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O'NEIL-TYPE INEQUALITIES FOR CONVOLUTIONS IN ANISOTROPIC LORENTZ SPACES

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Communicated by V.I. Burenkov

Key words: Young-O'Neil inequality, anisotropic Lorentz spases, convolution.

AMS Mathematics Subject Classification: 44A35, 46E30, 47G10.

Abstract. In this paper we study the boundedness of convolutions in the anisotropic Lorentz spaces.

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1 Introduction

R. O'Neil [16] investigated the boundedness of the following convolution operator

$$Af(y) = \int\limits_{\mathbb{R}^n} K(y-x)f(x)dx, \;\; y \in \mathbb{R}^n,$$

in the Lorentz spaces $L_{p,q} \equiv L_{p,q}(\mathbb{R}^n)$.

In particular, for $1 < p, r, q < \infty, \; 0 < h_1, h_2, h_3 \leq \infty, \; 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, and $\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}$ he obtained the following inequality:

$$\|K * f\|_{L_{q,h_1}} \leq c\|f\|_{L_{p,h_2}}\|K\|_{L_{r,h_3}}, \tag{1.1}$$

where $c > 0$ depends only on the numerical parametres.

This result was further developed in the works of R. Hunt [10], L. Yap [18], A. Blozinski [5], [6], [7], and other authors. See also the recent works [14], [15], [17].

In the paper of E. Nursultanov and S. Tikhonov [13] another method was used to obtain the generalization of inequality (1.1), which also covers the limiting cases. In particular, for $1 \leq h_1, h_2, h_3 \leq \infty$, and $\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}$ they proved that

a) if $1 < p < \infty$, then $\|K * f\|_{L_{p,h_1}} \leq c\|f\|_{L_{p,h_2}}\|K^{**}\|_{L_{1,h_3}}$,

b) if $p = 1$, then $\|(K * f)^{**}\|_{L_{1,h_1}} \leq c\|f^{**}\|_{L_{1,h_2}}\|K^{**}\|_{L_{1,h_3}}$,

c) if $p = \infty$, then $\|K * f\|_{L_{\infty,h_1}} \leq 4\|f\|_{L_{\infty,h_2}}\|K^{**}\|_{L_{1,h_3}}$,

where $c > 0$ depends only on the numerical parametrs, $K^{**}(t) = \frac{1}{t}\int\limits_0^t K^*(s)ds$, K^* is the decreasing rearrangement of K, and $L_{\infty,h}$ is the space introduced by C. Bennett and R. DeVore [2] (see also [1], [8]).

In the paper of M. Křepela [11] the boundedness of the convolution operator between weighted Lorentz spaces $\Gamma^p(v)$ and $\Gamma^q(w)$ was characterized for the range of parameters $p, q \in [1,\infty]$, or $p \in (0,1)$ and $q \in \{1,\infty\}$, or $p=\infty$ and $q \in (0,1)$. He provided Young-type convolution inequalities of the form

$$\|f*g\|_{\Gamma^q(w)} \le C\|f\|_{\Gamma^p(v)}\|g\|_Y, \quad f \in \Gamma^p(v), \ g \in Y,$$

characterizing the optimal rearrangement-invariant space Y for which the inequality is satisfied.

In the paper of D.V. Gorbachev, V.I. Ivanov, and S.Y. Tikhonov [9] the Young inequality was proved for a convolution defined by a generalized translation operator.

The aim of this paper is to investigate the Young-O'Neil-type inequality in the anisotropic Lorentz spaces.

2 Anisotropic Lorentz spaces $L_{\bar p,\bar q}([0,1]^2)$

Let $\bar p = (p_1,p_2)$, $\bar q = (q_1,q_2)$ be such that if $1 \le p_i < \infty$, then $1 \le q_i \le \infty$ and if $p_i = \infty$, then $q_i = \infty$, where $i = 1,2$.

Let $L_{\bar p,\bar q}([0,1]^2)$ be the anisotropic Lorentz space (see [12]), which is defined as the set of measurable functions $f(x_1,x_2)$ with respect to each variable with finite norm

$$\|f\|_{L_{\bar p,\bar q}} = \left(\int_0^1 \left(\int_0^1 \left(t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}} f^{*_1,*_2}(t_1,t_2)\right)^{q_1} \frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}} \frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}, \quad 1 \le q_i < \infty \ (i=1,2).$$

Here $f^{*_1,*_2}(t_1,t_2)$ is the multivariate decreasing rearrangement of $f(x_1,x_2)$ obtained by applying decreasing rearrangement $f^{*_1}(t_1,x_2)$ of $f(x_1,x_2)$ with respect to the first variable x_1, under fixed second variable x_2 and then with respect to x_2, under fixed first variable t_1 of $f^{*_1}(t_1,x_2)$. As usual the expression $\left(\int_0^1 (G(s))^q \frac{ds}{s}\right)^{\frac{1}{q}}$ for $q=\infty$ is understood as $\sup\limits_{s>0} G(s)$.

By C, C_i, c we will denote positive constants that may be different in different contexts and depend only on parameters p_i and q_i. We will write $F \asymp G$ if $F \le C_1 G$ and $G \le C_2 F$.

Let f be a measurable function locally integrable in $[0,1]^2$. We define the function $f^{\star_1,\star_2}(t_1,t_2)$ in $[0,1]^2$ as follows

$$f^{\star_1,\star_2}(t_1,t_2) = \frac{1}{t_1 t_2}\int_0^{t_2}\int_0^{t_1} f^{*_1,*_2}(s_1,s_2)ds_1 ds_2. \tag{2.1}$$

It has following properties.

Property 2.1.

$$f^{\star_1,\star_2}(t_1,t_2) = \sup_{\substack{|e_2|=t_2 \\ e_2\subset[0,1]}} \frac{1}{|e_2|}\int_{e_2} \sup_{\substack{|e_1|=t_1 \\ e_1\subset[0,1]}} \frac{1}{|e_1|}\int_{e_1} |f(x_1,x_2)|dx_1 dx_2, \tag{2.2}$$

where the suprema are taken over all compact sets $e_i \subset [0,1]$, whose measure $|e_i| = t_i$ $(i=1,2)$;

Proof. In the one-dimensional case, $f^\star$ can be written as (see [2], p. 53)

$$f^\star(t) = \sup_{\substack{|e|=t \\ e\subset[0,1]}} \frac{1}{|e|}\int_e |f(x)|dx.$$

Then

$$\sup_{\substack{|e_2|=t_2\\ e_2\subset[0,1]}} \frac{1}{|e_2|}\int_{e_2} \sup_{\substack{|e_1|=t_1\\ e_1\subset[0,1]}} \frac{1}{|e_1|}\int_{e_1} |f(x_1,x_2)|dx_1dx_2 = \sup_{\substack{|e_2|=t_2\\ e_2\subset[0,1]}} \frac{1}{|e_2|}\int_{e_2} |f^{\star_1}(t_1,x_2)|dx_2$$

$$= f^{\star_1,\star_2}(t_1,t_2).$$

□

Property 2.2.

$$f^{\star_1,\star_2}(t_1,t_2) \geq f^{*_1,*_2}(t_1,t_2). \tag{2.3}$$

Proof. Using formula (2.1) we have

$$f^{\star_1,\star_2}(t_1,t_2) = \frac{1}{t_1t_2}\int_0^{t_2}\int_0^{t_1} f^{*_1,*_2}(s_1,s_2)ds_1ds_2 \geq \frac{1}{t_1t_2}\int_0^{t_2} f^{*_1,*_2}(t_1,s_2)\int_0^{t_1} ds_1ds_2$$

$$= \frac{1}{t_2}\int_0^{t_2} f^{*_1,*_2}(t_1,s_2)ds_2 \geq \frac{1}{t_2} f^{*_1,*_2}(t_1,t_2)\int_0^{t_2} ds_2 = f^{*_1,*_2}(t_1,t_2).$$

□

Property 2.3.

$$\int_0^1\int_0^1 f(x_1,x_2)g(x_1,x_2)dx_1dx_2 \leq \int_0^1\int_0^1 f^{*_1,*_2}(x_1,x_2)g^{*_1,*_2}(x_1,x_2)dx_1dx_2. \tag{2.4}$$

