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EXTENSION AND DECOMPOSITION METHOD FOR DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS

I.N. Parasidis

Communicated by R. Oinarov

Key words: differential and Fredholm integro-differential equations, nonlocal integral boundary conditions, decomposition of operators, correct operators, exact solutions.

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Abstract. A direct method for finding exact solutions of differential or Fredholm integro-differential equations with nonlocal boundary conditions is proposed. We investigate the abstract equations of the form $Bu = Au - gF(Au) = f$ and $B_1u = A^2u - qF(Au) - gF(A^2u) = f$ with abstract nonlocal boundary conditions $\Phi(u) = N\Psi(Au)$ and $\Phi(u) = N\Psi(Au)$, $\Phi(Au) = DF(Au) + N\Psi(A^2u)$, respectively, where q, g are vectors, D, N are matrices, F, Φ, Ψ are vector-functions. In this paper:

1. we investigate the correctness of the equation $Bu = f$ and find its exact solution,
2. we investigate the correctness of the equation $B_1u = f$ and find its exact solution,
3. we find the conditions under which the operator B_1 has the decomposition $B_1 = B^2$, i.e. B_1 is a quadratic operator, and then we investigate the correctness of the equation $B^2u = f_1$ and find its exact solution.

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1 Introduction

Boundary value problems (BVPs) for differential and integro-differential equations (IDEs) with nonlocal boundary conditions arise in various fields of mechanics, physics, biology, biotechnology, chemical engineering, medical science, finance and others. (see [6], [8], [1], [2], [7], [13], [15], [16], [20], [30], [31], [32], [34], [36].) More precisely, elasticity problems, heat and mass transfer problems, diffraction problems, underground water flow and population dynamics. Perhaps the first known problem which has been reduced to IDE

$$a_1y^{iv}(t) + y(t) = -a_2\int_{-1}^{1} K(t,x)y^{iv}(x)dx$$

in XVII century is Proctor problem of the equilibrium of an elastic beam. Fredholm integro-differential equations with nonlocal integral boundary conditions and ordinary differential operator, probably, first have been considered by J.D. Tamarkin [35]. Problems with nonlocal boundary conditions for elliptic equations first have been investigated by A. Bitsadze, A. Samarskii [5], BVP for parabolic equations with nonlocal integral boundary conditions by J.R. Cannon [7], L.I. Kamynin [15], N.I. Ionkin [13] and others. Later these investigations for the Laplace, Poisson and heat equations have been extended by V.A. Il'in and E.I. Moiseev [12], and others [3], [14], [29]. Nonlocal

BVP involving integral conditions for hyperbolic equations were studied in [28]. Nonlocal BVP for the ordinary differential equations with integral boundary conditions have been studied in [4]. The problem of the existence of solutions for nonlocal BVP is the subject of many papers [4], [11], [14], [22], [25], [26], [29]. Finding an exact solution of BVP for Fredholm IDE has been considered in [37] and [27]. This problem is hard. The solutions are obtained in most cases by numerical methods [10]. The necessary and sufficient solvability conditions of the abstract operator equations:

$$\begin{aligned} Bu &= Au - Qu, \quad Qu = gF(Au), && (1.1)\\ D(B) &= \{u \in D(A) : \Phi(u) = N\Psi(Au)\} \\ B_1 u &= A^2u - Q_1u, \quad Q_1u = qF(Au) + gF(A^2u), && (1.2)\\ D(B_1) &= \left\{u \in D(A^2) : \Phi(u) = N\Psi(Au), \quad \Phi(Au) = DF(Au) + N\Psi(A^2u)\right\}, \end{aligned}$$

and their exact solutions are given in the present work in a closed form. These equations are useful for solving Fredholm IDEs with nonlocal boundary conditions, when A is a differential operator, Q, Q_1 are integral operators with separable kernels, for example

$$Qu(x) = gF(Au) = \sum_{k=1}^{n} \int_\Omega g_k(x)h_k(t)Au(t)dt.$$

Problems (1.1) and (1.2) arise naturally from Dezin-Oinarov extensions of linear operators [9], [22], which are not restrictions of a maximal operator, unlike the classical M. Krein, Von. Neuman extensions [19], [21] in a Hilbert space and Otelbaev-Kokebaev-Shynybekov extensions [17] in a Banach space. In [33] by numerical methods are investigated IDEs of types (1.1), (1.2), for example

$$z'(x) + z(x) + \lambda \int_0^1 y[z'(y) + z(y)]dy = 0, \quad z(0) = z_0,$$

which by substitution $u(x) = z(x) - z_0$ is reduced to Problem (1.1), where $Au = u'(x) + u(x)$, $F(Au) = \int_0^1 y[u'(y) + u(y)]dy$, $g = -\lambda$, $N = \mathbf{0}$, $\Phi(u) = u(0) = 0$, $f = -\frac{1}{2}\lambda z_0$.

This work is a generalization of the papers [22], [24], [25], [26], where integral boundary conditions have not been considered. The main results of this paper are Theorems 3.2 and 4.1, where the problem $B_1u = f$ is solved by the extension method and by the decomposition method, if $B_1 = B^2$. The reader can easily verify that the last method is essentially simpler and more convenient since the solvability condition for the equation $B_1u = f$ with a quadratic operator $B_1 = B^2$ coincides with the solvability condition for $Bu = f$ and the solution of the equation $B_1u = f$ is obtained by applying twice the formula for the solution of the equation $Bu = f$. We recognize that the probability that B_1 is quadratic is low, but if this is true, then the solving of the equation $B_1u = f$ is greatly simplified by the decomposition method. Note that the extension method is a generalization of the direct method which is presented in [37]. The essential ingredient in our approach is the extension of the main idea in [22].

The paper is organized as follows. In Section 2 we recall some basic terminology and notation for operators. In Section 3 we prove the main general results for equations of type $Bu = f$ and $B_1u = f$ using the extension method. In Section 4 by the decomposition method we prove the main general results for equations of type $B^2u = f$. Finally, in Section 5 we discuss some examples of differential and integro-differential equations with integral boundary conditions which show the usefulness of our results.

2 Terminology and notation

Let X, Y be complex Banach spaces and X^* be the adjoint space of X, i.e. the set of all complex-valued linear bounded functionals on X. We denote by $f(x)$ the value of f on x. We write $D(A) \subset X$ and $R(A) \subset Y$ for the domain and the range of the operator A, respectively. An operator A_2 is said to be an *extension* of an operator A_1, or A_1 is said to be a *restriction* of A_2, briefly $A_1 \subset A_2$, if $D(A_2) \supseteq D(A_1)$ and $A_1 x = A_2 x$, for all $x \in D(A_1)$. An operator $A : X \to Y$ is called *closed* if for every sequence x_n in $D(A)$ such that $x_n \to x_0$ in X and $Ax_n \to f_0$ in Y, it follows that $x_0 \in D(A)$ and $Ax_0 = f_0$. An operator A is called *maximal* if $R(A) = Y$ and $\ker A \neq \{0\}$. An operator $\widehat{A} : X \to Y$ is called *correct* if $R(\widehat{A}) = Y$ and the inverse $\widehat{A}^{-1}$ exists and is continuous on Y. An operator $\widehat{A}$ is called a *correct restriction* of the maximal operator A if it is a correct operator and $\widehat{A} \subset A$. If $\Psi_i \in X^*, i = 1, \ldots, n$, then we denote by $\Psi = \mathrm{col}(\Psi_1, \ldots, \Psi_n)$ and $\Psi(x) = \mathrm{col}(\Psi_1(x), \ldots, \Psi_n(x))$. Let $g = (g_1, \ldots, g_m)$ be a vector of X^m. We will denote by $\Psi(g)$ the $n \times m$ matrix whose i, j-th entry $\Psi_i(g_j)$ is the value of functional Ψ_i on element g_j. Note that $\Psi(gC) = \Psi(g)C$, where C is a $m \times k$ constant matrix. We will also denote by 0_{nm} the zero $n \times m$ matrix, by 0_n the zero and by I_n the identity $n \times n$ matrices. By $\vec{0}$ we will denote the zero column vector.

3 Extension methods for ordinary differential and Fredholm IDE

Lemma 3.1. *Let* A_i, B_i, C_i, D *be* $n \times n$ *matrices, where* $i = 1, 2, 3$ *and* $G = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$. *Then the following properties of determinants hold true:*

$$\det \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = \det \begin{pmatrix} A_1 + DB_1 & A_2 + DB_2 & A_3 + DB_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}, \tag{3.1}$$

$$\det \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = \det \begin{pmatrix} A_1 & A_2 + A_3 D & A_3 \\ B_1 & B_2 + B_3 D & B_3 \\ C_1 & C_2 + C_3 D & C_3 \end{pmatrix}. \tag{3.2}$$

Proof. Let $F = \begin{pmatrix} I_n & -D & 0_n \\ 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \end{pmatrix}$. Then $F^{-1} = \begin{pmatrix} I_n & D & 0_n \\ 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \end{pmatrix}$ and

$$\det G = \det F^{-1} G F = \det \begin{pmatrix} A_1 + DB_1 & A_2 + DB_2 & A_3 + DB_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}.$$

So (3.1) holds. Let now $F = \begin{pmatrix} I_n & 0_n & 0_n \\ 0_n & I_n & 0_n \\ 0_n & D & I_n \end{pmatrix}$. Then $F^{-1} = \begin{pmatrix} I_n & 0_n & 0_n \\ 0_n & I_n & 0_n \\ 0_n & -D & I_n \end{pmatrix}$ and

$$\det G = \det F^{-1} G F = \det \begin{pmatrix} A_1 & A_2 + A_3 D & A_3 \\ B_1 & B_2 + B_3 D & B_3 \\ C_1 & C_2 + C_3 D & C_3 \end{pmatrix}.$$

So (3.2) holds and Lemma 3.1 is proved. □

Remark 1. Let Γ be the matrix obtained from G by multiplying from the left a row of G by the matrix D and then adding it to another row, or by multiplying from the right a column of G by the matrix D and then adding it to another column of G. Then $\det G = \det \Gamma$.

