ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2019, Volume 10, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http://publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 515
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

At the end of year 2019 there is 10th anniversary of the activities of the Eurasian Mathematical Journal. Volumes EMJ 10-4 and EMJ 11-1 are dedicated to this event.

ANDREI ANDREEVICH SHKALIKOV

(to the 70th birthday)

Andrei Andreevich Shkalikov, corresponding member of the Russian Academy of Sciences, an outstanding mathematician with a wide range of interests, a remarkable person, professor of the Faculty of Mechanics and Mathematics of the M.V. Lomonosov Moscow State University was born on November 19, 1949.

Andrei Andreevich is a leading specialist in the theory of operators and their applications, especially to problems of mechanics and mathematical physics. He is well known for his work in the theory of functions and in the theory of spaces with indefinite metrics. He is also a specialist in the theory of entire and meromorphic functions and its applications to operator theory.

Andrei Andreevich is known for solving a number of difficult problems that for many years remained unsolved. His work on the basic properties of systems of root functions of differential operators is well known worldwide. He gave a justification for Mandelstam's hypothesis about the existence of solutions satisfying the radiation principle. He solved, in general form, the Rayleigh problem on the reflection of waves from a periodic surface, obtained a solution to the Sobolev problem on the stability of the motion of a top with a cavity filled with liquid. His contribution to the construction of an abstract theory of the Orr-Sommerfeld problem is invaluable. He obtained a description of the limiting spectral portraits for a large class of functions describing the profiles of fluid flows. He is one of the founders of the modern theory of differential operators, coefficients of which are distributions, and inverse problems for such operators.

Andrei Andreevich has been a plenary speaker at many international conferences. He conducts fruitful scientific work and collaborates with many international mathematical research centers.

Andrei Andreevich is an author of more than 130 scientific publications. Among his pupils there are more than 20 Candidates of Sciences and 6 Doctors of Sciences. The results obtained by A.A. Shkalikov, his pupils, collaborators and followers gained worldwide recognition.

Professor Shkalikov is also an outstanding organizer. Under his supervision, many international conferences were held. In particular, conferences dedicated to the memory of I.G. Petrovsky, I.M. Gelfand, S.M. Nikol'skii, B.M. Levitan, anniversary conferences of V.A. Sadovnichy, and others.

Andrei Andreyevich is a deputy editor-in-chief of the journals Mathematical Notes, Moscow University Mathematics Bulletin, Moscow University Mechanics Bulletin, and a member of the editorial boards of the Russian Mathematical Surveys, Proceedings of the Moscow Mathematical Society and other journals, including the Eurasian Mathematical Journal.

The Editorial Board of the Eurasian Mathematical Journal cordially congratulates Andrei Andreevich on the occasion of his 70th birthday and wishes him good health, and new achievements in mathematics and mathematical education.

GOOD NEWS: EMJ IS NOW AN SJR Q2 JOURNAL

Recently the lists were published of all mathematical journals included in 2018 SCImago Journal Rank (SJR) quartiles Q1 (385 journals), Q2 (430 journals), Q3 (445 journals), and Q4 (741 journals), and Scopus CiteScore quartiles Q1 (443 journals), Q2 (375 journals), Q3 (348 journals), and Q4 (283 journals).

With great pleasure we inform our readers and authors that the Eurasian Mathematical Journal was included in the most popular scientific ranking database SJR in quartile Q2, currently the only mathematical journal in the Republic of Kazakhstan and Central Asia. The SJR data for the Eurasian Mathematical Journal (2018) is as follows:

550. Eurasian Mathematical Journal (Kazakhstan), Q2, SJR=0.624.
(550 is the number in the list of all Q1 - Q4 journals.)

The SJR indicator is calculated by using the data of the Scopus Database of the Elsevier, the modern publishing business founded in 1880. It uses a sophisticated formula, taking into account various characteristics of journals and journals publications. This formula and related comments can be viewed on the web-page

http : //www.scimagojr.com/journalrank.php.

Some other SJR Q2 mathematical journals:
490. Studia Mathematica (Poland), SJR=0.706,
492. Comptes Rendus Mathematique (France), SJR=0.704,
522. Journal of Mathematical Physics (USA), SJR=0.667,
540. Doklady Mathematics (Russia), SJR=0.636,
570. Journal of Mathematical Sciences (Japan), SJR=0.602,
662. Journal of Applied Probability (UK), SJR=0.523,
733. Mathematical Notes (Russia), SJR=0.465,
791. Canadian Mathematical Bulletin (Canada), SJR=0.433.

Our journal ranks:

7726th place in the list of 31971 scientific journals, representing all subjects and all regions, included in this database (in the first 25% of journals of this category),

225th place in the list of 2519 scientific journals, representing all subjects, of the Asiatic region, included in this database (in the first 10% of journals of this category),

550th place in the list of 2011 mathematical journals, representing all regions, included in this database (in the first 30% of journals of this category),

19th place in the list of 165 mathematical journals of the Asiatic region, included in this database (in the first 15% of journals of this category).

On a separate page the SJR statistics for the Eurasian Mathematical Journal is attached.

Recall that the Eurasian Mathematical Journal started its work in 2010 (see [1]-[3]) and was first included in SJR indicator in 2014 (Q4, SJR=0.101, see [4], [5], [6]). So, the ambitious plan set in [6] was implemented and even essentially exceeded.

As for the Scopus CiteScore indicator, it uses another sophisticated formula, differently taking into account various characteristics journals publications. This formula and related comments can be viewed on the web-page

https : //ru.service.elsevier.com/app/answers/detail/a$_i$d/19266/supporthub/scopus.

In this indicator the Eurasian Mathematical Journal was included in quartile Q3. The CiteScore data for the Eurasian Mathematical Journal (2018) is as follows:

333. Eurasian Mathematical Journal (Kazakhstan), Q3, CiteScore = 0,41
(333 is the number in the list of only Q3 journals.)

