ satisfies boundary condition (3.1) and continuity conditions (3.2). Therefore, in accordance with Theorem 3.1, the function $\widetilde{x}(t)$ satisfies equation (1.1) as well, i.e. the function $\widetilde{x}(t)$ is a solution to problem (1.1), (3.1). $\square$

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