Lemma 3.2. *Let X be a Banach space, $\mathbf{A} : X \to X$ a linear closed operator and $A = \mathbf{A}^m$. Then:*
(i) the sets $\widehat{X}_A = \left(D(A), ||\cdot||_{\widehat{X}_A}\right)$ and $\widehat{X}_{A^2} = \left(D(A^2), ||\cdot||_{\widehat{X}_{A^2}}\right)$, are Banach spaces, where

$$||u||_{\widehat{X}_A} = \sum_{i=0}^{m} ||\mathbf{A}^i u||_X, \qquad ||u||_{\widehat{X}_{A^2}} = \sum_{i=0}^{2m} ||\mathbf{A}^i u||_X, \tag{3.3}$$

(ii) if the functionals $\Phi_i \in \widehat{X}_A^*$, $i = 1, ..., m$, then the functionals Φ_{m+i} defined by

$$\Phi_{m+i}(u) = \Phi_i(Au),\ i = 1, ..., m, \quad u \in D(A^2) \tag{3.4}$$

belong to $\widehat{X}_{A^2}^*$.

Proof. (i) Let $(u_k) \subset \widehat{X}_A$ be a fundamental sequence, i.e. $||u_k - u_l||_{\widehat{X}_A} \to 0, \quad k, l \to \infty$. Then (3.3) yelds

$$||u_k - u_l||_{\widehat{X}_A} = ||u_k - u_l||_X + ||\mathbf{A}(u_k - u_l)||_X + ... + ||\mathbf{A}^m(u_k - u_l)||_X \to 0, k, l \to \infty, \tag{3.5}$$

which implies that

$$||u_k - u_l||_X \to 0, \quad ||\mathbf{A}(u_k - u_l)||_X \to 0, ..., ||\mathbf{A}^m(u_k - u_l)||_X \to 0, \quad k, l \to \infty.$$

Since X is a Banach space, there exist the elements $y_0, y_1, y_2, ..., y_m \in X$ such that
$u_l \to y_0, \quad \mathbf{A}u_l \to y_1, \quad \mathbf{A}^2 u_l \to y_2, ..., \mathbf{A}^m u_l \to y_m, \quad l \to \infty.$
Then, since $\mathbf{A}$ is closed, it follows that for $l \to \infty$:

$$\begin{array}{llll}
u_l \to y_0, & \mathbf{A}u_l \to y_1 \Rightarrow & y_0 \in D(\mathbf{A}), & \mathbf{A}y_0 = y_1, \\
\mathbf{A}u_l \to y_1, & \mathbf{A}^2 u_l \to y_2 \Rightarrow & y_1 \in D(\mathbf{A}), & \mathbf{A}y_1 = y_2, \\
\mathbf{A}^2 u_l \to y_2, & \mathbf{A}^3 u_l \to y_3 \Rightarrow & y_2 \in D(\mathbf{A}), & \mathbf{A}y_2 = y_3, \\
\dots & \dots & \dots & \dots \\
\mathbf{A}^{m-1} u_l \to y_{m-1}, & \mathbf{A}^m u_l \to y_m \Rightarrow & y_{m-1} \in D(\mathbf{A}), & \mathbf{A}y_{m-1} = y_m.
\end{array}$$

Thus $y_0 \in D(\mathbf{A}^m)$ and $\mathbf{A}^m y_0 = y_m$. Further from (3.5) we obtain

$$\begin{array}{ll}
\lim_{l\to\infty}(& ||u_k - u_l||_X + ||\mathbf{A}(u_k - u_l)||_X + ... + ||\mathbf{A}^m(u_k - u_l)||_X) \\
= & ||u_k - u_0||_X + ||\mathbf{A}(u_k - u_0)||_X + ... + ||\mathbf{A}^m(u_k - u_0)||_X = ||u_k - u_0||_{\widehat{X}_A}.
\end{array}$$

The last equation yelds $u_k \to y_0,\ k \to \infty$. Hence $\widehat{X}_A$ is a Banach space. Similarly it is proved that $\widehat{X}_{A^2}$ is a Banach space.
(ii) Let $\Phi_i \in \widehat{X}_A^*$, $i = 1, ..., m$. Then there exist numbers k_i such that for all $u \in D(A)$ hold the inequalities $|\Phi_i(u)| \leq k_i ||u||_{\widehat{X}_A}$, $i = 1, ..., m$. This relation for all $u \in D(A^2)$ implies that

$$\begin{array}{l}
|\Phi_i(Au)| \leq k_i ||Au||_{\widehat{X}_A} \leq k_i(||u||_X + ||\mathbf{A}u||_X + ... + ||\mathbf{A}^{m-1}u||_X + ||\mathbf{A}^m u||_X + ... + ||\mathbf{A}^{2m}u||_X) \\
= k_i ||u||_{\widehat{X}_{A^2}},\ i = 1, ..., m.
\end{array}$$

Thus $|\Phi_{m+i}(u)| = |\Phi_i(Au)| \leq k_i ||u||_{\widehat{X}_{A^2}}$ and $\Phi_{m+i} \in \widehat{X}_{A^2}^*$, $\quad i = 1, ..., m$. $\square$

Remark 2. Consider the first order differential operator with smooth coefficients $\mathbf{A}u = a(x)u'(x) + b(x)u(x)$ and the operator $A = \mathbf{A}^m$. Then:
(i) for $\mathbf{A} : C[a, b] \to C[a, b]$, $D(\mathbf{A}) = \{u \in C^1[a, b]\}$, we have
$D(A) = \{u \in C^m[a, b]\}$, $\widehat{X}_A = C^m[a, b]$, $\widehat{X}_{A^2} = C^{2m}[a, b]$,
(ii) for $\mathbf{A} : L_2(a, b) \to L_2(a, b)$, $D(\mathbf{A}) = \{u \in \widehat{W}_2^1(a, b)\}$, we have
$D(A) = \{u \in \widehat{W}_2^m(a, b)\}$, $\widehat{X}_A = \widehat{W}_2^m(a, b)$, $\widehat{X}_{A^2} = \widehat{W}_2^{2m}(a, b)$, where $||u||_{\widehat{W}_2^1} = ||u||_{L_2} + ||u'||_{L_2}$,
(iii) we will use below the spaces $\widehat{X}_A$, $\widehat{X}_{A^2}$ as the domains of the functionals $\Phi_1, ..., \Phi_m$ and $\Phi_1, ..., \Phi_{2m}$.

Lemma 3.3. *Let X, Y be Banach spaces, $A : X \overset{on}{\to} Y$ be a linear operator, $z_1, ..., z_m$ a basis of* $\ker A$ *and $\widehat{A}$ the correct restriction of A defined by*

$$\widehat{A} \subset A, \quad D(\widehat{A}) = \{x \in D(A) : \Phi(x) = \vec{0}\}, \tag{3.6}$$

where the components of the vector-functional $\Phi = col(\Phi_1, \dots, \Phi_m)$ belong to $\widehat{X}_A^*$ and are biorthogonal to $z_1, ..., z_m$. Then A is closed and so A is a maximal operator.

Proof. Let $x_n \in D(A)$, $x_n \to x$, $n \to \infty$ and $Ax_n \to y$, $n \to \infty$. Since $D(A) = D(\widehat{A}) \oplus \ker A$ [17] and $R(\widehat{A}) = Y$, for every $x_n \in D(A)$ there exist $y_n \in Y$ and $\bar{z}_n = a_{1_n} z_1 + ... + a_{m_n} z_m \in \ker A$, $a_{i_n} \in \mathbf{C}$, $i = 1, ..., m$, such that

$$x_n = \widehat{A}^{-1} y_n + a_{1_n} z_1 + ... + a_{m_n} z_m, \tag{3.7}$$

From (3.7) we obtain

$$\begin{aligned} \Phi_i(x_n) &= a_{i_n}, \ i = 1, ..., m, \\ x_n &= \widehat{A}^{-1} y_n + \Phi_1(x_n) z_1 + ... + \Phi_m(x_n) z_m. \end{aligned}$$

From the last equation, since $\widehat{A}^{-1}, \Phi_1, ..., \Phi_m$ are bounded, it follows that

$$x = \widehat{A}^{-1} y + \Phi_1(x) z_1 + ... + \Phi_m(x) z_m \in D(A).$$

Then $Ax = y$. So A is closed and hence, A is a maximal operator. □

Remark 3. From (3.6) it is easy to verify that for $A, \widehat{A} : X \to X$

$$\widehat{A}^2 \subset A^2, \quad D(\widehat{A}^2) = \{x \in D(A^2) : \Phi(x) = \vec{0}, \quad \Phi(Ax) = \vec{0}\}. \tag{3.8}$$

Theorem 3.1. *Let X, Y be complex Banach spaces, $A : X \to Y$ be a maximal closed linear operator with finite dimensional kernel $\mathbf{z} = (z_1, ..., z_m)$, which is a basis of* $\ker A$, *the components of vector-functional $\Phi = col(\Phi_1, \dots, \Phi_m)$, $\Psi = col(\Psi_1, \dots, \Psi_n)$, $F = col(F_1, \dots, F_s)$ belong to $\widehat{X}_A^*$, Y^*, Y^*, respectively, and $\widehat{A}$ the correct restriction of A defined by (3.6). Suppose also that N is a $m \times n$ matrix, $\Phi_1, \dots, \Phi_m$ are biorthogonal to $z_1, ..., z_m$ and that the components of a vector $g = (g_1, \dots, g_s) \in Y^s$ are linearly independent. Then:*
(i) The operator B defined by

$$\begin{aligned} Bu = & \quad Au - gF(Au) = f, \quad f \in Y, \\ D(B) = & \quad \{u \in D(A) : \Phi(u) = N\Psi(Au)\} \end{aligned} \tag{3.9}$$

is injective if and only if

$$det\, W = det[I_s - F(g)] \neq 0. \tag{3.10}$$

(ii) If B is injective, then B is correct and for all $f \in Y$ the unique solution of (3.9) is given by

$$u = B^{-1} f = \widehat{A}^{-1} f + \left[\widehat{A}^{-1} g + \mathbf{z} N \Psi(g)\right] W^{-1} F(f) + \mathbf{z} N \Psi(f). \tag{3.11}$$