Some other Scopus CiteScore Q3 mathematical journals:

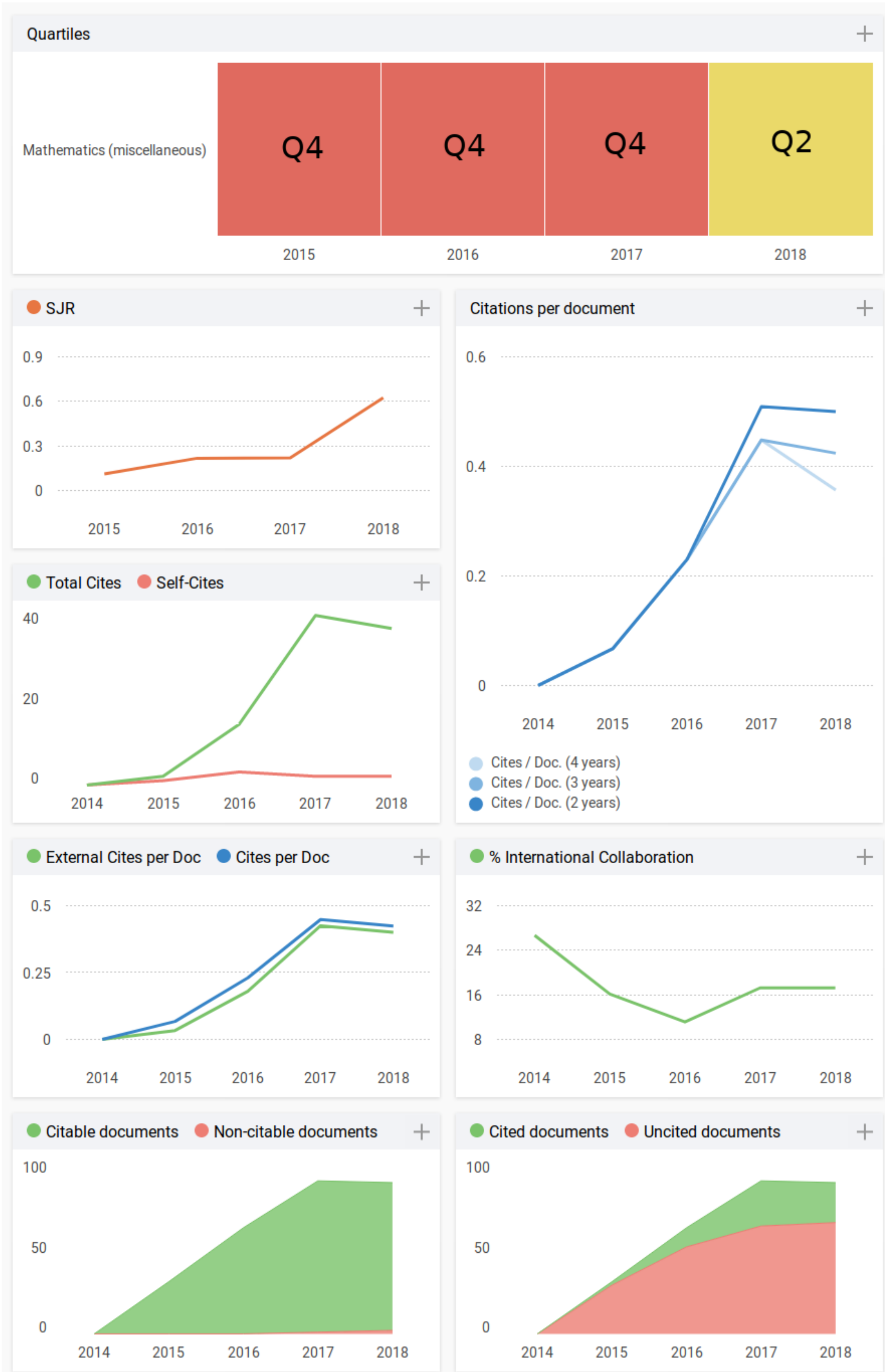
320. Czechoslovak Mathematical Journal (Czech Republic), CiteScore = 0.44,
321. Italian Journal of Pure and Applied Mathematics (Italy), CiteScore = 0.44,
323. Studia Scientiarum Mathematicarum Hungarica (Hungary), CiteScore = 0.44,
332. Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie (Romania), CiteScore = 0.41,
334. Indian Journal of Pure and Applied Mathematics (India), CiteScore = 0.41,
33. Transactions of the Moscow Mathematical Society (Russia), CiteScore = 0.41,
337. Illinois Journal of Mathematics (USA), CiteScore = 0.40,
339. Publications de l'Institut Mathematique (France), CiteScore = 0.40.

Our main current aim is to preserve the status of an SJR Q2 journal and of a Scopus CiteScore Q3 journal.

We hope that all respected members of the international Editorial Board, reviewers, current authors of our journal, representing more than 35 countries, and future authors will provide high quality publications in the EMJ which will allow to achieve this aim.

V.I. Burenkov, K.N. Ospanov, T.V. Tararykova, A.M. Temirkhanova.

Extract from http://www.scimagojr.com/journalrank.php
EURASIAN MATHEMATICAL JOURNAL



References

[1] B. Abdrayim, Opening address by the rector of L.N. Gumilyov Eurasian National University, Eurasian Math. J. 1 (2010), no. 1, 5.

[2] B. Abdrayim, Opening address by the rector of L.N. Gumilyov Eurasian National University, Eurasian Math. J. 1 (2010), no. 1, 6 (in Kazakh).

[3] B. Abdrayim, Opening address by the rector of L.N. Gumilyov Eurasian National University, Eurasian Math. J. 1 (2010), no. 1, 7 (in Russian).

[4] To the authors, reviewers, and readers of the Eurasian Mathematical Journal, Eurasian Math. J. 5 (2014), no. 2, 6.

[5] Eurasian Mathematical Journal is indexed in Scopus, Eurasian Math. J. 5 (2014), no. 3, 6–8.

[6] V.I. Burenkov, E.D. Nursultanov, T.Sh. Kalmenov, R. Oinarov, M. Otelbaev, T.V. Tararykova, A.M. Temirkhanova, EMJ: from Scopus Q4 to Scopus Q3 in two years?!, Eurasian Math. J. 7 (2016), no. 3, 6.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 10, Number 4 (2019), 14 – 23

NECESSARY AND SUFFICIENT CONDITIONS OF COMPACTNESS OF CERTAIN EMBEDDINGS OF SOBOLEV SPACES

V.I. Burenkov, T.V. Tararykova

Communicated by M.L. Goldman

Key words: Sobolev spaces, pre-compact sets, embeddings of Sobolev spaces.

AMS Mathematics Subject Classification: 46B25, 46B50, 47B38.

Abstract. Necessary and sufficient conditions on an open set $\Omega \subset \mathbb{R}^n$ are obtained ensuring that for $l, m \in \mathbb{N}_0, m < l$ the embedding $\overset{\circ}{W}{}^l_\infty(\Omega) \subset W^m_\infty(\Omega)$ is compact, where $W^m_\infty(\Omega)$ is the Sobolev space and $\overset{\circ}{W}{}^l_\infty(\Omega)$ is the closure in $W^l_\infty(\Omega)$ of the space of all infinitely continuously differentiable functions on Ω with supports compact in Ω.