Proof. Using the Hardy-Littlewood theorem (see [2]), we have

$$\int_0^1\int_0^1 f(x_1,x_2)g(x_1,x_2)dx_1dx_2 \leq \int_0^1\int_0^1 f^{*_1}(x_1,x_2)g^{*_1}(x_1,x_2)dx_1dx_2$$

$$\leq \int_0^1\int_0^1 f^{*_1,*_2}(x_1,x_2)g^{*_1,*_2}(x_1,x_2)dx_1dx_2.$$

□

Lemma 2.1 (Two-dimensional analogue of Hardy's inequality)**.** *Let* $1 < p_i < \infty$, $1 \leq q_i \leq \infty$, $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ $(i = 1, 2)$, *and* f *be a non-negative measurable function on* $[0,1]^2$, *then*

$$\begin{aligned} &\left(\int_0^1\left(\int_0^1\left(\frac{t_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2}}}{t_1t_2}\int_0^{t_2}\int_0^{t_1} f(s_1,s_2)ds_1ds_2\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\ &\leq p_1'p_2'\left(\int_0^1\left(\int_0^1\left(t_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2}} f(t_1,t_2)\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \end{aligned} \tag{2.5}$$

and

$$
\begin{gathered}
\left(\int_0^1\left(\int_0^1\left(t_1^{1-\frac{1}{p_1}}t_2^{1-\frac{1}{p_2}}\int_{t_2}^1\int_{t_1}^1 f(s_1,s_2)\frac{ds_1}{s_1}\frac{ds_2}{s_2}\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
\le p_1'p_2'\left(\int_0^1\left(\int_0^1\left(t_1^{1-\frac{1}{p_1}}t_2^{1-\frac{1}{p_2}}f(t_1,t_2)\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}.
\end{gathered}
\tag{2.6}
$$

Proof. Applying the generalized Minkowski inequality and sequentially in two variables the Hardy inequality (see [3]), we have

$$
\begin{aligned}
&\left(\int_0^1\left(\int_0^1\left(\frac{t_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2}}}{t_1t_2}\int_0^{t_2}\int_0^{t_1} f(s_1,s_2)ds_1ds_2\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&=\left(\int_0^1 t_2^{q_2\left(\frac{1}{p_2}-1\right)}\left\|t_1^{\frac{1}{p_1}-\frac{1}{q_1}-1}\int_0^{t_2}\int_0^{t_1} f(s_1,s_2)ds_1ds_2\right\|_{L_{q_1}}^{q_2}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&\le\left(\int_0^1 t_2^{q_2\left(\frac{1}{p_2}-1\right)}\left(\int_0^{t_2}\left\|t_1^{\frac{1}{p_1}-\frac{1}{q_1}-1}\int_0^{t_1} f(s_1,s_2)ds_1\right\|_{L_{q_1}}ds_2\right)^{q_2}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&=\left(\int_0^1 t_2^{q_2\left(\frac{1}{p_2}-1\right)}\left(\int_0^{t_2}\left(\int_0^1\left(t_1^{\frac{1}{p_1}-1}\int_0^{t_1} f(s_1,s_2)ds_1\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{1}{q_1}}ds_2\right)^{q_2}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&\le p_1'\left(\int_0^1\left(t_2^{\frac{1}{p_2}-1}\int_0^{t_2}\left(\int_0^1\left(t_1^{\frac{1}{p_1}}f(t_1,s_2)\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{1}{q_1}}ds_2\right)^{q_2}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&\le p_1'p_2'\left(\int_0^1 t_2^{\frac{1}{p_2}}\left(\int_0^1\left(t_1^{\frac{1}{p_1}}f(t_1,t_2)\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&=p_1'p_2'\left(\int_0^1\left(\int_0^1\left(t_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2}}f(t_1,t_2)\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}.
\end{aligned}
$$

Inequality (2.6) is proved similarly to inequality (2.5)

$$
\begin{aligned}
&\left(\int_0^1\left(\int_0^1\left(t_1^{1-\frac{1}{p_1}}t_2^{1-\frac{1}{p_2}}\int_{t_2}^1\int_{t_1}^1 f(s_1,s_2)\frac{ds_1}{s_1}\frac{ds_2}{s_2}\right)^{q_1}\frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}} \\
&\qquad=\left(\int_0^1 t_2^{\left(1-\frac{1}{p_2}\right)q_2}\left\|t_1^{1-\frac{1}{p_1}-\frac{1}{q_1}}\int_{t_2}^1\int_{t_1}^1 f(s_1,s_2)\frac{ds_1}{s_1}\frac{ds_2}{s_2}\right\|_{L_{q_1}}^{q_2}\frac{dt_2}{t_2}\right)^{\frac{1}{q_2}}
\end{aligned}
$$

$$\leq \left(\int_0^1 t_2^{\left(1-\frac{1}{p_2}\right)q_2} \left(\int_{t_2}^1 \left\| t_1^{1-\frac{1}{p_1}-\frac{1}{q_1}} \int_{t_1}^1 \frac{f(s_1,s_2)}{s_2} \frac{ds_1}{s_1} \right\|_{L_{q_1}} ds_2 \right)^{q_2} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}}$$

$$= \left(\int_0^1 t_2^{\left(1-\frac{1}{p_2}\right)q_2} \left(\int_{t_2}^1 \left(\int_0^1 \left(t_1^{1-\frac{1}{p_1}} \int_{t_1}^1 \frac{f(s_1,s_2)}{s_2} \frac{ds_1}{s_1} \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{1}{q_1}} ds_2 \right)^{q_2} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}}$$

$$\leq p_1' \left(\int_0^1 t_2^{\left(1-\frac{1}{p_2}\right)q_2} \left(\int_{t_2}^1 \left(\int_0^1 \left(t_1^{1-\frac{1}{p_1}} \frac{f(t_1,s_2)}{s_2} \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{1}{q_1}} ds_2 \right)^{q_2} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}}$$

$$= p_1' \left(\int_0^1 \left(t_2^{1-\frac{1}{p_2}} \int_{t_2}^1 \left(\int_0^1 \left(t_1^{1-\frac{1}{p_1}} f(t_1,s_2) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{1}{q_1}} \frac{ds_2}{s_2} \right)^{q_2} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}}$$

$$\leq p_1' p_2' \left(\int_0^1 \left(t_2^{1-\frac{1}{p_2}} \left(\int_0^1 \left(t_1^{1-\frac{1}{p_1}} f(t_1,t_2) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{1}{q_1}} \right)^{q_2} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}}$$

$$= p_1' p_2' \left(\int_0^1 \left(\int_0^1 \left(t_1^{1-\frac{1}{p_1}} t_2^{1-\frac{1}{p_2}} f(t_1,t_2) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}}.$$

□

Lemma 2.2 (see [4], p. 21)**.** *Let* $1 < p_i < \infty$, $\dfrac{1}{p_i} + \dfrac{1}{p^{TM}{}_i} = 1$ $(i = 1,2)$, $f \in L_{\bar{p}}\big([0,1]^2\big)$. *Then*

$$\|f\|_{L_{\bar{p}}} = \sup_{\substack{\|g\|_{L_{\bar{p}'}}=1 \\ 0 \leq g \downarrow}} \int_0^1 \int_0^1 f(t_1,t_2) g(t_1,t_2) dt_1 dt_2.$$

Lemma 2.3. *Let* $1 < p_i < \infty$, $1 \leq q_i \leq \infty$, $\dfrac{1}{p_i} + \dfrac{1}{p_i'} = 1$, $\dfrac{1}{q_i} + \dfrac{1}{q_i'} = 1$ $(i = 1,2)$. *Then*

$$\frac{1}{p^{TM}{}_1 p^{TM}{}_2} \|f\|_{L_{\bar{p},\bar{q}}} \leq \sup_{\substack{\|g\|_{L_{\bar{p}',\bar{q}'}}=1 \\ 0 \leq g \downarrow}} \int_0^1 \int_0^1 f^{*_1,*_2}(t_1,t_2) g(t_1,t_2) dt_1 dt_2 \leq \|f\|_{L_{\bar{p},\bar{q}}}.$$

Proof. We consider the quantity

$$J_{\bar{p},\bar{q}} = \sup_{0 \leq g \downarrow} \frac{\displaystyle\int_0^1 \int_0^1 f^{*_1,*_2}(t_1,t_2) g(t_1,t_2) dt_1 dt_2}{\left(\displaystyle\int_0^1 \left(\int_0^1 \left(t_1^{\frac{1}{p_1'}} t_2^{\frac{1}{p_2'}} g(t_1,t_2) \right)^{q^{\mathrm{TM}}{}_1} \frac{dt_1}{t_1} \right)^{\frac{q^{\mathrm{TM}}{}_2}{q^{\mathrm{TM}}{}_1}} \frac{dt_2}{t_2} \right)^{\frac{1}{q^{\mathrm{TM}}{}_2}}},$$

where the supremum is taken over all nonnegative functions $g(t_1,t_2)$, which are nonincreasing with respect to each variable.