Proof. (i). Let $det\, W \neq 0$ and $u \in \ker B$. Then $Bu = Au - gF(Au) = 0$, $\Phi(u) = N\Psi(Au)$ and $[I_s - F(g)]F(Au) = \vec{0}$. From the last equations it follows that $F(Au) = \vec{0}$, $Bu = Au = 0$ and $\Phi(u) = \vec{0}$. Hence $u \in D(\widehat{A})$, $\widehat{A}u = 0$, $u = 0$, i.e. $\ker B = \{0\}$ and B is an injective operator. To prove the converse, let $det\, W = 0$. Then there exists a vector $\vec{c} = col(c_1, ..., c_s) \neq \vec{0}$ such that $W\vec{c} = 0$. Note that $g\vec{c} \neq 0$ because $g_1, \ldots, g_s$ is a linearly independent set and that the element $u_0 = [\widehat{A}^{-1}g + \mathbf{z}N\Psi(g)]\vec{c} \neq \vec{0}$, otherwise $g = \vec{0}$. For u_0 we obtain

$$\Phi(u_0) - N\Psi(Au_0) = N\Psi(g)\vec{c} - N\Psi(g)\vec{c} = \vec{0},$$
$$Bu_0 = \widehat{A}u_0 - gF(\widehat{A}u_0) = g\vec{c} - gF(g)\vec{c} = g[I_n - F(g)]\vec{c} = gW\vec{c} = g\vec{0} = 0.$$

Hence $u_0 \in D(B)$ and $u_0 \in \ker B$. So $\ker B \neq \{0\}$ and B is not injective. So statement (i) holds. (ii) Let $det\, W \neq 0$. Since $\mathbf{z} \in [\ker A]^m$, $\Phi(\mathbf{z}) = I_m$, problem (3.9) can be written as

$$\begin{aligned} Bu &= A\,(u - \mathbf{z}N\Psi(Ax)) - gF(Au) = f, \quad f \in Y, \\ D(B) &= \{u \in D(A) : \Phi(u - \mathbf{z}N\Psi(Au)) = 0\}. \end{aligned} \tag{3.12}$$

Then, using the fact that $\widehat{A} \subset A$, we obtain
$u - \mathbf{z}N\Psi(Au) \in D(\widehat{A})$, $Bu = \widehat{A}\,(u - \mathbf{z}N\Psi(Au)) - gF(Au) = f$
and for every $u \in D(B)$ and $f \in Y$ from (3.9) and (3.12) we obtain

$$\begin{aligned} &[I_s - F(g)]F(Au) = F(f), \\ &F(Au) = W^{-1}F(f), \\ &\Psi(Au) = \Psi(g)W^{-1}F(f) + \Psi(f), \\ &u - \mathbf{z}N\Psi(Au) - \widehat{A}^{-1}gF(Au) = \widehat{A}^{-1}f, \\ &u = B^{-1}f = \widehat{A}^{-1}f + \widehat{A}^{-1}gW^{-1}F(f) + \mathbf{z}N[\Psi(g)W^{-1}F(f) + \Psi(f)]. \end{aligned}$$

From the last equation for every $f \in Y$, follows the unique of solution (3.11) of (3.9). Because f in (3.11) is arbitrary, we obtain $R(B) = Y$. Since the operator $\widehat{A}^{-1}$ and the functionals $F_1, ..., F_s$, $\Psi_1, ..., \Psi_n$ are bounded, from (3.11) follows the boundedness of B^{-1}. Hence, the operator B is correct if and only if (3.10) holds and the unique solution of (3.9) is given by (3.11). □

From the previous theorem for $g = \vec{0}$, follows the next corollary which is useful for solving some class of differential equations with nonlocal boundary conditions.

Corollary 3.1. *Let the spaces X, Y, the operators $A, \widehat{A}$, the vector $\mathbf{z} = (z_1, ..., z_m)$ and vector-functionals Φ, Ψ and the matrix N be defined as in Theorem 3.1. Then the operator B defined by*

$$Bu = Au = f, \quad f \in Y, \quad D(B) = \{u \in D(A) : \Phi(u) = N\Psi(Au)\} \tag{3.13}$$

is correct and for all $f \in Y$ the unique solution of (3.13) is given by

$$u = B^{-1}f = \widehat{A}^{-1}f + \mathbf{z}N\Psi(f). \tag{3.14}$$

Now we prove the main theorem, which is useful for solving some class of integro-differential equations with integral boundary conditions.

Theorem 3.2. *Let X be a Banach space, the vector $\mathbf{z}$, the operators $A, \widehat{A}$ and the vector functionals Φ, Ψ, F be defined as in Theorem 3.1, where $Y = X$ and the operator $B_1 : X \to X$ be defined by*

$$B_1 u = A^2u - qF(Au) - gF(A^2u) = f, \tag{3.15}$$
$$D(B_1) = \left\{u \in D(A^2) :\ \Phi(u) = N\Psi(Au),\ \Phi(Au) = DF(Au) + N\Psi(A^2u)\right\}. \tag{3.16}$$

Suppose also that the vectors q and g are linearly independent, $q = (q_1, \dots, q_s)$, $g = (g_1, \dots, g_s) \in X^s$, D is $m \times s$ and N is $m \times n$ matrices. Then:
(i) the operator B_1 corresponding to problem (3.15)-(3.16) is injective if and only if

$$\det L = \det \begin{pmatrix} 0_{sn} & -F(\mathbf{z})N & K_1 & -F(\widehat{A}^{-1}g) \\ I_n & -\Psi(\mathbf{z})N & -K_2 & -\Psi(\widehat{A}^{-1}g) \\ 0_n & I_n & -\Psi(q) & -\Psi(g) \\ 0_{sn} & 0_{sn} & -F(q) & I_s - F(g) \end{pmatrix} \neq 0, \tag{3.17}$$

or

$$\mathit{det}\, L_1 = \det \begin{pmatrix} K_1 - F(\mathbf{z})N\Psi(q) & -F(\mathbf{z})N\Psi(g) - F(\widehat{A}^{-1}g) \\ -F(q) & I_s - F(g) \end{pmatrix} \neq 0, \tag{3.18}$$

where $\det L = \pm \mathit{det}\, L_1$ *and* $K_1 = I_s - F(\mathbf{z})D - F(\widehat{A}^{-1}q)$, $K_2 = \Psi(\mathbf{z})D + \Psi(\widehat{A}^{-1}q)$,
(ii) if the operator B_1 is injective, then it is correct and the unique solution of (3.15)-(3.16) is given by

$$\begin{aligned} u = B_1^{-1}f = \widehat{A}^{-2}f \quad &+ \quad (\mathbf{z}N, \widehat{A}^{-1}\mathbf{z}N, \widehat{A}^{-1}\mathbf{z}D + \widehat{A}^{-2}q, \widehat{A}^{-2}g)L^{-1} \\ &\cdot \quad \mathit{col}\left(F(\widehat{A}^{-1}f), \Psi(\widehat{A}^{-1}f), \Psi(f), F(f)\right). \end{aligned} \tag{3.19}$$

Proof. (i) Let $\det L \neq 0$. Since $\Phi(\mathbf{z}) = I_m$, relations (3.16) can be represented as

$$\Phi(u - \mathbf{z}N\Psi(Au)) = 0,$$
$$\Phi(Au - \mathbf{z}DF(Au) - \mathbf{z}N\Psi(A^2u)) = 0,$$

which taking into account (3.6) imply that

$$u - \mathbf{z}N\Psi(Au) \in D(\widehat{A}), \tag{3.20}$$
$$Au - \mathbf{z}DF(Au) - \mathbf{z}N\Psi(A^2u) \in D(\widehat{A}). \tag{3.21}$$

Then, since $\mathbf{z} \in [\ker A]^m$, $\widehat{A} \subset A$ and $\widehat{A}$ is correct, from (3.15) we obtain

$$\begin{aligned} &\widehat{A}\left(Au - \mathbf{z}[DF(Au) + N\Psi(A^2u)]\right) - qF(Au) - gF(A^2u) = f, \\ &Au - \mathbf{z}[DF(Au) + N\Psi(A^2u)] - \widehat{A}^{-1}qF(Au) - \widehat{A}^{-1}gF(A^2u) = \widehat{A}^{-1}f, \\ &\widehat{A}\left(u - \mathbf{z}N\Psi(Au)\right) - \mathbf{z}[DF(Au) + N\Psi(A^2u)] - \widehat{A}^{-1}qF(Au) - \widehat{A}^{-1}gF(A^2u) \\ &\qquad = \widehat{A}^{-1}f, \\ &u - \mathbf{z}N\Psi(Au) - \widehat{A}^{-1}\mathbf{z}[DF(Au) + N\Psi(A^2u)] - \widehat{A}^{-2}qF(Au) - \widehat{A}^{-2}gF(A^2u) \\ &\qquad = \widehat{A}^{-2}f. \end{aligned}$$

Then, taking into account (3.15), we get

$$\begin{aligned} A^2u &= qF(Au) + gF(A^2u) + f, \\ Au &= \mathbf{z}[DF(Au) + N\Psi(A^2u)] + \widehat{A}^{-1}qF(Au) + \widehat{A}^{-1}gF(A^2u) + \widehat{A}^{-1}f, \\ u &= \mathbf{z}N\Psi(Au) + \widehat{A}^{-1}\mathbf{z}N\Psi(A^2u) + (\widehat{A}^{-1}\mathbf{z}D + \widehat{A}^{-2}q)F(Au) + \widehat{A}^{-2}gF(A^2u) \\ &+ \widehat{A}^{-2}f. \end{aligned} \tag{3.22}$$