DOI: https://doi.org/10.32523/2077-9879-2019-10-4-14-23

1 Introduction

Let $n, l \in \mathbb{N}, 1 \le p \le \infty$ and $\Omega \subset \mathbb{R}^n$ be a non-empty open set. We consider the Sobolev $W^l_p(\Omega)$ of all functions $f \in L_1^{\rm loc}(\Omega)$ for which for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \cdots \alpha_n = l$ the weak derivatives $D^\alpha_w f$ exist on Ω and

$$\|f\|_{W^l_p(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|=l} \|D^\alpha_w f\|_{L_p(\Omega)} < \infty\,,$$

and the Sobolev spaces $\overset{\circ}{W}{}^l_p(\Omega)$ which are the closures in $W^l_p(\Omega)$ of the space $C_0^\infty(\Omega)$ of all infinitely continuously differentiable functions on Ω with supports compact in Ω. (If $l = 0$ it is meant that $W^0_p(\Omega) = L_p(\Omega)$.)

Assume that $m \in \mathbb{N}_0$ and $m < l$. For any open set $\Omega \subset \mathbb{R}^n$ the embedding

$$\overset{\circ}{W}{}^l_p(\Omega) \subset W^m_p(\Omega) \tag{1.1}$$

is continuous. If $m = 0$ this is obvious. If $m \ge 1$, let, for any $f \in \overset{\circ}{W}{}^l_p(\Omega)$, $\overset{\circ}{f}$ denote the extension of f by 0 outside Ω. Then, by the well-known theorems on estimates for intermediate derivatives (see, for example, books [12], [1], [2]), for any function $f \in \overset{\circ}{W}{}^l_p(\Omega)$

$$\|f\|_{W^m_p(\Omega)} = \|\overset{\circ}{f}\|_{W^m_p(\mathbb{R}^n)} \le c_1 \|\overset{\circ}{f}\|_{W^l_p(\mathbb{R}^n)} = c_1 \|f\|_{W^l_p(\Omega)}\,, \tag{1.2}$$

where $c_1 > 0$ depends only on n, p and l.

The compactness of this embedding is, in particular, important in some areas of spectral theory of elliptic differential operators. It is one of the assumptions required to ensure the existence of eigenvalues and the min-max formula for the eigenvalues for certain spectral Dirichlet boundary values problems for such operators. (See, for example, [9], [7], [4], [5], [6], [3], [8].)

Clearly embedding (1.1) is compact if the embedding $W_p^l(\Omega) \subset W_p^m(\Omega)$ is compact. Its compactness holds under rather weak assumptions on the regularity of Ω. In [2] (Chatper 4) its compactness is proved for $m = 0$ for bounded sets with quasi-continuous boundaries. However, it may happen that this embedding is not compact, even does not hold, but embedding (1.1) is compact (see Example 1 in Section 2.1).

In this paper we consider the case $p = \infty$. As for $1 \le p < \infty$ we only make some comments.

The aim of this paper is to present necessary and sufficient conditions on Ω ensuring that the embedding

$$\overset{\circ}{W}{}^l_\infty(\Omega) \subset W_\infty^m(\Omega) \tag{1.3}$$

is compact, that is the embedding operator $I : \overset{\circ}{W}{}^l_\infty(\Omega) \to W_\infty^m(\Omega)$ ($If = f$ for $f \in \overset{\circ}{W}{}^l_\infty(\Omega)$) is compact.

2 Main results

Let, for $x \in \mathbb{R}^n, r > 0$, $B(x,r)$ denote the open ball centered at x of radius r. We shall use the following description of pre-compact sets in $W_\infty^l(\Omega)$ (see [2], page 168).

Lemma 2.1. *Let $n, l \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a non-empty open set.*

A subset $S \subset W_\infty^l(\Omega)$ is pre-compact if

1) the set S is bounded in $W_\infty^l(\Omega)$,

2)

$$\lim_{\delta \to 0^+} \sup_{f \in S} \|f\|_{W_\infty^l(\Omega \setminus \Omega_\delta)} = 0,$$

where $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$,

3)

$$\lim_{h \to 0} \sup_{f \in S} \|f(x+h) - f(x)\|_{W_\infty^l(\Omega_{|h|})} = 0,$$

and

4) if Ω is unbounded, then also

$$\lim_{r \to +\infty} \sup_{f \in S} \|f\|_{W_\infty^l(\Omega \setminus B(0,r))} = 0.$$

Remark 1. This lemma also holds for the spaces $W_p^l(\Omega)$ for $1 \le p < \infty$.

Theorem 2.1. *Let $n, l \in \mathbb{N}, m \in \mathbb{N}_0, l > m$ and $\Omega \subset \mathbb{R}^n$ be a non-empty open set.*

1) If the Lebesgue measure of Ω is finite, then embedding (1.3) is compact.

2) If the Lebesgue measure of Ω is infinite, then embedding (1.3) is compact if and only if Ω satisfies the following condition:

$$\lim_{r \to +\infty} \sup_{x \in \Omega \setminus B(0,r)} \operatorname{dist}(x, \partial\Omega) = 0\,. \tag{2.1}$$

or, equivalently,

$$\Omega \text{ does not contain an infinite family of disjoint open balls of the same radius } \varrho > 0. \tag{2.2}$$

Proof. Step 1. (*Proof of equivalence of conditions* (2.1) *and* (2.2) *for unbounded open sets* Ω.)

Step 1.1. Assume that condition (2.2) is not satisfied, that is there exist $\varrho > 0$ and $x_k \in \Omega, k \in \mathbb{N}$, such that $B(x_k, \varrho) \subset \Omega$ and $B(x_k, \varrho) \cap B(x_m, \varrho), k, m \in \mathbb{N}, k \neq m$. Then $\lim\limits_{k\to\infty} x_k = \infty$. Indeed, otherwise the sequence $\{x_k\}_{k\in\mathbb{N}}$ is bounded, hence there exists a subsequence $\{x_{k_s}\}_{s\in\mathbb{N}}$ and $x_0 \in \bar{\Omega}$ such that $\lim\limits_{s\to\infty} x_{k_s} = x_0$. Therefore, there exists $s_0 \in \mathbb{N}$ such that $x_{k_s} \in B(x_0, \frac{\varrho}{2})$ for all $s \in \mathbb{N}, s \geq s_0$. Consequently, $B(x_{k_s}, \varrho) \supset B(x_0, \frac{\varrho}{2})$, which contradicts the assumption that the $B(x_k, \varrho), k \in \mathbb{N}$, are disjoint.