By the Hölder inequality, we have

$$\int_0^1\int_0^1 f^{*_1,*_2}(t_1,t_2)g(t_1,t_2)dt_1dt_2 = \int_0^1\int_0^1 f^{*_1,*_2}(t_1,t_2)g^{*_1,*_2}(t_1,t_2)dt_1dt_2$$

$$= \int_0^1\int_0^1 \left(t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}} f^{*_1,*_2}(t_1,t_2)\right)\left(t_1^{\frac{1}{p'_1}} t_2^{\frac{1}{p'_2}} g^{*_1,*_2}(t_1,t_2)\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

$$\le \|f\|_{L_{\bar p,\bar q}}\|g\|_{L_{\bar p^{\mathrm{TM}},\bar q^{\mathrm{TM}}}}.$$

Thus, we get

$$J_{\bar p,\bar q} \le \|f\|_{L_{\bar p,\bar q}}.$$

Let us prove the reverse inequality.

$$J_{\bar p,\bar q} \ge \sup_{\substack{g(t_1,t_2)=\int_{t_1}^1\int_{t_2}^1 h(s_1,s_2)ds_2ds_1\\ h\ge 0}} \frac{\int_0^1\int_0^1 f^{*_1,*_2}(t_1,t_2)g(t_1,t_2)dt_1dt_2}{\left(\int_0^1\left(\int_0^1\left(t_1^{\frac{1}{p'_1}}t_2^{\frac{1}{p'_2}}g(t_1,t_2)\right)^{q^{\mathrm{TM}}_1}\frac{dt_1}{t_1}\right)^{\frac{q^{\mathrm{TM}}_2}{q^{\mathrm{TM}}_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q^{\mathrm{TM}}_2}}}$$

$$= \sup_{h\ge 0}\frac{\int_0^1\int_0^1 f^{*_1,*_2}(t_1,t_2)\int_{t_1}^1\int_{t_2}^1 h(s_1,s_2)ds_2ds_1dt_1dt_2}{\left(\int_0^1\left(\int_0^1\left(t_1^{\frac{1}{p'_1}}t_2^{\frac{1}{p'_2}}\int_{t_1}^1\int_{t_2}^1 h(s_1,s_2)ds_2ds_1\right)^{q^{\mathrm{TM}}_1}\frac{dt_1}{t_1}\right)^{\frac{q^{\mathrm{TM}}_2}{q^{\mathrm{TM}}_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q^{\mathrm{TM}}_2}}}.$$

By Lemma 2.1, we obtain

$$\left(\int_0^1\left(\int_0^1\left(t_1^{\frac{1}{p'_1}}t_2^{\frac{1}{p'_2}}\int_{t_1}^1\int_{t_2}^1 h(s_1,s_2)ds_2ds_1\right)^{q^{\mathrm{TM}}_1}\frac{dt_1}{t_1}\right)^{\frac{q^{\mathrm{TM}}_2}{q^{\mathrm{TM}}_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q^{\mathrm{TM}}_2}}$$

$$\le c\left(\int_0^1\left(\int_0^1\left(t_1^{\frac{1}{p'_1}+1}t_2^{\frac{1}{p'_2}+1}h(t_1,t_2)\right)^{q^{\mathrm{TM}}_1}\frac{dt_1}{t_1}\right)^{\frac{q^{\mathrm{TM}}_2}{q^{\mathrm{TM}}_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{q^{\mathrm{TM}}_2}}$$

$$= c\left(\int_0^1\left(\int_0^1 h^{q^{\mathrm{TM}}_1}(t_1,t_2)t_1^{\alpha_1}t_2^{\alpha_2}dt_1\right)^{\frac{q^{\mathrm{TM}}_2}{q^{\mathrm{TM}}_1}}dt_2\right)^{\frac{1}{q^{\mathrm{TM}}_2}},$$

where $\alpha_i = q^{\mathrm{TM}}{}_1\left(\frac{1}{q_i}+\frac{1}{p'_i}\right), i=1,2$ and $c = p'_1p'_2$.

Hence, using the dual representation in anisotropic space $L_{\bar{q}}$ (Lemma 2.2) we obtain

$$J_{\bar{p},\bar{q}} \geq \frac{1}{c}\sup_{h\geq 0}\frac{\int\limits_0^1\int\limits_0^1 h(t_1,t_2)\int\limits_0^{t_1}\int\limits_0^{t_2} f^{*_1,*_2}(s_1,s_2)ds_2ds_1dt_1dt_2}{\left(\int\limits_0^1\left(\int\limits_0^1 h^{q^{\mathrm{TM}}{}_1}(t_1,t_2)t_1^{\alpha_1}t_2^{\alpha_2}dt_1\right)^{\frac{q^{\mathrm{TM}}{}_2}{q^{\mathrm{TM}}{}_1}}dt_2\right)^{\frac{1}{q^{\mathrm{TM}}{}_2}}}$$

$$= \frac{1}{c}\sup_{h\geq 0}\frac{\int\limits_0^1\int\limits_0^1 h(t_1,t_2)t_1^{\frac{\alpha_1}{q^{\mathrm{TM}}{}_1}}t_2^{\frac{\alpha_2}{q^{\mathrm{TM}}{}_1}}\int\limits_0^{t_1}\int\limits_0^{t_2} f^{*_1,*_2}(s_1,s_2)ds_2ds_1t_1^{-\frac{\alpha_1}{q^{\mathrm{TM}}{}_1}}t_2^{-\frac{\alpha_2}{q^{\mathrm{TM}}{}_1}}dt_1dt_2}{\left(\int\limits_0^1\left(\int\limits_0^1 h^{q^{\mathrm{TM}}{}_1}(t_1,t_2)t_1^{\alpha_1}t_2^{\alpha_2}dt_1\right)^{\frac{q^{\mathrm{TM}}{}_2}{q^{\mathrm{TM}}{}_1}}dt_2\right)^{\frac{1}{q^{\mathrm{TM}}{}_2}}}$$

$$= \frac{1}{c}\left\|t_1^{-\frac{\alpha_1}{q^{\mathrm{TM}}{}_1}}t_2^{-\frac{\alpha_2}{q^{\mathrm{TM}}{}_1}}\int\limits_0^{t_1}\int\limits_0^{t_2} f^{*_1,*_2}(s_1,s_2)ds_2ds_1\right\|_{L_{\bar{q}}}$$

$$= \frac{1}{c}\left(\int\limits_0^1\left(\int\limits_0^1\left(t_1^{-\frac{\alpha_1}{q^{\mathrm{TM}}{}_1}}t_2^{-\frac{\alpha_2}{q^{\mathrm{TM}}{}_1}}\int\limits_0^{t_1}\int\limits_0^{t_2} f^{*_1,*_2}(s_1,s_2)ds_2ds_1\right)^{q_1}dt_1\right)^{\frac{q_2}{q_1}}dt_2\right)^{\frac{1}{q_2}}$$

$$\geq \frac{1}{c}\left(\int\limits_0^1\left(\int\limits_0^1\left(t_1^{1-\frac{\alpha_1}{q^{\mathrm{TM}}{}_1}}t_2^{1-\frac{\alpha_2}{q^{\mathrm{TM}}{}_1}}f^{*_1,*_2}(t_1,t_2)\right)^{q_1}dt_1\right)^{\frac{q_2}{q_1}}dt_2\right)^{\frac{1}{q_2}}$$

$$= \frac{1}{c}\left(\int\limits_0^1\left(\int\limits_0^1\left(t_1^{\frac{1}{p_1}-\frac{1}{q_1}}t_2^{\frac{1}{p_2}-\frac{1}{q_2}}f^{*_1,*_2}(t_1,t_2)\right)^{q_1}dt_1\right)^{\frac{q_2}{q_1}}dt_2\right)^{\frac{1}{q_2}} = \frac{1}{c}\|f\|_{L_{\bar{p},\bar{q}}}.$$

□

Lemma 2.4. *Let* $1 < p_i < \infty, 1 \leq q_i \leq \infty$ $(i=1,2)$. *Then*

$$\|f\|_{L_{\bar{p},\bar{q}}} \leq \|f^{\star_1,\star_2}\|_{L_{\bar{p},\bar{q}}} \leq p'_1p'_2\|f\|_{L_{\bar{p},\bar{q}}}$$

Proof. Applying formulas (2.3) and (2.1) we have

$$\|f\|_{L_{\overline{p},\overline{q}}} = \left(\int_0^1 \left(\int_0^1 \left(t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}} f^{*_1,*_2}(t_1,t_2) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}}$$

$$\leq \left(\int_0^1 \left(\int_0^1 \left(t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}} f^{\star_1,\star_2}(t_1,t_2) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}} = \|f^{\star_1,\star_2}\|_{L_{\overline{p},\overline{q}}}$$

$$= \left(\int_0^1 \left(\int_0^1 \left(\frac{t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}}}{t_1 t_2} \int_0^{t_2} \int_0^{t_1} f^{*_1,*_2}(s_1,s_2) ds_1 ds_2 \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}}$$

$$\leq p_1' p_2' \left(\int_0^1 \left(\int_0^1 \left(t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}} f^{*_1,*_2}(t_1,t_2) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}} = p_1' p_2' \left\|f\right\|_{L_{\overline{p},\overline{q}}},$$

where we have used Lemma 2.1. □

3 Main result

Let f and K be measurable functions on $[0,1]^2$ and $[-1,1]^2$ respectively. Let the integral

$$\int_0^1 \int_0^1 f(y_1,y_2) K(x_1 - y_1, x_2 - y_2) dy_1 dy_2, \tag{3.1}$$

exists for almost every $(x_1,x_2) \in [0,1]^2$ and it is called the convolution of the functions f, K, denoted by $f * K$.