Further, acting by vector-functionals F and Ψ, we arrive at the system

$$\begin{aligned}
F(Au) &= F(\mathbf{z})[DF(Au)+N\Psi(A^2u)]+F(\widehat{A}^{-1}q)F(Au)+F(\widehat{A}^{-1}g)F(A^2u)\\
&+ F(\widehat{A}^{-1}f),\\
\Psi(Au) &= \Psi(\mathbf{z})[DF(Au)+N\Psi(A^2u)]+\Psi(\widehat{A}^{-1}q)F(Au)+\Psi(\widehat{A}^{-1}g)F(A^2u)\\
&+ \Psi(\widehat{A}^{-1}f),\\
\Psi(A^2u) &= \Psi(q)F(Au)+\Psi(g)F(A^2u)+\Psi(f),\\
F(A^2u) &= F(q)F(Au)+F(g)F(A^2u)+F(f),
\end{aligned}$$

or

$$\begin{aligned}
&-F(\mathbf{z})N\Psi(A^2u)+[I_s-F(\mathbf{z})D-F(\widehat{A}^{-1}q)]F(Au)-F(\widehat{A}^{-1}g)F(A^2u)\\
&\qquad = F(\widehat{A}^{-1}f),\\
&\Psi(Au)-\Psi(\mathbf{z})N\Psi(A^2u)-[\Psi(\mathbf{z})D+\Psi(\widehat{A}^{-1}q)]F(Au)-\Psi(\widehat{A}^{-1}g)F(A^2u)\\
&\qquad = \Psi(\widehat{A}^{-1}f),\\
&\Psi(A^2u)-\Psi(q)F(Au)-\Psi(g)F(A^2u)=\Psi(f),\\
&-F(q)F(Au)+[I_s-F(g)]F(A^2u)=F(f).
\end{aligned}$$

Denoting by $K_1=I_s-F(\mathbf{z})D-F(\widehat{A}^{-1}q)$, $K_2=\Psi(\mathbf{z})D+\Psi(\widehat{A}^{-1}q)$, $W=I_s-F(g)$ from the above equations we get the system

$$\begin{pmatrix} 0_{sn} & -F(\mathbf{z})N & K_1 & -F(\widehat{A}^{-1}g)\\ I_n & -\Psi(\mathbf{z})N & -K_2 & -\Psi(\widehat{A}^{-1}g)\\ 0_n & I_n & -\Psi(q) & -\Psi(g)\\ 0_{sn} & 0_{sn} & -F(q) & W \end{pmatrix}\begin{pmatrix}\Psi(Au)\\ \Psi(A^2u)\\ F(Au)\\ F(A^2u)\end{pmatrix}=\begin{pmatrix}F(\widehat{A}^{-1}f)\\ \Psi(\widehat{A}^{-1}f)\\ \Psi(f)\\ F(f)\end{pmatrix}. \tag{3.23}$$

Denoting the $(2n+2s)\times(2n+2s)$ matrix on the left by L and using property (3.1), where $D=F(\mathbf{z})N$, we obtain

$$\begin{aligned}
\det L &= \pm\det\begin{pmatrix} -F(\mathbf{z})N & K_1 & -F(\widehat{A}^{-1}g)\\ I_n & -\Psi(q) & -\Psi(g)\\ 0_{sn} & -F(q) & I_s-F(g)\end{pmatrix}\\
&= \pm\det\begin{pmatrix} 0_{sn} & -F(\mathbf{z})N\Psi(q)+K_1 & -F(\mathbf{z})N\Psi(g)-F(\widehat{A}^{-1}g)\\ I_n & -\Psi(q) & -\Psi(g)\\ 0_{sn} & -F(q) & I_s-F(g)\end{pmatrix}\\
&= \pm\det\begin{pmatrix} I_s-F(\widehat{A}^{-1}q)-F(\mathbf{z})[D+N\Psi(q)] & -F(\mathbf{z})N\Psi(g)-F(\widehat{A}^{-1}g)\\ -F(q) & I_s-F(g)\end{pmatrix}\\
&= \pm\det L_1.
\end{aligned}$$

Therefore $\det L=\pm\det L_1$. Let $u\in\ker B_1$. Then in systems (3.22), (3.23) $f=0$ and from (3.23) we get $Lcol\,(\Psi(Au),\Psi(A^2u),F(Au),F(A^2u))=\vec{0}$, which, since $\det L\neq 0$, yields $\Psi(Au)=\Psi(A^2u)=\vec{0}$, $F(Au)=F(A^2u)=\vec{0}$. Substitution of these values into (3.15), (3.16) implies $B_1u=A^2u=0$, $\Phi(u)=\Phi(Au)=\vec{0}$. Taking into account (3.8) we acquire $u\in D(\widehat{A}^2)$ and $B_1u=\widehat{A}^2u=0$. By hypothesis $\widehat{A}$ is correct and so $\widehat{A}^2$ is also correct and $u=0$. We thus obtain $\ker B_1=\{0\}$, i.e. B_1 is injective. To prove the converse, let $\det L=0$. Then $\det L_1=0$ and there exists a vector $\vec{c}=col(\vec{c}_1,\vec{c}_2)\neq\vec{0}$, $\vec{c}_1=col(c_{11},...,c_{1s})$, $\vec{c}_2=col(c_{21},...,c_{2s})$ such that $L_1\vec{c}=\vec{0}$.

Consider the element

$$\begin{aligned} u_0 &= \widehat{A}^{-2}(q\vec{c}_1 + g\vec{c}_2) + \widehat{A}^{-1}\mathbf{z}\left[(D + N\Psi(q))\,\vec{c}_1 + N\Psi(g)\vec{c}_2\right] \\ &+ \mathbf{z}N\left[\Psi(\widehat{A}^{-1}q) + \Psi(\mathbf{z})\,(D + N\Psi(q))\right]\vec{c}_1 \\ &+ \mathbf{z}N\left[\Psi(\widehat{A}^{-1}g) + \Psi(\mathbf{z})N\Psi(g)\right]\vec{c}_2. \end{aligned} \tag{3.24}$$

Observe that $u_0 \neq 0$, otherwise $q\vec{c}_1 + g\vec{c}_2 = 0$. But the last equality is true if and only if $\vec{c}_1 = \vec{c}_2 = \vec{0}$, since the vectors q, g are linearly independent. Then

$$\begin{aligned} \Phi(u_0) &= N\left[\Psi(\widehat{A}^{-1}q) + \Psi(\mathbf{z})\,(D + N\Psi(q))\right]\vec{c}_1 + N\left[\Psi(\widehat{A}^{-1}g) + \Psi(\mathbf{z})N\Psi(g)\right]\vec{c}_2, \\ Au_0 &= \widehat{A}^{-1}(q\vec{c}_1 + g\vec{c}_2) + \mathbf{z}[D + N\Psi(q)]\vec{c}_1 + \mathbf{z}N\Psi(g)\vec{c}_2, \\ A^2u_0 &= q\vec{c}_1 + g\vec{c}_2, \\ \Psi(Au_0) &= \Psi(\widehat{A}^{-1}q)\vec{c}_1 + \Psi(\widehat{A}^{-1}g)\vec{c}_2 + \Psi(\mathbf{z})[D + N\Psi(q)]\vec{c}_1 + \Psi(\mathbf{z})N\Psi(g)\vec{c}_2, \\ \Phi(Au_0) &= [D + N\Psi(q)]\vec{c}_1 + N\Psi(g)\vec{c}_2, \\ F(Au_0) &= F(\widehat{A}^{-1}q)\vec{c}_1 + F(\widehat{A}^{-1}g)\vec{c}_2 + F(\mathbf{z})[D + N\Psi(q)]\vec{c}_1 + F(\mathbf{z})N\Psi(g)\vec{c}_2, \\ F(A^2u_0) &= F(q)\vec{c}_1 + F(g)\vec{c}_2, \\ \Psi(A^2u_0) &= \Psi(q)\vec{c}_1 + \Psi(g)\vec{c}_2. \end{aligned} \tag{3.25}$$

It is evident that u_0 satisfies the first boundary condition of (3.16). Substituting the above values into the second boundary condition of (3.16), we get

$$\begin{aligned} [D &+ N\Psi(q)]\vec{c}_1 + N\Psi(g)\vec{c}_2 - D\{F(\widehat{A}^{-1}q)\vec{c}_1 + F(\widehat{A}^{-1}g)\vec{c}_2 + F(\mathbf{z})[D + N\Psi(q)]\vec{c}_1 \\ &+ F(\mathbf{z})N\Psi(g)\vec{c}_2\} - N\Psi(q)\vec{c}_1 - N\Psi(g)\vec{c}_2 \\ &= (D, 0_{ms})\begin{pmatrix} I_s - F(\widehat{A}^{-1}q) - F(\mathbf{z})[D + N\Psi(q)] & -F(\mathbf{z})N\Psi(g) - F(\widehat{A}^{-1}g) \\ -F(q) & I_s - F(g) \end{pmatrix}\begin{pmatrix} \vec{c}_1 \\ \vec{c}_2 \end{pmatrix} \\ &= (D, 0_{ms})L_1\vec{c} = (D, 0_{ms})\vec{0} = \vec{0}. \end{aligned} \tag{3.26}$$

So $u_0 \in D(B_1)$. We will show that $u_0 \in \ker B_1$.