So, for any $r > 0$ there exists $k(r) \in \mathbb{N}$ such that $x_{k(r)} \in \Omega \setminus B(0, r)$, hence, for any $r > 0$

$$\sup_{x\in\Omega\setminus B(0,r)} \text{dist } (x, \partial\Omega) \geq \text{dist } (x_{k(r)}, \partial\Omega) \geq \text{dist } (x_{k(r)}, \partial B(x_{k(r)}, \varrho) = \varrho\,.$$

Therefore, condition (2.1) is not satisfied.

Step 1.2. Assume that condition (2.1) is not satisfied. Since the function $\sup\limits_{x\in\Omega\setminus B(0,r)} \text{dist}(x, \partial\Omega)$ is non-increasing, it follows that for some $\varrho > 0$ for any $r > 0$

$$\sup_{x\in\Omega\setminus B(0,r)} \text{dist}(x, \partial\Omega) \geq \varrho\,. \tag{2.3}$$

Taking here $r = \varrho$ we get that there exists $x_1 \in \Omega$ such that $\text{dist}(x, \partial\Omega) \geq \frac{\varrho}{2} \Longleftrightarrow B(x_1, \frac{\varrho}{2}) \subset \Omega$ and $|x_1| \geq \varrho$. Next by (2.3) with $r = \varrho + |x_1$ we get that there exists $x_2 \in \Omega$ such that $\text{dist}(x, \partial\Omega) \geq \frac{\varrho}{2} \Longleftrightarrow B(x_2, \frac{\varrho}{2}) \subset \Omega$ and $|x_2| \geq \varrho + |x_1|$, and so on. So, there exist $x_k \in \Omega, k \in \mathbb{N}$, such that

$$B\Big(x_2, \frac{\varrho}{2}\Big) \subset \Omega \quad \text{and} \quad |x_{k+1}| \geq \varrho + |x_k|\,.$$

Hence, for any $m \in \mathbb{N}, m > k$

$$|x_m| \geq \varrho + |x_{m-1}| \geq 2\varrho + |x_{m-2}| \geq \cdots \geq (m-k)\varrho + |x_k|$$

and

$$|x_m - x_k| \geq |x_m| - |x_k| \geq (m-k)\varrho\,.$$

Let $z \in B(x_m, \frac{\varrho}{2})$. Then

$$|z - x_k| = |x_m - x_k + z - x_m| \geq |x_m - x_k| - |z - x_m| > (m-k)\varrho - \frac{\varrho}{2} \geq \frac{\varrho}{2}\,,$$

therefore, $z \notin B(x_k, \frac{\varrho}{2}) \Longrightarrow B(x_m, \frac{\varrho}{2}) \cap B(x_k, \frac{\varrho}{2}) = \varnothing$.

Step 2. (*Proof of the necessity of condition* (2.2) for open sets Ω of infinite Lebesgue measure.)

Assume that condition (2.2) is not satisfied, then there exist $0 < \varrho \leq 1$ and an infinite family of disjoint open balls $B(x_k, \varrho) \subset \Omega, k \in \mathbb{N}$.

Let $\varphi \in C^\infty(\mathbb{R}^n), \varphi \not\equiv 0$ be such that $\text{supp}\,\varphi \subset B(0,1)$ and $S = \{f_k\}_{k\in\mathbb{N}}$, where $f_k(x) = \varphi\big(\frac{x-x_k}{\varrho}\big), x \in \Omega, k \in \mathbb{N}$. Then $f_k \in \overset{\circ}{W}{}^l_\infty(\Omega)$ and for any $k \in \mathbb{N}$

$$\|f_k\|_{W^l_\infty(\Omega)} = \Big\|\varphi\Big(\frac{x-x_k}{\varrho}\Big)\Big\|_{L_\infty(\Omega)} + \sum_{|\alpha|=l} \Big\|D^\alpha\varphi\Big(\frac{x-x_k}{\varrho}\Big)\Big\|_{L_\infty(\Omega)}$$

$$= \|\varphi\|_{L_\infty(B(0,1))} + \varrho^{-l} \sum_{|\alpha|=l} \|D^\alpha\varphi\|_{L_\infty(B(0,1))} \leq \varrho^{-l}\|\varphi\|_{W^l_\infty(B(0,1))}\,.$$

Also for any $k \in \mathbb{N}$

$$\|f_k\|_{W^l_\infty(\Omega)} \geq \Big\|\varphi\Big(\frac{x-x_k}{\varrho}\Big)\Big\|_{L_\infty(\Omega)} = \|\varphi\|_{L_\infty(B(0,1))}\,.$$

So, the set S is bounded in $W^l_\infty(\Omega)$, but does not contain any subsequence convergent in $W^l_\infty(\Omega)$, because $\operatorname{supp} f_k \cap \operatorname{supp} f_s = \varnothing$ for $k, s \in \mathbb{N}, k \neq s$, and

$$\|f_k - f_s\|_{W^l_\infty(\Omega)} \geq \|f_k\|_{W^l_\infty(\Omega)} \geq \|\varphi\|_{L_\infty(B(0,1))} .$$

Thus, embedding (1.3) is not compact.

Step 3. (*Proof of the compactness of embedding* (1.3) *for unbounded open sets* Ω *satisfying condition* (2.1).)

Let S be a bounded set in $\overset{\circ}{W}{}^l_\infty(\Omega)$. It is required to prove that it is pre-compact in $W^m_\infty(\Omega)$. So, it suffices to prove that for S conditions 1) – 4) of Lemma 2.1 are satisfied.

Step 3.1. (*Proof of condition* 1) *for any non-empty open set* Ω.) Inequality (1.2) implies that

$$\sup_{f\in S} \|f\|_{W^m_\infty(\Omega)} \leq c_1 \sup_{f\in S} \|f\|_{W^l_\infty(\Omega)} ,$$

which implies condition 1).

Step 3.2. (*Proof of condition* 2) *for any non-empty open set* Ω.)