Lemma 3.1. *Let f and g be measurable on $[0,1]^2$ and K be measurable on $[-1,1]^2$ functions. Then*

$$\int_0^1 \int_0^1 g(t_1,t_2)(f*K)^{*_1,*_2}(t_1,t_2) dt_1 dt_2$$

$$\leq \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \int_0^1 \int_0^1 f^{*_1,*_2}(s_1,s_2) K^{\star_1,\star_2}(\max(s_1,t_1),\max(s_2,t_2)) ds_1 ds_2 dt_1 dt_2.$$

Proof. Using properties (2.2), (2.3), (2.4), we obtain

$$I = \int_0^1 \int_0^1 g(t_1,t_2)(f*K)^{*_1,*_2}(t_1,t_2) dt_1 dt_2$$

$$\leq \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2)(f*K)^{\star_1,\star_2}(t_1,t_2) dt_1 dt_2$$

$$= \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \sup_{\substack{|e_2|=t_2 \\ e_2 \subset [0,1]}} \frac{1}{|e_2|} \int_{e_2} \sup_{\substack{|e_1|=t_1 \\ e_1 \subset [0,1]}} \frac{1}{|e_1|} \int_{e_1} \left|(f*K)(x_1,x_2)\right| dx_1 dx_2 dt_1 dt_2.$$

By the definition of the supremum, for any $\varepsilon > 0$ there exists a compact $e_1(t_1,x_2) \subset [0,1]$ with measure $|e_1(t_1,x_2)| = t_1$ such that

$$\sup_{|e_1|=t_1} \int_{e_1} |(f*K)(x_1,x_2)|dx_1 \le (1+\varepsilon) \int_{e_1(t_1,x_2)} |(f*K)(x_1,x_2)|dx_1.$$

Hence

$$I \le (1+\varepsilon) \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \sup_{|e_2|=t_2} \frac{1}{|e_2|} \int_{e_2} \frac{1}{|e_1(t_1,x_2)|} \int_{e_1(t_1,x_2)} \big|(f*K)(x_1,x_2)\big| dx_1dx_2dt_1dt_2$$

$$= (1+\varepsilon) \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \sup_{|e_2|=t_2} \frac{1}{|e_2|} \int_{e_2} \frac{1}{|e_1(t_1,x_2)|}$$

$$\times \int_{e_1(t_1,x_2)} \left| \int_0^1 \int_0^1 f(y_1,y_2) K(x_1-y_1,x_2-y_2) dy_1dy_2 \right| dx_1dx_2dt_1dt_2$$

$$\le (1+\varepsilon) \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \sup_{|e_2|=t_2} \int_0^1 \int_0^1 \big|f(y_1,y_2)\big| \frac{1}{|e_2|} \int_{e_2} \frac{1}{|e_1(t_1,x_2)|}$$

$$\times \int_{e_1(t_1,x_2)} \big|K(x_1-y_1,x_2-y_2)\big| dx_1dx_2dy_1dy_2dt_1dt_2$$

$$\le (1+\varepsilon) \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \sup_{|e_2|=t_2} \int_0^1 \int_0^1 f^{*_1,*_2}(s_1,s_2)$$

$$\times \left(\frac{1}{|e_2|} \int_{e_2} \frac{1}{|e_1(t_1,x_2)|} \int_{e_1(t_1,x_2)} \big|K(x_1-\cdot,x_2-\cdot)\big| dx_1dx_2 \right)^{\star_1,\star_2} (s_1,s_2) ds_1ds_2dt_1dt_2$$

$$= (1+\varepsilon) \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \sup_{|e_2|=t_2} \int_0^1 \int_0^1 f^{*_1,*_2}(s_1,s_2) \sup_{\substack{|\omega_2|=s_2 \\ \omega_2 \subset [0,1]}} \frac{1}{|\omega_2|} \int_{\omega_2} \sup_{\substack{|\omega_1|=t_1 \\ \omega_1 \subset [0,1]}} \frac{1}{|\omega_1|}$$

$$\times \int_{\omega_1} \left| \frac{1}{|e_2|} \int_{e_2} \frac{1}{|e_1(t_1,x_2)|} \int_{e_1(t_1,x_2)} K(x_1-y_1,x_2-y_2) dy_1dy_2 \right| dx_1dx_2ds_1ds_2dt_1dt_2.$$

Similarly, there is a compact $\omega_1(s_1,y_2) \subset [0,1]$ with the measure $|\omega_1(s_1,y_2)| = s_1$ such that

$$I \le (1+\varepsilon)^2 \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \sup_{|e_2|=t_2} \int_0^1 \int_0^1 f^{*_1,*_2}(s_1,s_2) \sup_{\substack{|\omega_2|=s_2 \\ \omega_2 \subset [0,1]}} \frac{1}{|\omega_2|} \int_{\omega_2} \frac{1}{|\omega_1(s_1,y_2)|}$$

$$\times \int_{\omega_1(s_1,y_2)} \left| \frac{1}{|e_2|} \int_{e_2} \frac{1}{|e_1(t_1,x_2)|} \int_{e_1(t_1,x_2)} K(x_1-y_1,x_2-y_2) dy_1dy_2 \right| dx_1dx_2ds_1ds_2dt_1dt_2$$

$$\le (1+\varepsilon)^2 \int_0^1\int_0^1 g^{*_1,*_2}(t_1,t_2) \sup_{|e_2|=t_2} \int_0^1\int_0^1 f^{*_1,*_2}(s_1,s_2) \sup_{\substack{|\omega_2|=s_2\\ \omega_2\subset[0,1]}} \frac{1}{|\omega_2|}\int_{\omega_2} \frac{1}{|e_2|}\int_{e_2} \frac{1}{|\omega_1(s_1,y_2)|}$$

$$\times \int_{\omega_1(s_1,y_2)} \frac{1}{|e_1(t_1,x_2)|} \int_{e_1(t_1,x_2)} |K(x_1-y_1,x_2-y_2)| dx_1dy_1dx_2dy_2ds_1ds_2dt_1dt_2.$$

Since in this inequality ε is an arbitrary positive number, the desired inequality follows

$$I \le \int_0^1\int_0^1 g^{*_1,*_2}(t_1,t_2) \sup_{|e_2|=t_2} \int_0^1\int_0^1 f^{*_1,*_2}(s_1,s_2) \sup_{\substack{|\omega_2|=s_2\\ \omega_2\subset[0,1]}} \frac{1}{|\omega_2|}\int_{\omega_2} \frac{1}{|e_2|}\int_{e_2} \frac{1}{|\omega_1(s_1,y_2)|}$$

$$\times \int_{\omega_1(s_1,y_2)} \frac{1}{|e_1(t_1,x_2)|} \int_{e_1(t_1,x_2)} |K(x_1-y_1,x_2-y_2)| dx_1dy_1dx_2dy_2ds_1ds_2dt_1dt_2.$$