$$\begin{aligned} B_1u_0 &= A^2u_0 - qF(Au_0) - gF(A^2u_0) = q\vec{c}_1 + g\vec{c}_2 - q[F(\widehat{A}^{-1}q)\vec{c}_1 + F(\widehat{A}^{-1}g)\vec{c}_2 \\ &+ F(\mathbf{z})(D + N\Psi(q))\vec{c}_1 + F(\mathbf{z})N\Psi(g)\vec{c}_2] - g[F(q)\vec{c}_1 + F(g)\vec{c}_2] \\ &= (q, g)\begin{pmatrix} I_s - F(\widehat{A}^{-1}q) - F(\mathbf{z})[D + N\Psi(q)] & -F(\mathbf{z})N\Psi(g) - F(\widehat{A}^{-1}g) \\ -F(q) & I_s - F(g) \end{pmatrix}\begin{pmatrix} \vec{c}_1 \\ \vec{c}_2 \end{pmatrix} \\ &= (q, g)L_1\vec{c} = (q, g)\vec{0} = 0. \end{aligned}$$

So there exists a nonzero element $u_0 \in D(B_1)$ and $u_0 \in \ker B_1$. This means that B_1 is not injective. Hence the operator B_1 is injective if and only if $\det L \neq 0$.
(ii) Since $\det L \neq 0$, system (3.23) for all $f \in X$ has the unique solution of the form

$$col\left(\Psi(Au), \Psi(A^2u), F(Au), F(A^2u)\right) = L^{-1}col\left(F(\widehat{A}^{-1}f), \Psi(\widehat{A}^{-1}f), \Psi(f), F(f)\right) \tag{3.27}$$

and the operator B_1 is injective. Substituting (3.27) into (3.22) we obtain the unique solution (3.19) of problem (3.15)-(3.16). In the above solution the element f is arbitrary. Consequently, $R(B_1) = X$. Since the operators $\widehat{A}^{-2}, \widehat{A}^{-1}$ and the functionals F and Ψ are bounded, from (3.19) follows the boundedness of B_1^{-1}, i.e. the operator B_1 is correct. □

The next corollary follows from the above theorem for $q = g = \vec{0}$ and is useful for solving some class of differential equations with nonlocal boundary conditions.

Corollary 3.2. *Let the space X, the operators $A, \widehat{A}$, the vectors $\mathbf{z}$, Φ, Ψ, F and matrices D, N be defined as in Theorem 3.2 and the operator $B_1 : X \to X$ be defined by*

$$\begin{aligned} B_1 u &= A^2 u = f, \\ D(B_1) &= \left\{ u \in D(A^2) : \Phi(u) = N\Psi(Au), \quad \Phi(Au) = DF(Au) + N\Psi(A^2u) \right\}. \end{aligned} \tag{3.28}$$

Then:
(i) the operator B_1 corresponding to problem (3.28) is injective if and only if

$$det\, V = \det[I_s - F(\mathbf{z})D] \neq 0, \tag{3.29}$$

(ii) if the operator B_1 is injective, then it is correct and the unique solution of (3.28) is given by

$$u = B_1^{-1} f = \widehat{A}^{-2} f + \ (\mathbf{z}N, \widehat{A}^{-1}\mathbf{z}N, \widehat{A}^{-1}\mathbf{z}D) L_0 col\left(F(\widehat{A}^{-1}f), \Psi(\widehat{A}^{-1}f), \Psi(f)\right) \tag{3.30}$$

where

$$L_0 = \begin{pmatrix} \Psi(\mathbf{z})DV^{-1} & I_n & \Psi(\mathbf{z})\left[I_m + DV^{-1}F(\mathbf{z})\right]N \\ 0_{ns} & 0_n & I_n \\ V^{-1} & 0_{sn} & V^{-1}F(\mathbf{z})N \end{pmatrix}. \tag{3.31}$$

Proof. (i) For $g = q = \vec{0}$ from (3.17) and (3.18) it immediately follows that

$$\det L = \det \begin{pmatrix} 0_{sn} & -F(\mathbf{z})N & V & 0_s \\ I_n & -\Psi(\mathbf{z})N & -\Psi(\mathbf{z})D & 0_{ns} \\ 0_n & I_n & 0_{ns} & 0_{ns} \\ 0_{sn} & 0_{sn} & 0_s & I_s \end{pmatrix} \neq 0 \tag{3.32}$$

and (3.29), respectively. It is easy to verify that the inverse matrix for L is

$$L^{-1} = \begin{pmatrix} \Psi(\mathbf{z})DV^{-1} & I_n & \Psi(\mathbf{z})\left[I_m + DV^{-1}F(\mathbf{z})\right]N & 0_{ns} \\ 0_{ns} & 0_n & I_n & 0_{ns} \\ V^{-1} & 0_{sn} & V^{-1}F(\mathbf{z})N & 0_s \\ 0_s & 0_{sn} & 0_{sn} & I_s \end{pmatrix}.$$

From (3.19) for $g = q = \vec{0}$ it follows the unique solution of (3.28) has the form

$$\begin{aligned} u = B_1^{-1} f = \widehat{A}^{-2} f \quad &+ \quad (\mathbf{z}N, \widehat{A}^{-1}\mathbf{z}N, \widehat{A}^{-1}\mathbf{z}D, \mathbf{0})L^{-1} \\ &\cdot \quad col\left(F(\widehat{A}^{-1}f), \Psi(\widehat{A}^{-1}f), \Psi(f), F(f)\right). \end{aligned} \tag{3.33}$$

It is easy to verify that

$$\begin{aligned} &(\mathbf{z}N, \widehat{A}^{-1}\mathbf{z}N, \widehat{A}^{-1}\mathbf{z}D, \mathbf{0})L^{-1} col\left(F(\widehat{A}^{-1}f), \Psi(\widehat{A}^{-1}f), \Psi(f), F(f)\right) \\ &= (\mathbf{z}N, \widehat{A}^{-1}\mathbf{z}N, \widehat{A}^{-1}\mathbf{z}D) L_0\, col\left(F(\widehat{A}^{-1}f), \Psi(\widehat{A}^{-1}f), \Psi(f)\right). \end{aligned} \tag{3.34}$$

Hence, from (3.33) follows (3.30). $\square$

4 Decomposition method for ordinary differential and integro-differential equations

Theorem 4.1. *Let the space X, the operators $A, \widehat{A}$, the vectors $\mathbf{z}, q, g, \Phi, \Psi, F$, matrices D, N be defined as in Theorem 3.2 and the operators $B, B_1 : X \to X$ be defined by*

$$\begin{aligned} Bu &= Au - gF(Au), \qquad D(B) = \{u \in D(A) : \Phi(u) = N\Psi(Au)\}, && (4.1)\\ B_1 u &= A^2 u - qF(Au) - gF(A^2 u), && (4.2)\\ D(B_1) &= \left\{u \in D(A^2) :\, \Phi(u) = N\Psi(Au), \quad \Phi(Au) = DF(Au) + N\Psi(A^2 u)\right\}, \end{aligned}$$

respectively, where the components of the vector g are linearly independent. Then:
(i) the operator B_1 is decomposed in $B_1 = B^2$ (i.e. B_1 is quadratic) if

$$\begin{aligned} g \in D(A)^n, \quad q &= Ag - gF(Ag), \quad D = \Phi(g) - N\Psi(Ag), \qquad \textit{where} && (4.3)\\ B^2 u &= A^2 u - [Ag - gF(Ag)]F(Au) - gF(A^2 u), && (4.4)\\ D(B^2) &= \{u \in D(A^2) : \Phi(u) = N\Psi(Au), \quad \Phi(Au)\\ &= [\Phi(g) - N\Psi(Ag)]F(Au) + N\Psi(A^2 u)\}, && (4.5) \end{aligned}$$

(ii) if the vectors q, g and matrices D, N satisfy (4.3), then the operator B_1 corresponding to the problem

$$\begin{aligned} B_1 u &= A^2 u - qF(Au) - gF(A^2 u) = f, \quad f \in X, && (4.6)\\ D(B_1) &= \left\{u \in D(A^2) :\, \Phi(u) = N\Psi(Au), \quad \Phi(Au) = DF(Au) + N\Psi(A^2 u)\right\} \end{aligned}$$

is injective if and only if

$$det\, W = det[I_s - F(g)] \neq 0, \tag{4.7}$$

(iii) if the vectors q, g and matrices D, N satisfy (4.3) and $det\, W \neq 0$, then the operator B_1 corresponding to (4.6) is correct and the unique solution of the problem (4.6) is

$$\begin{aligned} u = B_1^{-1} f &= \widehat{A}^{-2} f + YF(\widehat{A}^{-1} f) + \mathbf{z}N\Psi(\widehat{A}^{-1} f) + [\widehat{A}^{-1} Y + YF(V)\\ &+ \mathbf{z}N\Psi(Y)]F(f) + [\widehat{A}^{-1}\mathbf{z} + YF(\mathbf{z}) + \mathbf{z}N\Psi(\mathbf{z})]N\Psi(f) \qquad \textit{or} && (4.8)\\ u = B_1^{-1} f &= \widehat{A}^{-1}\tilde{f} + YF(\tilde{f}) + \mathbf{z}N\Psi(\tilde{f}), \qquad \textit{where} && (4.9)\\ \tilde{f} &= \widehat{A}^{-1} f + YF(f) + \mathbf{z}N\Psi(f), \quad Y = [\widehat{A}^{-1} g + \mathbf{z}N\Psi(g)]W^{-1}. && (4.10) \end{aligned}$$

Proof. (i) First we prove formula (4.5). Denote by

$$\widetilde{D} = \{u \in D(A^2) : \Phi(u) = N\Psi(Au), \quad \Phi(Au) = [\Phi(g) - N\Psi(Ag)]F(Au) + N\Psi(A^2 u)\}.$$