Let $\delta > 0$, $x \in \Omega \setminus \Omega_\delta$, and let $y \in \partial\Omega$ be such that $\operatorname{dist}(x, \partial\Omega) = |x - y|$. Moreover, let $f \in \overset{\circ}{W}{}^l_\infty(\Omega)$ and $f_k \in C_0^\infty(\Omega), k \in \mathbb{N}$, be such that $\lim\limits_{k\to\infty} \|f_k - f\|_{W^l_\infty(\Omega)} = 0$. Clearly, $\overset{\circ}{f}_k \in C_0^\infty(\mathbb{R}^n)$. Since $[x, y) \subset \Omega \setminus \Omega_\delta$, it follows that for some $t \in (0, 1)$

$$f_k(x) = \overset{\circ}{f}_k(x) - \overset{\circ}{f}_k(y) = \sum_{j=1}^{n} \frac{\partial \overset{\circ}{f}_k}{\partial x_j}(y + t(x - y))(x_j - y_j)$$

$$= \sum_{j=1}^{n} \frac{\partial f_k}{\partial x_j}(y + t(x - y))(x_j - y_j) , \tag{2.4}$$

hence, for all $x \in \Omega \setminus \Omega_\delta$

$$|f_k(x)| \leq \sum_{j=1}^{n} \Big|\frac{\partial f_k}{\partial x_j}(y + t(x - y))\Big||x_j - y_j| \leq \delta \sum_{j=1}^{n} \Big\|\frac{\partial f_k}{\partial x_j}\Big\|_{L_\infty(\Omega\setminus\Omega_\delta)}$$

and

$$\|f_k\|_{L_\infty(\Omega\setminus\Omega_\delta)} \leq \delta \sum_{j=1}^{n} \Big\|\frac{\partial f_k}{\partial x_j}\Big\|_{L_\infty(\Omega\setminus\Omega_\delta)} .$$

For similar reasons for any $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$

$$\|D^\alpha f_k\|_{L_\infty(\Omega\setminus\Omega_\delta)} \leq \delta \sum_{j=1}^{n} \Big\|\frac{\partial D^\alpha f_k}{\partial x_j}\Big\|_{L_\infty(\Omega\setminus\Omega_\delta)} .$$

Hence,

$$\|f_k\|_{W^m_\infty(\Omega\setminus\Omega_\delta)} = \|f_k\|_{L_\infty(\Omega\setminus\Omega_\delta)} + \sum_{|\alpha|=l} \|D^\alpha f_k\|_{L_\infty(\Omega\setminus\Omega_\delta)}$$

$$\leq \delta \Big(\sum_{j=1}^{n} \Big\|\frac{\partial f_k}{\partial x_j}\Big\|_{L_\infty(\Omega\setminus\Omega_\delta)} + \sum_{|\alpha|=m} \sum_{j=1}^{n} \Big\|\frac{\partial D^\alpha f_k}{\partial x_j}\Big\|_{L_\infty(\Omega\setminus\Omega_\delta)}\Big)$$

$$\leq \delta \Big(\sum_{j=1}^{n} \Big\|\frac{\partial f_k}{\partial x_j}\Big\|_{L_\infty(\Omega)} + n \sum_{|\beta|=m+1} \Big\|\frac{\partial D^\beta f_k}{\partial x_j}\Big\|_{L_\infty(\Omega)}\Big) .$$

By applying inequality (1.2) with m replaced by 1 and by $m+1$ we get that for some $c_2 > 0$, depending only on l, m, n, for any $\delta > 0$ and $k \in \mathbb{N}$

$$\|f_k\|_{W^m_\infty(\Omega\setminus\Omega_\delta)} \le c_2\,\delta\,\|f_k\|_{W^l_\infty(\Omega)}\,. \tag{2.5}$$

By passing to the limit as $k \to \infty$ it follows that for any $f \in \overset{\circ}{W}{}^l_\infty(\Omega)$

$$\|f\|_{W^m_\infty(\Omega\setminus\Omega_\delta)} \le c_2\,\delta\,\|f\|_{W^l_\infty(\Omega)}$$

and

$$\sup_{f\in S}\|f\|_{W^m_\infty(\Omega\setminus\Omega_\delta)} \le c_2\,\delta\,\sup_{f\in S}\|f\|_{W^l_\infty(\Omega)}\,,$$

which implies condition 2).

Step 3.3. (*Proof of condition* 3) for any non-empty open set Ω.)

By Corollary 7 of Chapter 3 of [2], page 103, for any $f \in \overset{\circ}{W}{}^l_\infty(\Omega)$

$$\|f(x+h)-f(x)\|_{W^m_\infty(\Omega_{|h|})} = \|f(x+h)-f(x)\|_{L_\infty(\Omega_{|h|})}$$

$$+\sum_{|\alpha|=m}\|D^\alpha f(x+h)-D^\alpha f(x)\|_{L_\infty(\Omega_{|h|})}$$

$$\le |h|\Big(\sum_{j=1}^n\Big\|\frac{\partial f}{\partial x_j}\Big\|_{L_\infty(\Omega\setminus\Omega_\delta)}+\sum_{|\alpha|=m}\sum_{j=1}^n\Big\|\frac{\partial D^\alpha f}{\partial x_j}\Big\|_{L_\infty(\Omega\setminus\Omega_\delta)}\Big)$$

$$\le |h|\Big(\sum_{j=1}^n\Big\|\frac{\partial f}{\partial x_j}\Big\|_{L_\infty(\Omega)}+n\sum_{|\beta|=m+1}\Big\|\frac{\partial D^\beta f}{\partial x_j}\Big\|_{L_\infty(\Omega)}\Big)\,.$$

By applying inequality (1.2) with m replaced by 1 and by $m+1$ we get that for the same $c_2 > 0$ as in inequality (2.5) for any $h \in \mathbb{R}^n$

$$\|f(x+h)-f(x)\|_{W^m_\infty(\Omega_{|h|})} \le c_2\,|h|\,\|f\|_{W^l_\infty(\Omega)}\,.$$

Hence,

$$\sup_{f\in S}\|f(x+h)-f(x)\|_{W^m_\infty(\Omega_{|h|})} \le c_2\,|h|\,\sup_{f\in S}\|f\|_{W^l_\infty(\Omega)}\,,$$

which implies condition 3).

If Ω is a non-empty bounded open set, then conditions 1) - 3) of Lemma 2.1 are satisfied, hence, the set S is pre-compact in $W^m_\infty(\Omega)$ and embedding (1.3) is compact.

Step 3.4. (*Proof of condition* 4) *for any unbounded open set* Ω *satisfying condition* (2.1).)