We consider

$$D(s_1,t_1,x_2,y_2) = \frac{1}{|\omega_1(s_1,y_2)|} \int_{\omega_1(s_1,y_2)} \frac{1}{|e_1(t_1,x_2)|} \int_{e_1(t_1,x_2)} |K(x_1-y_1,x_2-y_2)| dx_1dy_1$$

$$\le \begin{cases} \dfrac{1}{\omega_1(s_1,y_2)} \displaystyle\int_{\omega_1(s_1,y_2)} \sup_{|e_1|=t_1} \frac{1}{|e_1|}\int_{e_1} |K(x_1,x_2-y_2)| dx_1dy_2, & s_1\le t_1,\\ \dfrac{1}{e_1(t_1,x_2)} \displaystyle\int_{e(t_1,x_2)} \sup_{|\omega_1|=s_1} \frac{1}{|\omega_1|}\int_{\omega_1} |K(y_1,x_2-y_2)| dy_1dx_2, & s_1>t_1. \end{cases}$$

$$\le \begin{cases} \displaystyle\sup_{|e_1|=t_1} \frac{1}{|e_1|}\int_{e_1} |K(x_1,x_2-y_2)| dx_1, & s_1\le t_1,\\ \displaystyle\sup_{|\omega_1|=s_1} \frac{1}{|\omega_1|}\int_{\omega_1} |K(y_1,x_2-y_2)| dy_1, & s_1>t_1. \end{cases}$$

Now we estimate the quantity

$$\Phi(t_1,s_1,t_2,s_2) = \frac{1}{|\omega_2|}\int_{\omega_2} \frac{1}{|e_2|}\int_{e_2} \frac{1}{|\omega_1(s_1,y_2)|} \int_{\omega_1(s_1,y_2)} \frac{1}{|e_1(t_1,x_2)|}$$

$$\times \int_{e_1(t_1,x_2)} |K(x_1-y_1,x_2-y_2)| dx_1dy_1dx_2dy_2$$

$$\le \begin{cases} \displaystyle\sup_{|e_2|=t_2} \frac{1}{|e_2|}\int_{e_2} \sup_{|e_1|=t_1} \frac{1}{|e_1|}\int_{e_1} |K(x_1,x_2)| dx_1dx_2, & s_1\le t_1, s_2\le t_2,\\ \displaystyle\sup_{|\omega_2|=s_2} \frac{1}{|\omega_2|}\int_{\omega_2} \sup_{|e_1|=t_1} \frac{1}{|e_1|}\int_{e_1} |K(x_1,y_2)| dx_1dy_2, & s_1\le t_1, s_2>t_2,\\ \displaystyle\sup_{|e_2|=t_2} \frac{1}{|e_2|}\int_{e_2} \sup_{|\omega_1|=s_1} \frac{1}{|\omega_1|}\int_{\omega_1} |K(y_1,x_2)| dy_1dx_2, & s_1>t_1, s_2\le t_2,\\ \displaystyle\sup_{|\omega_2|=s_2} \frac{1}{|\omega_2|}\int_{\omega_2} \sup_{|\omega_1|=s_1} \frac{1}{|\omega_1|}\int_{\omega_1} |K(y_1,y_2)| dy_1dy_2, & s_1>t_1, s_2>t_2 \end{cases}$$

$$= K^{\star_1,\star_2}(\max(s_1,t_1),\max(s_2,t_2)).$$

Thus,

$$I \leq \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \int_0^1 \int_0^1 f^{*_1,*_2}(s_1,s_2) K^{\star_1,\star_2}(\max(s_1,t_1),\max(s_2,t_2)) ds_1 ds_2 dt_1 dt_2.$$

$\square$

Theorem 3.1. *Let* $1 < q_i < \infty$, $1 \leq p_i, r_i, h_i, \xi_i, \eta_i < \infty$, *and* $1 + \frac{1}{q_i} = \frac{1}{p_i} + \frac{1}{r_i}$, $\frac{1}{h_i} = \frac{1}{\xi_i} + \frac{1}{\eta_i}$ $(i = 1,2)$. *Suppose that* f *and* K *are respectively measurable on* $[0,1]^2$ *and* $[-1,1]^2$ *functions such that* $f^{\star_1,\star_2} \in L_{\bar{p},\bar{\xi}}([0,1]^2)$ *and* $K^{\star_1,\star_2} \in L_{\bar{r},\bar{\eta}}([0,1]^2)$. *Then* $f * K \in L_{\bar{q},\bar{h}}([0,1]^2)$ *and*

$$\|f * K\|_{L_{\bar{q},\bar{h}}} \leq 4(q_1' q_2')^2 \|f^{\star_1,\star_2}\|_{L_{\bar{p},\bar{\xi}}} \|K^{\star_1,\star_2}\|_{L_{\bar{r},\bar{\eta}}}. \tag{3.2}$$

Proof. By Lemmas 2.3 and 3.1, we have

$$\begin{aligned}
\|f * K\|_{L_{\bar{q},\bar{h}}} &\leq q_1' q_2' \sup_{\|g\|_{L_{\bar{q}^{\mathrm{TM}},\bar{h}^{\mathrm{TM}}}}=1} \int_0^1 \int_0^1 g(t_1,t_2)(K * f)^{*_1,*_2}(t_1,t_2) dt_1 dt_2 \\
&\leq q_1' q_2' \sup_{\|g\|_{L_{\bar{q}^{\mathrm{TM}},\bar{h}^{\mathrm{TM}}}}=1} \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \int_0^1 \int_0^1 f^{*_1,*_2}(s_1,s_2) \\
&\times K^{\star_1,\star_2}(\max(s_1,t_1),\max(s_2,t_2)) ds_1 ds_2 dt_1 dt_2 \\
&= q_1' q_2' \sup_{\|g\|_{L_{\bar{q}^{\mathrm{TM}},\bar{h}^{\mathrm{TM}}}}=1} \int_0^1 \int_0^1 g^{*_1,*_2}(t_1,t_2) \left(\int_0^{t_1} \int_0^{t_2} f^{*_1,*_2}(s_1,s_2) K^{\star_1,\star_2}(t_1,t_2) ds_1 ds_2 \right. \\
&+ \int_0^{t_1} \int_{t_2}^1 f^{*_1,*_2}(s_1,s_2) K^{\star_1,\star_2}(t_1,s_2) ds_1 ds_2 \\
&+ \int_{t_1}^1 \int_0^{t_2} f^{*_1,*_2}(s_1,s_2) K^{\star_1,\star_2}(s_1,t_2) ds_1 ds_2 \\
&+ \left. \int_{t_1}^1 \int_{t_2}^1 f^{*_1,*_2}(s_1,s_2) K^{\star_1,\star_2}(s_1,s_2) ds_1 ds_2 \right) dt_1 dt_2.
\end{aligned}$$

Changing the order of integration and taking into account the definition of $f^{\star_1,\star_2}$, we get

$$\begin{aligned}
\|f * K\|_{L_{\bar{q},\bar{h}}} &\leq q_1' q_2' \sup_{\|g\|_{L_{\bar{q}',\bar{h}'}}=1} \int_0^1 \int_0^1 K^{\star_1,\star_2}(t_1,t_2) \left(g^{*_1,*_2}(t_1,t_2) \int_0^{t_1} \int_0^{t_2} f^{*_1,*_2}(s_1,s_2) ds_1 ds_2 \right. \\
&+ \int_0^{t_1} f^{*_1,*_2}(s_1,t_2) ds_1 \int_0^{t_2} g^{*_1,*_2}(t_1,s_2) ds_2 + \int_0^{t_1} g^{*_1,*_2}(s_1,t_2) ds_1 \\
&\times \left. \int_0^{t_2} f^{*_1,*_2}(t_1,s_2) ds_2 + f^{*_1,*_2}(t_1,t_2) \int_0^{t_1} \int_0^{t_2} g^{*_1,*_2}(s_1,s_2) ds_1 ds_2 \right) dt_1 dt_2 \\
&= q_1' q_2' \sup_{\|g\|_{L_{\bar{q}^{\mathrm{TM}},\bar{h}^{\mathrm{TM}}}}=1} \int_0^1 \int_0^1 t_1 t_2 K^{\star_1,\star_2}(t_1,t_2) \Big(g^{*_1,*_2}(t_1,t_2) f^{\star_1,\star_2}(t_1,t_2)
\end{aligned}$$

$$
\begin{aligned}
&+ f^{\star_1,*_2}(t_1,t_2)g^{*_1,\star_2}(t_1,t_2) + g^{\star_1,*_2}(t_1,t_2)f^{*_1,\star_2}(t_1,t_2) \\
&+ f^{*_1,*_2}(t_1,t_2)g^{\star_1,\star_2}(t_1,t_2)\Big)dt_1dt_2 \\
&\leq q_1'q_2' \sup_{\|g\|_{L_{\bar{q}\mathrm{TM},\bar{h}\mathrm{TM}}}=1} \int_0^1\int_0^1 t_1t_2K^{\star_1,\star_2}(t_1,t_2)g^{\star_1,\star_2}(t_1,t_2)f^{\star_1,\star_2}(t_1,t_2)dt_1dt_2 \\
&= q_1'q_2' \sup_{\|g\|_{L_{\bar{q}\mathrm{TM},\bar{h}\mathrm{TM}}}=1} \int_0^1\int_0^1 t_1^{\frac{1}{p_1}-\frac{1}{\xi_1}}t_2^{\frac{1}{p_2}-\frac{1}{\xi_2}}f^{\star_1,\star_2}(t_1,t_2)t_1^{\frac{1}{r_1}-\frac{1}{\eta_1}}t_2^{\frac{1}{r_2}-\frac{1}{\eta_2}}K^{\star_1,\star_2}(t_1,t_2) \\
&\times t_1^{\frac{1}{q_1'}-\frac{1}{h_1'}}t_2^{\frac{1}{q_2'}-\frac{1}{h_2'}}g^{\star_1,\star_2}(t_1,t_2)dt_1dt_2.
\end{aligned}
$$