Let $u \in D(B^2)$ and $g \in D(A)^n$. Then, by definition, $u \in D(B)$ and $Bu \in D(B)$, which by (4.1) implies that $u \in D(A)$, $\Phi(u) = N\Psi(Au)$ and $Bu \in D(A)$, $\Phi(Bu) = N\Psi(ABu)$. From $Bu = Au - gF(Au) \in D(A)$ it follows that $u \in D(A^2)$. Further from the equation $\Phi(Bu) = N\Psi(ABu)$ is implied that $u \in \widetilde{D}$.
Conversely, let $u \in \widetilde{D}$, this means that $u \in D(A^2)$, $\Phi(u) = N\Psi(Au)$ and $\Phi(Au) - [\Phi(g) - N\Psi(Ag)]F(Au) = N\Psi(A^2 u)$. Then $u \in D(B)$, $Bu \in D(A)$ and $\Phi(Au) - \Phi(g)F(Au) = N\Psi(A^2 u) + N\Psi(Ag)F(Au)$, which implies that $\Phi(Bu) = N\Psi(ABu)$ or $Bu \in D(B)$. Hence $u \in D(B^2)$ and so (4.5) holds. Now we prove formula (4.4). Let $u \in D(B^2)$, $y = Bu$, $g \in D(A)^n$. Then

$$\begin{aligned} B^2 u &= By = Ay - gF(Ay) = ABu - gF(ABu)\\ &= A[Au - gF(Au)] - gF\left(A[Au - gF(Au)]\right)\\ &= A^2 u - AgF(Au) - gF(A^2 u) + gF(Ag)F(Au), \end{aligned}$$

which yields (4.4). By comparing (4.2) with (4.4) it is easy to verify that $B_1 u = B^2 u$ if (4.3) holds. (ii) Let (4.3) hold and $\det W \neq 0$. By statement (i), $B_1 = B^2$, so $D(B_1) = D(B^2)$. Since $\Phi(z) = I_m$, relations (4.5) can be represented as

$$\begin{aligned}
&\Phi\left(u - \mathbf{z}N\Psi(Au)\right) = 0,\\
&\Phi\left(Au - [g - \mathbf{z}N\Psi(Ag)]F(Au) - \mathbf{z}N\Psi(A^2u)\right) = 0,
\end{aligned}$$

which taking into account (3.6) implies that

$$u - \mathbf{z}N\Psi(Au) \in D(\widehat{A}), \quad Au - [g - \mathbf{z}N\Psi(Ag)]F(Au) - \mathbf{z}N\Psi(A^2u) \in D(\widehat{A}).$$

Then from (4.4), since $z \in [\ker A]^m$, we obtain

$$\begin{aligned}
&\widehat{A}\left(Au - gF(Au) + \mathbf{z}N[\Psi(Ag)F(Au) - \Psi(A^2u)]\right) + g[F(Ag)F(Au) - F(A^2u)] = f,\\
&Au - gF(Au) + \mathbf{z}N[\Psi(Ag)F(Au) - \Psi(A^2u)] + \widehat{A}^{-1}g[F(Ag)F(Au) - F(A^2u)]\\
&\qquad = \widehat{A}^{-1}f,\\
&\widehat{A}\left(u - \mathbf{z}N\Psi(Au)\right) - gF(Au) + \mathbf{z}N[\Psi(Ag)F(Au) - \Psi(A^2u)] + \widehat{A}^{-1}g[F(Ag)F(Au)\\
&\qquad - F(A^2u)] = \widehat{A}^{-1}f,\\
&u - \mathbf{z}N\Psi(Au) - \widehat{A}^{-1}gF(Au) + \widehat{A}^{-1}\mathbf{z}N[\Psi(Ag)F(Au) - \Psi(A^2u)]\\
&\qquad + \widehat{A}^{-2}g[F(Ag)F(Au) - F(A^2u)] = \widehat{A}^{-2}f.
\end{aligned}$$

Futher, taking into account (4.4), we get

$$\begin{aligned}
&A^2u = [Ag - gF(Ag)]F(Au) + gF(A^2u) + f,\\
&Au = gF(Au) - \mathbf{z}N[\Psi(Ag)F(Au) - \Psi(A^2u)] - \widehat{A}^{-1}g[F(Ag)F(Au) - F(A^2u)] + \widehat{A}^{-1}f,\\
&u = \mathbf{z}N\Psi(Au) + \widehat{A}^{-1}gF(Au) - \widehat{A}^{-1}\mathbf{z}N[\Psi(Ag)F(Au) - \Psi(A^2u)] - \widehat{A}^{-2}g[F(Ag)F(Au)\\
&\qquad - F(A^2u)] + \widehat{A}^{-2}f.
\end{aligned}$$

Acting by vector-functionals F and Ψ on the previous equations we get

$$\begin{aligned}
&F(Au) = F(g)F(Au) - F(\mathbf{z})N[\Psi(Ag)F(Au) - \Psi(A^2u)] - F(\widehat{A}^{-1}g)[F(Ag)F(Au)\\
&\qquad - F(A^2u)] + F(\widehat{A}^{-1}f),\\
&\Psi(Au) = \Psi(g)F(Au) - \Psi(\mathbf{z})N[\Psi(Ag)F(Au) - \Psi(A^2u)] - \Psi(\widehat{A}^{-1}g)[F(Ag)F(Au)\\
&\qquad - F(A^2u)] + \Psi(\widehat{A}^{-1}f),\\
&\Psi(A^2u) = [\Psi(Ag) - \Psi(g)F(Ag)]F(Au) + \Psi(g)F(A^2u) + \Psi(f),\\
&F(A^2u) = [F(Ag) - F(g)F(Ag)]F(Au) + F(g)F(A^2u) + F(f),
\end{aligned}$$

or

$$\begin{aligned}
&[I_s - F(g) + F(\mathbf{z})N\Psi(Ag) + F(\widehat{A}^{-1}g)F(Ag)]F(Au) - F(\widehat{A}^{-1}g)F(A^2u)\\
&\qquad - F(\mathbf{z})N\Psi(A^2u) = F(\widehat{A}^{-1}f),\\
&\Psi(Au) - [\Psi(g) - \Psi(\mathbf{z})N\Psi(Ag) - \Psi(\widehat{A}^{-1}g)F(Ag)]F(Au) - \Psi(\widehat{A}^{-1}g)F(A^2u)\\
&\qquad - \Psi(\mathbf{z})N\Psi(A^2u) = \Psi(\widehat{A}^{-1}f),\\
&\Psi(A^2u) - [\Psi(Ag) - \Psi(g)F(Ag)]F(Au) - \Psi(g)F(A^2u) = \Psi(f),\\
&[F(Ag) - F(g)F(Ag)]F(Au) + [F(g) - I_n]F(A^2u) = -F(f).
\end{aligned}$$

Denoting

$$D_1 = -[\Psi(g) - \Psi(\mathbf{z})N\Psi(Ag) - \Psi(\widehat{A}^{-1}g)F(Ag)],$$

$$D_2 = W + F(\mathbf{z})N\Psi(Ag) + F(\widehat{A}^{-1}g)F(Ag),$$

from the above equations we get

$$\begin{pmatrix} 0_{sn} & -F(\mathbf{z})N & D_2 & -F(\widehat{A}^{-1}g) \\ I_n & -\Psi(\mathbf{z})N & D_1 & -\Psi(\widehat{A}^{-1}g) \\ 0_n & I_n & \Psi(g)F(Ag) - \Psi(Ag) & -\Psi(g) \\ 0_{sn} & 0_{sn} & WF(Ag) & -W \end{pmatrix} \begin{pmatrix} \Psi(Au) \\ \Psi(A^2u) \\ F(Au) \\ F(A^2u) \end{pmatrix} = \begin{pmatrix} F(\widehat{A}^{-1}f) \\ \Psi(\widehat{A}^{-1}f) \\ \Psi(f) \\ -F(f) \end{pmatrix}. \tag{4.11}$$

Denote the matrix on the left by L_2, then

$$\begin{aligned} \det L_2 &= \det \begin{pmatrix} 0_{sn} & -F(\mathbf{z})N & D_2 & -F(\widehat{A}^{-1}g) \\ I_n & -\Psi(\mathbf{z})N & D_1 & -\Psi(\widehat{A}^{-1}g) \\ 0_n & I_n & \Psi(g)F(Ag) - \Psi(Ag) & -\Psi(g) \\ 0_{sn} & 0_{sn} & WF(Ag) & -W \end{pmatrix} \\ &= \pm\det \begin{pmatrix} -F(\mathbf{z})N & W + F(\mathbf{z})N\Psi(Ag) + F(\widehat{A}^{-1}g)F(Ag) & -F(\widehat{A}^{-1}g) \\ I_n & \Psi(g)F(Ag) - \Psi(Ag) & -\Psi(g) \\ 0_{sn} & WF(Ag) & -W \end{pmatrix}. \end{aligned}$$

Now, using property (3.2), where $D = F(Ag)$, we obtain

$$\det L_2 = \pm\det \begin{pmatrix} -F(\mathbf{z})N & W + F(\mathbf{z})N\Psi(Ag) & -F(\widehat{A}^{-1}g) \\ I_n & -\Psi(Ag) & -\Psi(g) \\ 0_{sn} & 0_s & -W \end{pmatrix}.$$

By property (3.1), where $D = F(\mathbf{z})N$, we get

$$\begin{aligned} \det L_2 &= \pm\det \begin{pmatrix} 0_{sn} & W & -F(\widehat{A}^{-1}g) - F(\mathbf{z})N\Psi(g) \\ I_n & -\Psi(Ag) & -\Psi(g) \\ 0_{sn} & 0_s & -W \end{pmatrix} \\ &= \pm\det \begin{pmatrix} W & -F(\widehat{A}^{-1}g) - F(\mathbf{z})N\Psi(g) \\ 0_s & -W \end{pmatrix} \\ &= \pm \begin{pmatrix} W^{-1} & 0_s \\ 0_s & W^{-1} \end{pmatrix} \begin{pmatrix} W & -F(\widehat{A}^{-1}g) - F(\mathbf{z})N\Psi(g) \\ 0_s & -W \end{pmatrix} \begin{pmatrix} W & 0_s \\ 0_s & W \end{pmatrix} \\ &= \pm \begin{pmatrix} I_s & W^{-1}[-F(\widehat{A}^{-1}g) - F(\mathbf{z})N\Psi(g)] \\ 0_s & -I_s \end{pmatrix} \begin{pmatrix} W & 0_s \\ 0_s & W \end{pmatrix} = \pm|W|^2. \end{aligned}$$

So, $\det L_2 = \pm|W|^2$. Let $u \in \ker B_1$. Then in system (4.11) $f = 0$ and from (4.11) we get $L_2 col\,(\Psi(Au), \Psi(A^2u), F(Au), F(A^2u)) = \vec{0}$, which, since $\det L_2 = \pm|W|^2 \neq 0$, yields $\Psi(Au) = \Psi(A^2u) = \vec{0}$, $F(Au) = F(A^2u) = \vec{0}$. Substitution of these values into (4.4), (4.5) implies that $B_1u = B^2u = A^2u = 0$, $\Phi(u) = \Phi(Au) = \vec{0}$. Taking into account (3.8) we acquire $u \in D(\widehat{A}^2)$ and $B_1u = \widehat{A}^2u = 0$. By hypothesis $\widehat{A}$ is correct and so $\widehat{A}^2$ is also correct and $u = 0$. Thus $\ker B_1 = \{0\}$ and B_1 is injective.