Let $r > 0$, $x \in \Omega\setminus B(0,r)$, and let y, f, f_k be the same as in Step 2.2. Since $[x,y) \subset \Omega$, it follows that for some $t \in (0,1)$ equality (2.4) holds for all $x \in \Omega\setminus B(0,r)$. Hence,

$$|f_k(x)| \le \sum_{j=1}^n\Big|\frac{\partial f_k}{\partial x_j}(y+t(x-y))\Big||x_j-y_j| \le |x-y|\sum_{j=1}^n\Big\|\frac{\partial f_k}{\partial x_j}\Big\|_{L_\infty(\Omega)}$$

$$= \operatorname{dist}(x,\partial\Omega)\sum_{j=1}^n\Big\|\frac{\partial f_k}{\partial x_j}\Big\|_{L_\infty(\Omega)}$$

and

$$\|f_k\|_{L_\infty(\Omega\setminus B(0,r))} \le \sup_{x\in\Omega\setminus B(0,r)}\operatorname{dist}(x,\partial\Omega)\sum_{j=1}^n\Big\|\frac{\partial f_k}{\partial x_j}\Big\|_{L_\infty(\Omega)}\,.$$

For similar reasons for any $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$

$$\|D^\alpha f_k\|_{L_\infty(\Omega\setminus B(0,r))} \leq \sup_{x\in\Omega\setminus B(0,r)} \operatorname{dist}(x,\partial\Omega) \sum_{j=1}^{n} \Big\| \frac{\partial D^\alpha f_k}{\partial x_j} \Big\|_{L_\infty(\Omega)} .$$

Hence,

$$\|f_k\|_{W^m_\infty(\Omega\setminus B(0,r))} \leq \sup_{x\in\Omega\setminus B(0,r)} \operatorname{dist}(x,\partial\Omega) \Big(\sum_{j=1}^{n} \Big\| \frac{\partial f_k}{\partial x_j} \Big\|_{L_\infty(\Omega)} + \sum_{|\alpha|=m} \sum_{j=1}^{n} \Big\| \frac{\partial D^\alpha f_k}{\partial x_j} \Big\|_{L_\infty(\Omega)} \Big)$$

$$\leq \sup_{x\in\Omega\setminus B(0,r)} \operatorname{dist}(x,\partial\Omega) \Big(\sum_{j=1}^{n} \Big\| \frac{\partial f_k}{\partial x_j} \Big\|_{L_\infty(\Omega)} + n \sum_{|\beta|=m+1} \Big\| \frac{\partial D^\beta f_k}{\partial x_j} \Big\|_{L_\infty(\Omega)} \Big) .$$

By applying inequality (1.2) with m replaced by 1 and by $m+1$ we get that for the same $c_2 > 0$ as in inequality (2.5), for any $r > 0$ and $k \in \mathbb{N}$

$$\|f_k\|_{W^m_\infty(\Omega\setminus B(0,r))} \leq c_2 \sup_{x\in\Omega\setminus B(0,r)} \operatorname{dist}(x,\partial\Omega)\, \|f_k\|_{W^l_\infty(\Omega)} . \tag{2.6}$$

By passing to the limit as $k \to \infty$ it follows that for any $f \in \overset{\circ}{W}{}^l_\infty(\Omega)$

$$\|f\|_{W^m_\infty(\Omega\setminus B(0,r))} \leq c_2 \sup_{x\in\Omega\setminus B(0,r)} \operatorname{dist}(x,\partial\Omega)\, \|f\|_{W^l_\infty(\Omega)} ,$$

hence,

$$\sup_{f\in S} \|f\|_{W^m_\infty(\Omega\setminus B(0,r))} \leq c_2 \sup_{x\in\Omega\setminus B(0,r)} \operatorname{dist}(x,\partial\Omega) \sup_{f\in S} \|f\|_{W^l_\infty(\Omega)} ,$$

which implies condition 4).

So, all four conditions of Lemma 2.1 are satisfied, hence, the set S is pre-compact in $W^m_\infty(\Omega)$ and embedding (1.3) is compact.

Step 4. (*Proof of the compactness of embedding* (1.3) *open set* Ω *of finite Lebesgue measure.*)

If Ω is bounded, then according to Lemma 2.1 it suffices to verify only conditions 1) - 3). Their validity follow fy Steps 3.1 - 3.3 (they are proved for any open sets Ω), hence, embedding (1.3) is compact.

If Ω is unbounded and is of finite Lebesgue measure, then condition (2.2) is satisfied. Indeed, otherwise there exist $\varrho > 0$ and disjoint open balls $B(x_k, \varrho) \subset \Omega, k \in \mathbb{N}$, and

$$\operatorname{meas}\Omega \geq \sum_{k=1}^{\infty} \operatorname{meas} B(x_k,\varrho) = +\infty ,$$

therefore, by Step 3 embedding (1.3) is compact. □

Remark 2. The proof of Step 3.2 also works for the pair of spaces $\overset{\circ}{W}{}^l_p(\Omega)$ and $W^m_p(\Omega)$ for any $1 \leq p < \infty$. Indeed, the functions $f_k, k \in \mathbb{N}$, constructed in Step 2, also belong to $\overset{\circ}{W}{}^l_p(\Omega)$ for any $1 \leq p < \infty$. Furthermore, for any $k \in \mathbb{N}$

$$\|f_k\|_{W^l_p(\Omega)} = \varrho^{\frac{n}{p}} \|\varphi\|_{L_p(B(0,1))} + \varrho^{\frac{n}{p}-l} \sum_{|\alpha|=l} \|D^\alpha\varphi\|_{L_p(B(0,1))} \leq \max\{1, \varrho^{\frac{n}{p}-l}\} \|\varphi\|_{W^l_p(B(0,1))}$$

and for any $k, s \in \mathbb{N}, k \neq s$,

$$\|f_k - f_s\|_{W^l_\infty(\Omega)} \geq \|f_k\|_{W^l_p(\Omega)} \geq \varrho^{\frac{n}{p}} \|\varphi\|_{L_p(B(0,1))} ,$$

which implies that embedding (1.3) is not compact if condition (2.2) is not satisfied.

Remark 3. The proofs of Steps 3.1 and 3.3 also work for the pair of spaces $\overset{\circ}{W}{}_p^{\,l}(\Omega)$ and $W_p^m(\Omega)$ for any $1 \le p < \infty$. (One just needs to replace the parameter ∞ in $\overset{\circ}{W}{}_\infty^{\,l}(\Omega)$ and $W_\infty^m(\Omega)$ by p.)

Remark 4. Condition (2.2) can be satisfied for some open sets of infinite Lebesgue measure, hence, embedding (1.3) can be compact for some open sets of infinite Lebesgue measure. We give two examples.