Applying Hölder's inequality and Lemma 2.4, we derive

$$
\begin{aligned}
\|f*K\|_{L_{\bar{q},\bar{h}}} &\leq q_1'q_2' \sup_{\|g\|_{L_{\bar{q}',\bar{h}'}}=1} \|f^{\star_1,\star_2}\|_{L_{\bar{p},\bar{\xi}}}\|K^{\star_1,\star_2}\|_{L_{\bar{r},\bar{\eta}}}\|g^{\star_1,\star_2}\|_{L_{\bar{q}',\bar{h}'}} \\
&\leq (q_1'q_2')^2 \sup_{\|g\|_{L_{\bar{q}',\bar{h}'}}=1} \|f^{\star_1,\star_2}\|_{L_{\bar{p},\bar{\xi}}}\|K^{\star_1,\star_2}\|_{L_{\bar{r},\bar{\eta}}}\|g\|_{L_{\bar{q}',\bar{h}'}} \\
&= (q_1'q_2')^2\|f^{\star_1,\star_2}\|_{L_{\bar{p},\bar{\xi}}}\|K^{\star_1,\star_2}\|_{L_{\bar{r},\bar{\eta}}}.
\end{aligned}
$$

□

Remark 1. This theorem also covers the limiting cases when at least one of the parameters p_i, r_i is equal to 1. In the case $p_i > 1$, $r_i > 1$, the functions $f^{\star_1,\star_2}$ and $K^{\star_1,\star_2}$ in (3.2) can be replaced by the functions f and K, respectively.

Let us give an example showing the sharpness of the result of Theorem 2.1, where in inequality (3.2) for $1 < q_i = p_i < \infty$ the factor $\|K^{\star_1,\star_2}\|_{L_{\bar{1},\bar{\eta}}}$ cannot be replaced by $\|K\|_{L_{\bar{1},\bar{\eta}}}$. That is, for $1 < q_i = p_i < \infty$ and $\dfrac{1}{h_i} = \dfrac{1}{\xi_i} + \dfrac{1}{\eta_i}$ the inequality

$$
\|f*K\|_{L_{\bar{q},\bar{h}}} \leq c\|f^{\star_1,\star_2}\|_{L_{\bar{q},\bar{\xi}}}\|K\|_{L_{\bar{1},\bar{\eta}}} \tag{3.3}
$$

where c>0 is independent of f and K, does not hold. Also in the case $1 < q_i = r_i < \infty$ and $\dfrac{1}{h_i} = \dfrac{1}{\xi_i} + \dfrac{1}{\eta_i}$ the norm $\|f^{\star_1,\star_2}\|_{L_{\bar{1},\bar{\xi}}}$ cannot be replaced by $\|f\|_{L_{\bar{1},\bar{\xi}}}$, that is, the following inequality

$$
\|f*K\|_{L_{\bar{q},\bar{h}}} \leq c\|f\|_{L_{\bar{1},\bar{\xi}}}\|K^{\star_1,\star_2}\|_{L_{\bar{q},\bar{\eta}}} \tag{3.4}
$$

where c>0 is independent of f and K, does not hold.

Example 1. Let $1 < q_1, q_2 < \infty$. For sufficiently large $N_1, N_2 \in \mathbb{N}$ we define

$$
f(t_1,t_2) = \left(\min\left(N_1, \frac{1}{t_1}\right)\right)^{\frac{1}{q_1}} \left(\min\left(N_2, \frac{1}{t_2}\right)\right)^{\frac{1}{q_2}}
$$

and

$$
K(t_1,t_2) = \min\left(N_1, \frac{1}{t_1}\right)\min\left(N_2, \frac{1}{t_2}\right).
$$

Then

$$
\|f^{\star_1,\star_2}\|_{L_{\bar{q},\bar{\xi}}}\|K\|_{L_{\bar{1},\bar{\eta}}} \approx (\ln N_1)^{\frac{1}{h_1}}(\ln N_2)^{\frac{1}{h_2}}.
$$

and

$$
\|f*K\|_{L_{\bar{q},\bar{h}}} \geq c_1(\ln N_1)^{1+\frac{1}{h_1}}(\ln N_2)^{1+\frac{1}{h_2}}
$$

Thus, (3.3) implies

$$(\ln N_1)(\ln N_2) \le c,$$

i.e. we arrive at a contradiction.

Proof. We have

$$\|f^{\star_1,\star_2}\|_{L_{\bar q,\bar \xi}} \approx \|f\|_{L_{\bar q,\bar \xi}} =$$

$$= \left(\int_0^1 \left(\int_0^1 \left(t_1^{\frac{1}{q_1}-\frac{1}{\xi_1}} t_2^{\frac{1}{q_2}-\frac{1}{\xi_2}} \left(\min\left(N_1, \frac{1}{t_1}\right)\right)^{\frac{1}{q_1}} \left(\min\left(N_2, \frac{1}{t_2}\right)\right)^{\frac{1}{q_2}} \right)^{\xi_1} dt_1 \right)^{\frac{\xi_2}{\xi_1}} dt_2 \right)^{\frac{1}{\xi_2}}$$

$$= \left(\int_0^1 \left(t_1^{\frac{1}{q_1}-\frac{1}{\xi_1}} \left(\min\left(N_1, \frac{1}{t_1}\right)\right)^{\frac{1}{q_1}} \right)^{\xi_1} dt_1 \right)^{\frac{1}{\xi_1}}$$

$$\times \left(\int_0^1 \left(t_2^{\frac{1}{q_2}-\frac{1}{\xi_2}} \left(\min\left(N_2, \frac{1}{t_2}\right)\right)^{\frac{1}{q_2}} \right)^{\xi_2} dt_2 \right)^{\frac{1}{\xi_2}}$$

$$= \left(\int_0^{\frac{1}{N_1}} \left(t_1^{\frac{1}{q_1}-\frac{1}{\xi_1}} N_1^{\frac{1}{q_1}} \right)^{\xi_1} dt_1 + \int_{\frac{1}{N_1}}^1 \left(t_1^{\frac{1}{q_1}-\frac{1}{\xi_1}} t_1^{-\frac{1}{q_1}} \right)^{\xi_1} dt_1 \right)^{\frac{1}{\xi_1}}$$

$$\times \left(\int_0^{\frac{1}{N_2}} \left(t_2^{\frac{1}{q_2}-\frac{1}{\xi_2}} N_2^{\frac{1}{q_2}} \right)^{\xi_2} dt_2 + \int_{\frac{1}{N_2}}^1 \left(t_2^{\frac{1}{q_2}-\frac{1}{\xi_2}} t_2^{-\frac{1}{q_2}} \right)^{\xi_2} dt_2 \right)^{\frac{1}{\xi_2}}$$

$$= \left(\frac{q_1}{\xi_1} + \ln N_1 \right)^{\frac{1}{\xi_1}} \left(\frac{q_2}{\xi_2} + \ln N_2 \right)^{\frac{1}{\xi_2}} \approx (\ln N_1)^{\frac{1}{\xi_1}} (\ln N_2)^{\frac{1}{\xi_2}}$$

and

$$\|K\|_{L_{\bar 1,\bar \eta}} = \left(\int_0^1 \left(\int_0^1 \left(t_1^{1-\frac{1}{\eta_1}} t_2^{1-\frac{1}{\eta_2}} \min\left(N_1, \frac{1}{t_1}\right) \min\left(N_2, \frac{1}{t_2}\right)\right)^{\eta_1} dt_1 \right)^{\frac{\eta_2}{\eta_1}} dt_2 \right)^{\frac{1}{\eta_2}}$$

$$= \left(\int_0^1 \left(t_1^{1-\frac{1}{\eta_1}} \min\left(N_1, \frac{1}{t_1}\right)\right)^{\eta_1} dt_1 \right)^{\frac{1}{\eta_1}} \left(\int_0^1 \left(t_2^{1-\frac{1}{\eta_2}} \min\left(N_2, \frac{1}{t_2}\right)\right)^{\eta_2} dt_2 \right)^{\frac{1}{\eta_2}}$$