To prove the converse, let $\det W = 0$. Then there exists a vector $\vec{c} = (c_1, ..., c_s) \neq \vec{0}$ such that $W\vec{c} = \vec{0}$. Consider the element $u_0 = \widehat{A}^{-1}g\vec{c} + \mathbf{z}N\Psi(g)\vec{c}$. Note that $u_0 \neq 0$, otherwise $\widehat{A}^{-1}g\vec{c} = \mathbf{z}N\Psi(g)\vec{c} = \vec{0}$ and $\vec{c} = \vec{0}$, since $g_1, \dots, g_s$ are linearly independent. Then $\Phi(u_0) = N\Psi(g)\vec{c}$, $\Phi(Au_0) =$

$\Phi(g)\vec{c}$, $\Psi(Au_0) = \Psi(g)\vec{c}$, $F(Au_0) = F(g)\vec{c}$, $F(A^2u_0) = F(Ag)\vec{c}$, $\Psi(A^2u_0) = \Psi(Ag)\vec{c}$. It is evident that u_0 satisfies the first boundary condition of (4.5). Substituting the above values into the second boundary condition of (4.5), we get

$$\begin{aligned}
&\Phi(g)\vec{c} - \Phi(g)F(g)\vec{c} + N\Psi(Ag)F(g)\vec{c} - N\Psi(Ag)\vec{c} = \vec{0},\\
&\Phi(g)[I_n - F(g)]\vec{c} - N\Psi(Ag)[I_n - F(g)]\vec{c} = \vec{0},\\
&\Phi(g)W\vec{c} - N\Psi(Ag)W\vec{c} = \vec{0}.
\end{aligned}$$

So, $u_0 \in D(B^2)$. Furthermore $u_0 \in \ker B^2$, since

$$B^2u_0 = A^2u_0 - [Ag - gF(Ag)]F(Au_0) - gF(A^2u_0) =$$

$$Ag\vec{c} - [Ag - gF(Ag)]F(g)\vec{c} - gF(Ag)\vec{c} =$$

$$Ag[I_n - F(g)]\vec{c} - gF(Ag)[I_n - F(g)]\vec{c} = AgW\vec{c} - gF(Ag)W\vec{c} = 0.$$

So, there exists a nonzero element $u_0 \in D(B^2)$ and $u_0 \in \ker B^2$. That means $B_1 = B^2$ is not injective. Thus $B_1 = B^2$ is injective if and only if $\det W \neq 0$.
(iii) Let the vectors q, g and matrix D satisfy (4.3) and $\det W \neq 0$. Then, by statement (ii), the operator B_1 is injective and problem $B_1u = f$ from (4.6) has a unique solution. We remind that by Theorem 3.1 the unique solution of equation (4.1) for all $f \in X$ is given by

$$u = B^{-1}f = \widehat{A}^{-1}f + \left[\widehat{A}^{-1}g + \mathbf{z}N\Psi(g)\right]W^{-1}F(f) + \mathbf{z}N\Psi(f). \tag{4.12}$$

Let $B_1u = B^2u = f$, where $f \in X$. Denoting $Y = \left[\widehat{A}^{-1}g + \mathbf{z}N\Psi(g)\right]W^{-1}$ and $\tilde{f} = Bu$, we get $B\tilde{f} = f$. Then by (3.14) the solution of this equation is given by

$$\tilde{f} = B^{-1}f = \widehat{A}^{-1}f + YF(f) + \mathbf{z}N\Psi(f). \tag{4.13}$$

Applying again (3.14), we find the solution of the problem $Bu = \tilde{f}$ or $B_1u = f$:

$$u = B_1^{-1}f = B^{-1}\tilde{f} = \widehat{A}^{-1}\tilde{f} + YF(\tilde{f}) + \mathbf{z}N\Psi(\tilde{f}). \tag{4.14}$$

So we obtained (4.9). Substituting the value of $\tilde{f}$ from (4.13) into (4.14), we get

$$\begin{aligned}
u &= B_1^{-1}f = B^{-2}f = B^{-1}\tilde{f} = \widehat{A}^{-2}f + \widehat{A}^{-1}YF(f) + \widehat{A}^{-1}\mathbf{z}N\Psi(f)\\
&+ Y[F(\widehat{A}^{-1}f) + F(Y)F(f) + F(\mathbf{z})N\Psi(f)]\\
&+ \mathbf{z}N[\Psi(\widehat{A}^{-1}f) + \Psi(Y)F(f) + \Psi(\mathbf{z})N\Psi(f)]\\
&= \widehat{A}^{-2}f + YF(\widehat{A}^{-1}f) + \mathbf{z}N\Psi(\widehat{A}^{-1}f) + [\widehat{A}^{-1}Y + YF(Y) + \mathbf{z}N\Psi(Y)]F(f)\\
&+ [\widehat{A}^{-1}\mathbf{z} + YF(\mathbf{z}) + \mathbf{z}N\Psi(\mathbf{z})]N\Psi(f).
\end{aligned} \tag{4.15}$$

Thus we obtain solution (4.8). In the above solutions f is arbitrary, consequently, $R(B_1) = X$. Since the operators $\widehat{A}^{-2}, \widehat{A}^{-1}$ and functionals F and Ψ are bounded, from (4.8) follows the boundedness of $B_1^{-1} = B^{-2}$, i.e. the operator B_1 is correct. □

The next corollary follows from the above theorem in the case $q = g = 0$ and is useful for solving some class of ordinary differential equations with nonlocal boundary conditions.

Corollary 4.1. *Let the space X, the operators $A, \widehat{A}$, the vector-functionals Φ, Ψ, F and matrix N be defined as in Theorem 4.1 and the operator $B_1 : X \to X$ be defined by*

$$\begin{aligned} B_1 u &= A^2 u = f, \\ D(B_1) &= \left\{ u \in D(A^2) : \Phi(u) = N\Psi(Au), \quad \Phi(Au) = N\Psi(A^2 u) \right\}. \end{aligned} \tag{4.16}$$

Then the operator B_1 corresponding to problem (4.16) is correct and the unique solution of (4.16) is given by

$$u = B_1^{-1} f = \widehat{A}^{-2} f + \mathbf{z} N\Psi(\widehat{A}^{-1} f) + [\widehat{A}^{-1}\mathbf{z} + \mathbf{z}N\Psi(\mathbf{z})]N\Psi(f). \tag{4.17}$$

Proof. For $g = q = \vec{0}$ from (4.7) it immediately follows that $\det W = \det I_n = 1 \neq 0$. By Theorem 4.1, the operator B_1 is correct. From (4.8) for $g = q = \vec{0}$ follows solution (4.17). $\square$

5 Examples

In the next example we use the extension method of Theorem 3.1.

Example 1. The integro-differential equation on $C[0,1]$

$$u'' - e^t \int_0^1 u''(x)\cos(\pi x)dx = -\pi^2 \sin(\pi t), \tag{5.1}$$

$$u(0) = \frac{1}{\pi}\int_0^1 x u''(x)dx, \quad u'(0) = -\frac{\pi+1}{5\pi}\int_0^1 (x+2)u''(x)dx, \tag{5.2}$$

with integral boundary conditions is correct and the unique solution of (5.1)-(5.2) is given by

$$u(t) = t + \sin(\pi t) - 1. \tag{5.3}$$

Proof. If we compare (5.1) with (3.9), it is natural to take $X = Y = C[0,1]$, $Au = \widehat{A}u = u''(t)$, $D(A) = \{u \in C^2[0,1]\}$, $D(\widehat{A}) = \{u \in D(A) : u(0) = u'(0) = 0\}$, $g = e^t$, $f = -\pi^2 \sin(\pi t)$, $m = 2$, $n = 1$, $\mathbf{z} = (1, t)$, $F(\widehat{A}u) = \int_0^1 u''(x)\cos(\pi x)dx$,

$$\begin{aligned} \Psi_1(\widehat{A}u) &= \int_0^1 x u''(x)dx, \quad \Psi_2(\widehat{A}u) = \int_0^1 (x+2)u''(x)dx, \\ \Psi &= \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad N = \begin{pmatrix} 1/\pi & 0 \\ 0 & -(\pi+1)/(5\pi) \end{pmatrix}, \qquad \Phi(u) = \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \\ \Phi_1(u) &= u(0), \quad \Phi_2(u) = u'(0), \quad Bu = Au - gF(\widehat{A}u), \quad D(B) = \{u(x) \in D(A) : \\ u(0) &= \frac{1}{\pi}\int_0^1 x u''(x)dx, \quad u'(0) = -\frac{\pi+1}{5\pi}\int_0^1 (x+2)u''(x)dx\}. \end{aligned}$$