Example 1. (*Infinite union of disjoint balls.*) Let

$$\Omega = \bigcup_{k=1}^{\infty} B(x_k, \varrho_k),$$

where the balls $B(x_k, \varrho_k), k \in \mathbb{N}$ are disjoint, $0 < \varrho_k < \infty$ and

$$\lim_{k \to \infty} \varrho_k = 0. \tag{2.7}$$

If condition (2.7) is not satisfied, then there exist $\varrho > 0$ and a subsequence $\{k_m\}_{m \in \mathbb{N}}$ such that $\varrho_{k_m} \ge \varrho, m \in \mathbb{N}$, hence, Ω contains the infinite family of disjoint balls $B(x_{k_m}, \varrho), m \in \mathbb{N}$. Therefore, by Theorem 2.1 embedding (1.3) is not compact.

Let condition (2.7) be satisfied. Assume, to the contrary, that there exists an infinite family of disjoint balls $B(y_m, \varrho) \subset \Omega, m \in \mathbb{N}$, of the same radius $\varrho > 0$. Since the balls $B(x_k, \varrho_k), k \in \mathbb{N}$, are disjoint, each ball $B(y_m, \varrho)$ is contained in a certain ball $B(x_{k_m}, \varrho_{k_m})$. Furthermore, there exists an infinite family of balls $B(x_{k_l}, \varrho_{k_l}), l \in \mathbb{N}$, containing $B(y_m, \varrho)$ for some $m \in \mathbb{N}$, since any ball $B(x_{k_l}, \varrho_{k_l})$, being finite, cannot contain infinitely many disjoint balls $B(y_m, \varrho)$. Hence, $\varrho_{k_l} \ge \varrho$ for any $l \in \mathbb{N}$ which contradicts (2.7). So, Ω does not contain an infinite family of disjoint balls of the same positive radius. Therefore, by Theorem 2.1 embedding (1.3) is compact.

If $\sum_{k=1}^{\infty} \varrho_k^n = \infty$, then $\operatorname{meas} \Omega = \infty$.

Note that the embedding $W_\infty^l(\Omega) \subset W_\infty^m(\Omega)$ for $l > m > 0$ does not hold. Indeed, let

$$f(x) = \sum_{k=1}^{\infty} f_k(x),\ x \in \Omega, \quad \text{where} \quad f_k(x) = \varrho_k^{1-l}(x_1 - x_{k1})^{l-1},\ x \in B(x_k, \varrho_k), k \in \mathbb{N}.$$

Then

$$\|f\|_{W_\infty^l(\Omega)} = \|f\|_{L_\infty(\Omega)} = \sup_{k \in \mathbb{N}} \|f_k\|_{L_\infty(B(x_k, \varrho_k))} = 1$$

and

$$\begin{aligned} \|f\|_{W_\infty^m(\Omega)} &= \|f\|_{L_\infty(\Omega)} + (l-1)\cdots(l-m)\varrho_k^{1-l}\|(x_1 - x_{k1})^{l-m-1}\|_{L_\infty(B(x_k, \varrho_k))} \\ &= 1 + (l-1)\cdots(l-m) \sup_{k \in \mathbb{N}} \varrho_k^{-m} = \infty \,. \end{aligned}$$

Example 2. (*Tube.*) Let

$$\Omega = \left\{ x \in \mathbb{R}^n : \ 0 < x_1 < \infty, \ (x_2^2 + \cdots + x_n^2)^{1/2} < \varphi(x_1) \right\},$$

where φ is a continuous positive non-increasing function on $[0, \infty)$ such that $\lim_{x_1 \to \infty} \varphi(x_1) = 0$.

Note that for any unbounded open set condition (2.1) is equivalent to the condition

$$\lim_{r \to +\infty} \sup_{x \in \Omega \setminus Q(0,r)} \operatorname{dist}(x, \partial\Omega) = 0 \,. \tag{2.8}$$

where $Q(0,r)=\{x\in\mathbb{R}^n:\,|x_j|<r,\,j=1,\dots,n\}$, because

$$\sup_{x\in\Omega\setminus B(0,r\sqrt{n})}\mathrm{dist}(x,\partial\Omega)\le\sup_{x\in\Omega\setminus Q(0,r)}\mathrm{dist}(x,\partial\Omega)\le\sup_{x\in\Omega\setminus B(0,r)}\mathrm{dist}(x,\partial\Omega)\,.$$

Clearly, for any $x\in\Omega$ under consideration, dist $(x,\Omega)\le\varphi(x_1))$, hence, for any $r>\varphi(0)$

$$\sup_{x\in\Omega\setminus Q(0,r)}\mathrm{dist}(x,\partial\Omega)\le\varphi(r)\,.$$

Therefore, condition (2.8) is satisfied and embedding (1.3) is compact for all considered Ω.

If $\int_0^\infty\varphi(x_1)^{n-1}dx_1=\infty$, then $\mathrm{meas}\,\Omega=\infty$.

Corollary 2.1. *For an arbitrary one-dimensional open set*

$$\Omega=\bigcup_{k=1}^{s}(a_k,b_k)\,,$$

where $s\in\mathbb{N}$ or $s=\infty$, embedding (1.3) *is compact if and only if $b_k-a_k<\infty$ for all $k\in\mathbb{N}$, $s\in\mathbb{N}$ or $s=\infty$ and*

$$\lim_{k\to\infty}(b_k-a_k)=0\,. \tag{2.9}$$

Proof. If, for some $k\in\mathbb{N}$, $b_k-a_k=\infty$, then Ω contains infinitely many disjoint intervals $(a_k+m,a_k+m+1),m\in\mathbb{N}$, hence, by Theorem 2.1 embedding (1.3) is not compact. If Ω is bounded, then by Theorem 2.1 embedding (1.3) is compact.

Finally, let $b_k-a_k<\infty$ for all $k\in\mathbb{N}$ and $s=\infty$. Then condition (2.9) is necessary and sufficient for the compactness of embedding (1.3) by Example 2. □

Remark 5. Let $1\le p\le\infty,l\in\mathbb{N}$ and $\Omega\subset\mathbb{R}^n$ be an open set. Often another variant of Sobolev spaces is used, namely the space $\widetilde{W}_p^l(\Omega)$ of all functions $f\in L_1^{\mathrm{loc}}(\Omega)$ for which for any multi-index $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$ with $|\alpha|=\alpha_1+\cdots\alpha_n=l$ the weak derivatives $D_w^\alpha f$ exist on Ω and

$$\|f\|_{\widetilde{W}_p^l(\Omega)}=\sum_{|\alpha|\le l}\|D_w^\alpha f\|_{L_p(\Omega)}<\infty\,,$$

and, respectively, the Sobolev spaces $\overset{\circ}{\widetilde{W}}{}_p^l(\Omega)$ which are the closures in $\widetilde{W}_p^l(\Omega)$ of the space $C_0^\infty(\Omega)$ of all infinitely continuously differentiable functions on Ω with supports compact in Ω.