$$= \left(\int_0^{\frac{1}{N_1}} \left(t_1^{1-\frac{1}{\eta_1}} N_1 \right)^{\eta_1} dt_1 + \int_{\frac{1}{N_1}}^1 \left(t_1^{1-\frac{1}{\eta_1}} t_1^{-1} \right)^{\eta_1} dt_1 \right)^{\frac{1}{\eta_1}}$$

$$\times \left(\int_0^{\frac{1}{N_2}} \left(t_2^{1-\frac{1}{\eta_2}} N_2 \right)^{\eta_2} dt_2 + \int_{\frac{1}{N_2}}^1 \left(t_2^{1-\frac{1}{\eta_2}} t_2^{-1} \right)^{\eta_2} dt_2 \right)^{\frac{1}{\eta_2}}$$

$$= \left(\frac{1}{\eta_1} + \ln N_1 \right)^{\frac{1}{\eta_1}} \left(\frac{1}{\eta_2} + \ln N_2 \right)^{\frac{1}{\eta_2}} \approx (\ln N_1)^{\frac{1}{\eta_1}} (\ln N_2)^{\frac{1}{\eta_2}}$$

$$= (\ln N_1)^{\frac{1}{h_1} - \frac{1}{\xi_1}} (\ln N_2)^{\frac{1}{h_2} - \frac{1}{\xi_2}} .$$

Next, we define

$$\phi(x_1, x_2) = \begin{cases} (K * f)(x_1, x_2), & (x_1, x_2) \in [a_1, 1] \times [a_2, 1] \\ 0, & (x_1, x_2) \notin [a_1, 1] \times [a_2, 1], \end{cases} \qquad a_i = \frac{1}{N_i}(e^{q_i} + 1), \quad i = 1, 2.$$

Then if $(x_1, x_2) \in [a_1, 1] \times [a_2, 1]$, we obtain

$$\phi(x_1, x_2) = (K * f)(x_1, x_2) \geq \int_{\frac{1}{N_2}}^{x_2 - \frac{1}{N_2}} \int_{\frac{1}{N_1}}^{x_1 - \frac{1}{N_1}} f(s_1, s_2) K(x_1 - s_1, x_2 - s_2) ds_1 ds_2$$

$$= \int_{\frac{1}{N_2}}^{x_2 - \frac{1}{N_2}} \int_{\frac{1}{N_1}}^{x_1 - \frac{1}{N_1}} \left(\min\left(N_1, \frac{1}{s_1} \right) \right)^{\frac{1}{q_1}} \left(\min\left(N_2, \frac{1}{s_2} \right) \right)^{\frac{1}{q_2}}$$

$$\times \min\left(N_1, \frac{1}{x_1 - s_1} \right) \min\left(N_2, \frac{1}{x_2 - s_2} \right) ds_1 ds_2$$

$$= \int_{\frac{1}{N_2}}^{x_2 - \frac{1}{N_2}} \int_{\frac{1}{N_1}}^{x_1 - \frac{1}{N_1}} \left(\frac{1}{s_1} \right)^{\frac{1}{q_1}} \left(\frac{1}{s_2} \right)^{\frac{1}{q_2}} \frac{ds_1 ds_2}{(x_1 - s_1)(x_2 - s_2)}$$

$$\geq \frac{1}{\left(x_1 - \frac{1}{N_1} \right)^{\frac{1}{q_1}} \left(x_2 - \frac{1}{N_2} \right)^{\frac{1}{q_2}}} \int_{\frac{1}{N_1}}^{x_1 - \frac{1}{N_1}} \frac{ds_1}{x_1 - s_1} \int_{\frac{1}{N_2}}^{x_2 - \frac{1}{N_2}} \frac{ds_2}{x_2 - s_2}$$

$$= \frac{N_1^{\frac{1}{q_1}} N_2^{\frac{1}{q_2}} \ln(N_1 x_1 - 1) \ln(N_2 x_2 - 1)}{(N_1 x_1 - 1)^{\frac{1}{q_1}} (N_2 x_2 - 1)^{\frac{1}{q_2}}}.$$

We note that for $\theta_1 > e^{q_1}$, we have

$$\left(\frac{\ln \theta_1 \ln \theta_2}{\theta_1^{\frac{1}{q_1}} \theta_2^{\frac{1}{q_2}}} \right)'_{\theta_1} = \frac{(q_1 - \ln \theta_1) \ln \theta_2}{q_1 \theta_1^{\frac{1}{q_1}+1} \theta_2^{\frac{1}{q_2}}} < 0,$$

and for $\theta_2 > e^{q_2}$, we get

$$\left(\frac{\ln \theta_1 \ln \theta_2}{\theta_1^{\frac{1}{q_1}} \theta_2^{\frac{1}{q_2}}} \right)'_{\theta_2} = \frac{\ln \theta_1 (q_2 - \ln \theta_2)}{q_2 \theta_1^{\frac{1}{q_1}} \theta_2^{\frac{1}{q_2}+1}} < 0.$$

Then we obtain that the function $\frac{\ln\theta_1 \ln\theta_2}{\theta_1^{\frac{1}{q_1}}\theta_2^{\frac{1}{q_2}}}$ is decreasing for each variable on $[e_1^q, 1]\times[e_2^q, 1]$. Hence,

$$\phi^{*_1,*_2}(t_1,t_2) \geq \frac{\ln(N_1(t_1+a_1)-1)\ln(N_2(t_2+a_2)-1)}{\left(t_1+a_1-\frac{1}{N_1}\right)^{\frac{1}{q_1}}\left(t_2+a_2-\frac{1}{N_2}\right)^{\frac{1}{q_2}}}, \quad (t_1,t_2)\in(0,1-a_1)\times(0,1-a_2).$$

Using this, we derive

$$\|f*K\|_{L_{\bar{q},\bar{h}}} \geq \left(\int_0^1\left(\int_0^1\left(t_1^{\frac{1}{q_1}-\frac{1}{h_1}}t_2^{\frac{1}{q_2}-\frac{1}{h_2}}\phi^{*_1,*_2}(t_1,t_2)\right)^{h_1}dt_1\right)^{\frac{h_2}{h_1}}dt_2\right)^{\frac{1}{h_2}}$$

$$\geq \left(\int_0^{1-a_2}\left(\int_0^{1-a_1}\left(t_1^{\frac{1}{q_1}-\frac{1}{h_1}}t_2^{\frac{1}{q_2}-\frac{1}{h_2}}\frac{\ln(N_1(t_1+a_1)-1)\ln(N_2(t_2+a_2)-1)}{\left(t_1+a_1-\frac{1}{N_1}\right)^{\frac{1}{q_1}}\left(t_2+a_2-\frac{1}{N_2}\right)^{\frac{1}{q_2}}}\right)^{h_1}dt_1\right)^{\frac{h_2}{h_1}}dt_2\right)^{\frac{1}{h_2}}$$

$$= \left(\int_0^{1-a_1}\left(t_1^{\frac{1}{q_1}}\frac{\ln(N_1(t_1+a_1)-1)}{\left(t_1+a_1-\frac{1}{N_1}\right)^{\frac{1}{q_1}}}\right)^{h_1}\frac{dt_1}{t_1}\right)^{\frac{1}{h_1}}$$

$$\times\left(\int_0^{1-a_2}\left(t_2^{\frac{1}{q_2}}\frac{\ln(N_2(t_2+a_2)-1)}{\left(t_2+a_2-\frac{1}{N_2}\right)^{\frac{1}{q_2}}}\right)^{h_2}\frac{dt_2}{t_2}\right)^{\frac{1}{h_2}}$$

$$\geq 2^{-\frac{1}{q_1}}\left(\int_0^{1-a_1}(\ln(N_1(t_1+a_1)-1))^{h_1}\frac{dt_1}{t_1}\right)^{\frac{1}{h_1}}$$

$$\times 2^{-\frac{1}{q_2}}\left(\int_0^{1-a_2}(\ln(N_2(t_2+a_2)-1))^{h_2}\frac{dt_2}{t_2}\right)^{\frac{1}{h_2}}$$

$$\asymp \left(\ln^{h_1+1}(N_1(t_1+a_1)-1)\right)^{\frac{1}{h_1}}\Big|_0^{1-a_1}\left(\ln^{h_2+1}(N_2(t_2+a_2)-1)\right)^{\frac{1}{h_2}}\Big|_0^{1-a_2}$$

$$\asymp (\ln N_1)^{1+\frac{1}{h_1}}(\ln N_2)^{1+\frac{1}{h_2}}. \qquad \square$$

The fact that inequality (3.4) does not hold can be proved in a similar way. It suffices to consider the following functions

$$f(t_1,t_2) = \min\left(N_1,\frac{1}{t_1}\right)\min\left(N_2,\frac{1}{t_2}\right)$$

and

$$K(t_1,t_2) = \left(\min\left(N_1,\frac{1}{|t_1|}\right)\right)^{\frac{1}{q_1}}\left(\min\left(N_2,\frac{1}{|t_2|}\right)\right)^{\frac{1}{q_2}}.$$

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References

[1] J. Bastero, M. Milman, F. Ruiz, *A note on $L(\infty, q)$ spaces and Sobolev embeddings.* Indiana Univ. Math. J. 52 (2003), 1215–1230.