Clearly, $\widetilde{W}_p^l(\Omega)\subset W_p^l(\Omega)$. In general, $\widetilde{W}_p^l(\Omega)\ne W_p^l(\Omega)$ (for counter-examples see, for example, books [10], [11]), but, under rather weak assumptions on the regularity of Ω, $\widetilde{W}_p^l(\Omega)=W_p^l(\Omega)$. For example, this equality holds for bounded open sets with quasi-resolved boundaries ([2], Chapter 3). However, due to inequality (1.2),

$$\overset{\circ}{\widetilde{W}}{}_p^l(\Omega)=\overset{\circ}{W}{}_p^l(\Omega) \tag{2.10}$$

for any open set Ω.

Remark 6. The problem of the compactness of the embedding

$$\overset{\circ}{\widetilde{W}}{}_\infty^l(\Omega)\subset\widetilde{W}_\infty^m(\Omega) \tag{2.11}$$

is very close to the problem of the compactness of embedding (1.3).

Lemma 2.1 and Theorem 2.1 are also valid if the space $W_\infty^l(\Omega)$ is replaced by the space $\widetilde{W}_\infty^l(\Omega)$.

Assume that conditions 1) − 4) of Lemma 2.1 are satisfied for a subset S of the space $\widetilde{W}^l_\infty(\Omega)$. Then they are satisfied for the spaces $W^s_\infty(\Omega)$ with $s = 1, ..., l$. By Lemma 2.1 the set S is pre-compact in the spaces $W^s_\infty(\Omega)$ for $s = 1, ..., m$. Hence, S is pre-compact in the space $\bigcap_{s=1}^m W^s_\infty(\Omega) = \widetilde{W}^m_\infty(\Omega)$.

Assume that the assumptions of Theorem 2.1 (which are independent of l and m) are satisfied, then the embeddings

$$\overset{\circ}{\widetilde{W}}{}^l_\infty(\Omega) = \overset{\circ}{W}{}^l_\infty(\Omega) \subset W^s_\infty(\Omega)\,, \quad s = 1, ..., m,$$

are compact, hence, the embedding

$$\overset{\circ}{\widetilde{W}}{}^l_\infty(\Omega) \subset \bigcap_{s=1}^m W^s_\infty(\Omega) = \widetilde{W}^m_\infty(\Omega)$$

is also compact.

The necessity of condition (2.1) follows since the compactness of embedding (2.11) implies the compactness of embedding (1.3), because

$$\overset{\circ}{W}{}^l_\infty(\Omega) = \overset{\circ}{\widetilde{W}}{}^l_\infty(\Omega) \subset \widetilde{W}^m_\infty(\Omega) \subset W^m_\infty(\Omega)\,.$$

Acknowledgments

The research of V.I. Burenkov was supported by the Russian Science Foundation (project no. 19-11-00087) and was carried out in the V.A. Steklov Mathematical Institute of the Russian Academy of Sciences.

References

[1] O.V. Besov, V.P. Il'in, S.M. Nikol'skii, *Integral representations of functions and embedding theorems* (in Russian), Nauka, Moscow, 1975. English transl. Halsted Press Book, 1978-1979.

[2] V.I. Burenkov, *Sobolev spaces on domains.* B.G. Teubner, Stuttgart-Leipzig, 1998, 312 pp.

[3] V.I. Burenkov, V. Goldshtein, A. Ukhlov, *Conformal spectral stability estimates for the Dirichlet Laplacian.* Mathematische Nachrichten 288 (2015), no. 16, 1822–1833.

[4] V.I. Burenkov, P.D. Lamberti, *Spectral stability estimates for elliptic operators in domain perturbation problems.* Eurasian Mathematical Journal 1 (2008), 11–21.

[5] V.I. Burenkov, P.D. Lamberti, *Spectral stability of higher order uniformly elliptic operators.* Sobolev spaces in mathematics. II, Int. Math. Ser. (N.Y.), 9, Springer, New York, 2009, 69–102.

[6] V.I. Burenkov, P.D. Lamberti, *Spectral stability of the p-Laplacian.* Nonlinear Anal. 71 (2009), no. 5-6, 2227–2235.

[7] V.I. Burenkov, P.D. Lamberti, M. Lanza de Cristoforis, Spectral stability of non-negative self-adjoint operators. Sovrem. Mat. Fundam. Napravl. 15 (2006), 76–111 (Russian). English transl. in Journal of Mathematical Sciences 149 (2006).

[8] V.I. Burenkov, B.Th. Tuyen, *On spectral stability problem for a pair of self-adjoint elliptic differential operators on bounded open sets*, Eurasian Mathematical Journal 10 (2019). no. 3. 84–88.

[9] E.B. Davies, *Spectral theory and differential operators.* Cambridge University Press, Cambridge, 1995

[10] V.G. Maz'ya, *Sobolev spaces*, LGU, Leningrad, 1984 in Russian); English transl., Springer-Verlag, Springer Series in Soviet Mathematics, 1985.

[11] V.G. Maz'ya, S.V. Poborchii, *Differentiable functions on bad domains*, World Scientific Publishing, 1997.

[12] S.M. Nikol'ski, *Approximation of functions of several variables and embedding theorems*, 1-st ed., "Nauka", Moscow, 1969 (in Russian); 2-nd ed., "Nauka", Moscow, 1977 (Russian); English transl. of 1-st ed., Springer-Verlag, 1975.

Victor Ivanovich Burenkov
S.M. Nikol'skii Institute of Mathematics
RUDN University
6 Miklukho Maklay St
117198 Moscow, Russia
and
V.A. Steklov Institute of Mathematics
Russian Academy of Sciences
42 Vavilov St
117966 Moscow, Russia
E-mail: burenkov@cardiff.ac.uk.

Tamara Vasil'evna Tararykova
S.M. Nikol'skii institute of Mathematics
RUDN University
6 Miklukho Maklay St
117198 Moscow, Russia
and
Cardiff School of Mathematics
Cardiff University
Senghennydd Road
CF24 4AG Cardiff, United Kingdom
E-mail: tararykovat@cf.ac.uk

Received: 01.09.2019