[2] C. Bennet, R. DeVore, R. Sharpley, *Weak-L^∞ and BMO.* Ann. Math. 113 (1981), no. 2, 601–611.

[3] C. Bennett, R. Sharpley, *Interpolation of operators.* Academic Press, San Diego, 1988.

[4] O.V. Besov, V.P. Il'in, S.M. Nikolskii, *Integral representations of functions and embedding theorems.* Nauka, M., 1975.

[5] A.P. Blozinski, *On a convolution theorem for $L(p, q)$ spases.* Trans. Am. Math. Soc. 164 (1972), 255–265.

[6] A.P. Blozinski, *Convolution of $L(p, q)$ functions.* Proc. Am. Math. Soc. 32 (1972), no. 1, 237–240.

[7] A.P. Blozinski, *Multivariate rearrangements and Banach function spaces with mixed norms.* Trans. Am. Math. Soc. 263 (1981), 149–167.

[8] M.J. Carro, A. Gogatishvili, J. Martin, L. Pick, *Functional properties of rearrangement invariant spaces defined in terms of oscilations.* J. Funct. Anal. 227 (2005), 335–404.

[9] D.V. Gorbachev, V.I. Ivanov, S.Y. Tikhonov, *Positive L_p-bounded Dunkl-type generalized translation operator and its applications.* Constructive Approximation 49 (2019), no. 3, 555–605.

[10] R.A. Hunt, *On $L(p, q)$ spases.* Enseign. Math. 12 (1966), no. 2, 249–276.

[11] M. Křepela, *Convolution in weighted Lorentz spaces of type Γ.* Math. Scand. 119 (2016), no. 1, 113–132.

[12] E.D. Nursultanov, *Interpolation theorems for anisotropic function spaces and their applications.* Dokl. Math. 69 (2004), no. 1, 16–19.

[13] E. Nursultanov, S. Tikhonov, *Convolution inequalities in Lorentz spaces.* J. Fourier Anal. Appl. 17 (2011), 486–505.

[14] E. Nursultanov, S. Tikhonov, N. Tleukhanova, *Norm inequalities for convolution operators.* C. R. Acad. Sci. Paris, I, 347 (2009), 1385–1388.

[15] E. Nursultanov, S. Tikhonov, N. Tleukhanova, *Norm convolution inequalities in Lebesgue spaces.* Rev. Mat. Iberoam., 34 (2018), no. 2, 811–838.

[16] R. O'Neil, *Convolution operators and $L(p, q)$ spases.* Duke Math. J. 30 (1963), 129–142.

[17] V.D. Stepanov, *Some topics in the theory of integral convolution operators.* Vladivostok, Dalnauka, 2000.

[18] L.Y.H. Yap, *Some remarks on convolution operators and $L(p, q)$ spases.* Duke Math. J. 36 (1969), 647–658.

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Events

5TH INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF MATHEMATICS AND COMPUTER SCIENCE: THEORY, METHODOLOGY, PRACTICE" (APRIL 18–20, 2019, YELETS, RUSSIA)

The XX century is marked by the enrichment of world science with outstanding achievements in the field of mathematics, solving many important problems that remain relevant in the modern world. Such problems include, in particular, the problems considered in fundamental works of academician S. Chaplygin. Based on his works new research paths were set, and serious applied problems were addressed in the fields of aerodynamics, gas dynamics, hydrodynamics, and mechanics. They were further intensively developed due to the achievements of contemporary information technology.

To commemorate his activities was organized a large-scale scientific event in the historic homeland of academician S. Chaplygin — the 5th international conference "Actual problems of mathematics and computer science: theory, methodology, practice", dedicated to the 150th anniversary of the birth of academician S. Chaplygin.

The Ivan Bunin Yelets State University (Russia), the Samarkand State University (Uzbekistan), the Higher School of Insurance and Finance (Bulgaria),the Khachatur Abovyan Armenian State Pedagogical University (Armenia), and the Scientific and Methodological Council on Mathematics of the Ministry of Science and Higher Education of Russia held through April 18–20, 2019 the 5th International Conference "Actual problems of mathematics and computer science: theory, methodology, practice" dedicated to the 150th anniversary of academician S. Chaplygin.

The conference marked the three major milestones related to the development of mathematical science in the Lipetsk region and in the oldest university center in the region — the Ivan Bunin Yelets State University.

1. April 2019 is the 150th anniversary of the birth of S. Chaplygin (1869–1942) — a well-known Russian scientist, academician of the Academy of Sciences of the USSR. S. Chaplygin is an outstanding representative of the Lipetsk region, whose surname is immortalized in the name of the city Chaplygin (previously Ranenburg) in the Lipetsk region.

2. 2019 year is the 80th anniversary of the foundation of the Faculty of Physics and Mathematics. It is the oldest faculty of the Ivan Bunin Yelets State University, where students of the scientific school of academician N. Zhukovsky were taught, whose famous representative was academician S. Chaplygin.

3. In October 2019 there will be 10 years since the organization of the Lipetsk Branch of the Scientific and Methodological Council for Mathematics of the Ministry of Science and Higher Education of Russia on the basis of the Ivan Bunin Yelets State University.

The main goals of the conference were the creation of conditions for international scientific communication of representatives of fundamental and applied areas in the field of mathematics, understanding the importance of scientific works of S. Chaplygin, the actualization of his scientific achievements, taking into account the rapid development of information technologies and their adaptation to modern mathematical education.

The plenary session of the conference was opened by the Rector of the Ivan Bunin Yelets State University Professor E. Gerasimova and continued by the President of the International Academy of the History of Science Professor S. Demidov (Moscow, Russia), who presented the talk "Pure and Applied Mathematics at the M.V. Lomonosov Moscow State University in the first half of the twentieth century: N. Luzin and S. Chaplygin".

Professor A. Soleev (Samarkand, Uzbekistan) devoted his talk to basic ideas and general provisions of the Power Geometry. Professor A. Soldatov (Moscow, Russia) focused on the consideration of the Dirichlet problem for equations of mixed type. In her talk Professor G. Zhukova (Moscow, Russia) discussed the dependence of solutions to singularly perturbed linear differential systems on a small parameter. The talk of Professors O. Masina (Yelets, Russia) and O. Druzhinina (Moscow, Russia) was devoted to the analysis of the known and developed by the authors approaches to the study of the stability of intelligent control systems. The talks of Professors V. Tikhomirov (Moscow, Russia), T. Sergeeva (Moscow, Russia) and E. Smirnov (Yaroslavl, Russia) addressed the issues of improving mathematical education, introducing novelty into the teaching process while maintaining the best traditions of high-quality teaching mathematics, laid by S. Chaplygin in his productive teaching activities.

The relevance of the event was noted in the talks of Professors A. Abylkasymova (Alma-Ata, Kazakhstan), A. Borovskikh (Moscow, Russia), S. Grozdev (Sofia, Bulgaria), M. Mkrtchyan (Yerevan, Armenia) and other scientists. At the end of the plenary session, talks were presented by the authors of this communication on the history of the Scientific and Methodological Council on Mathematics of the Ministry of Science and Higher Education of Russia and its contribution to the development of mathematics and its applications in Russian education, as well as on the activities of the Lipetsk Branch of the Scientific and Methodological Council.

The following sections were working at the conference: “Modern Directions in Mathematics”, “Applied problems of mathematics”, “Computer modeling, information technologies and systems”, “New theories, models and technologies of teaching mathematics and computer science at schools and universities”, “Actualization of the problems of the history of mathematics and mathematical education in modern conditions”.

At the conference there were more than 250 participants, including leading foreign specialists from Armenia, Bulgaria, Uzbekistan, Kazakhstan, well-known scientists from more than twenty regions of Russia, as well as young researchers. Overall, it was a successful conference, which helped to increase the scientific and innovative activity of the region, stimulated the participants to develop mathematics, information technologies and mathematical education.

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