The operator A, by Lemma 3.3, is closed and maximal. It is easy to verify that $z_1 = 1$, $z_2 = t$ are the linearly independent solutions of the equation $u''(t) = 0$, the set $\mathbf{z} = (1, t)$ is biorthogonal to (Φ_1, Φ_2) and $\widehat{A}u = f$ is the initial value problem which is correct and has the unique solution $\widehat{A}^{-1}f(t) = \int_0^t (t-x)f(x)dx$. For $f = -\pi^2\sin(\pi t)$ we compute $\widehat{A}^{-1}f(t) = -\pi^2\int_0^t (t-x)\sin(\pi x)dx = \sin(\pi t) - \pi t$. Then $F(f) = \int_0^1 f(x)\cos(\pi x)dx = -\pi^2\int_0^1 \sin(\pi x)\cos(\pi x)dx = 0$, $F(g) = \int_0^1 e^x\cos(\pi x)dx = -\frac{1}{\pi^2+1}(e+1)$, $\Psi_1(f) = \int_0^1 x f(x)dx = -\pi^2\int_0^1 x\sin(\pi x)dx = -\pi$, $\Psi_2(f) = \int_0^1 (x+2)f(x)dx = -\pi^2\int_0^1 (x+2)\sin(\pi x)dx = -5\pi$. From $|F(f)| \leq \int_0^1 |f(x)\cos(\pi x)|dx \leq ||f||_C$ it follows that $F \in C[0,1]^* = X^*$. In the same way it is proved that $\Psi_1, \Psi_2 \in X^*$. Since $Au = \mathbf{A}^2 u$, where $\mathbf{A}u = u'(t)$, we use Remark 2 and take $\widehat{X}_A = C^2[0,1]$. Then $|\Phi_1(u)| =$

$|u(0)| \le ||u||_C + ||u'||_C + ||u''||_C = ||u||_{\widehat{X}_A}$, implies $\Phi_1 \in \widehat{X}_A^*$. From $|\Phi_2(u)| = |u'(0)| \le ||u||_C + ||u'||_C + ||u''||_C = ||u||_{\widehat{X}_A}$, it follows that $\Phi_2 \in \widehat{X}_A^*$. Since $I_n - F(g) = 1 + \frac{1}{\pi^2+1}(e+1) \neq 0$ and $F(f) = 0$, problem (5.1)-(5.2), by Theorem 3.1, has the unique solution

$$u(t) = \widehat{A}^{-1}f + \mathbf{z}N\Psi(f) = \sin(\pi t) - \pi t - (1,t)\begin{pmatrix} 1/\pi & 0 \\ 0 & -(\pi+1)/(5\pi) \end{pmatrix}\begin{pmatrix} \pi \\ 5\pi \end{pmatrix},$$

which follows from (3.11) and yields (5.3). □

Now we use the extension method for the following differential equation with integral boundary conditions. We can solve this equation by Corollary 3.1, but it is easier to solve it by Corollary 3.2.
Example 2. The differential equation on $C[0,1]$

$$u'' = 24t + 6, \quad u(0) = 20\int_0^1 x^3u'(x)dx, \quad u'(0) = 30\int_0^1 x^2u'(x)dx + 20\int_0^1 x^3u''(x)dx, \tag{5.4}$$

is correct and the unique solution of (5.4) is given by the formula

$$u(t) = 4t^3 + 3t^2 - 27t - 71. \tag{5.5}$$

Proof. If we compare (5.4) with (3.28), it is natural to take

$$Au = \widehat{A}u = u',\ D(A) = C^1[0,1],\ D(\widehat{A}) = \{u(t) \in D(A):\ u(0) = 0\},$$

$$A^2u = \widehat{A}^2u = u'',\ D(A^2) = C^2[0,1],\ D(\widehat{A}^2) = \{u(t) \in D(A^2) : u(0) = u'(0) = 0\},$$

$$B_1 : C[0,1] \to C[0,1],\ B_1u = A^2u, \quad D(B_1) = \{u(t) \in D(A^2) : u(0) = 20\int_0^1 x^3u'(x)dx,$$

$$u'(0) = 30\int_0^1 x^2u'(x)dx + 20\int_0^1 x^3u''(x)dx\},$$

$$f(t) = 24t + 6,\ m = n = 1,$$

$$F(Au) = \int_0^1 x^2u'(x)dx, \quad \Phi(u) = u(0), \quad \Phi(Au) = u'(0),$$

$$\Psi(Au) = \int_0^1 x^3u'(x)dx, \quad \Psi(A^2u) = \int_0^1 x^3u''(x)dx,$$

$N = 20$, $D = 30$ and $\mathbf{z} = 1$, since $\Phi(\mathbf{z}) = \Phi(1) = 1$. Then $\Psi(f) = \int_0^1 x^3f(x)dx$. We remind that the matrices D, N are defined in Theorem 3.2 and $\mathbf{z} = 1$ is a linearly independent solution of the equation $Au = u'(t) = 0$. Since $Au = \mathbf{A}u = u'(t)$, we use Remark 2 and take $\widehat{X}_A = C^1[0,1]$. From $|\Phi(u)| = |u(0)| \le ||u||_C + ||u'||_C = ||u||_{C^1}$ for all $u(t) \in C^1[0,1]$ it follows that $\Phi \in \widehat{X}_A^*$. Note that $\Psi \in C[0,1]^*$, since $|\Psi(f)| = |\int_0^1 x^3f(x)dx| \le 0.25||f||_C$ for all $f(t) \in C[0,1]$. By analogy it is proved that $F \in C[0,1]^*$. The Cauchy problem $\widehat{A}u = u' = h$, $u(0) = 0$ is correct and $u(t) = \widehat{A}^{-1}h(t) = \int_0^t h(x)dx$ for $h \in C[0,1]$. Then $\widehat{A}^{-2}h(t) = \int_0^t (t-x)h(x)dx$ and now we find

$$\widehat{A}^{-1}\mathbf{z} = t, \quad \Psi(\mathbf{z}) = \frac{1}{4}, \quad F(\mathbf{z}) = \frac{1}{3},$$

$$\widehat{A}^{-1}f(t) = 12t^2 + 6t, \quad \widehat{A}^{-2}f(t) = 4t^3 + 3t^2, \quad \Psi(\widehat{A}^{-1}f) = \frac{16}{5},$$

$$F(\widehat{A}^{-1}f) = \frac{39}{10}, \quad \Psi(f) = \frac{63}{10}.$$

Substituting these values into (3.29), we get $\det V = \det[I_n - F(z)D] = 1 - 30(1/3) = -9 \neq 0$. So, by Corollary 3.2, the problem (5.4) is correct. Then (3.31) implies

$$L_0 = \begin{pmatrix} (30/4)(-1/9) & 1 & 5[1+30(-1/9)(1/3)] \\ 0 & 0 & 1 \\ -1/9 & 0 & (-1/9)(1/3)20 \end{pmatrix} = \begin{pmatrix} -5/6 & 1 & -5/9 \\ 0 & 0 & 1 \\ -1/9 & 0 & -20/27 \end{pmatrix}.$$

To find the unique solution of this problem we substitute the above values into (3.30) and get

$$\begin{aligned} u(t) = B_1^{-1} f &= 4t^3 + 3t^2 + (20, 20t, 30t) \begin{pmatrix} -5/6 & 1 & -5/9 \\ 0 & 0 & 1 \\ -1/9 & 0 & -20/27 \end{pmatrix} \begin{pmatrix} 39/10 \\ 16/5 \\ 63/10 \end{pmatrix} \\ &= 4t^3 + 3t^2 - 27t - 71. \end{aligned}$$

□

The next example we solve by the extension method of Theorem 3.2 and by the decomposition method of Theorem 4.1.

Example 3. The integro-differential equation on $C[0,1]$ with integral boundary conditions

$$\begin{gathered} u'' - \frac{1}{2}(2-t)\int_0^1 xu'(x)dx - t\int_0^1 xu''(x)dx = \frac{1}{3}(t+6), \\ u(0) = \int_0^1 u'(x)dx, \quad u'(0) = -\int_0^1 xu'(x)dx + \int_0^1 u''(x)dx \end{gathered} \tag{5.6}$$

is correct and the unique solution of (5.6) is given by the formula

$$u(t) = \frac{1}{48}(16t^3 + 135t^2 + 144t + 295). \tag{5.7}$$

Proof. If we compare (5.6) with (3.15)-(3.16), it is natural to take the operators $A, \widehat{A}$, the vectors $\mathbf{z}, \Phi$ as in Example 2,

$$q = \frac{1}{2}(2-t),\ g = t,\ f(t) = \frac{1}{3}(t+6) \in C[0,1],\ m = n = 1,\ F(Au) = \int_0^1 xu'(x)dx,$$

$$F(A^2u) = \int_0^1 xu''(x)dx, \quad \Psi(Au) = \int_0^1 u'(x)dx, \quad \Psi(A^2u) = \int_0^1 u''(x)dx,$$

$N = 1$, $D = -1$. Then $F(f) = \int_0^1 xf(x)dx$, $\Psi(f) = \int_0^1 f(x)dx$. Remind that the matrices D, N are defined in (3.16). We take $B_1 : C[0,1] \to C[0,1]$,

$$B_1 u = u'' - \frac{1}{2}(2-t)\int_0^1 xu'(x)dx - t\int_0^1 xu''(x)dx,$$

$$D(B_1) = \{u \in D(A^2) : u(0) = \int_0^1 u'(x)dx, \quad u'(0) = -\int_0^1 xu'(x)dx + \int_0^1 u''(x)dx\}.$$

It is easy to verify that $\Psi,\ F \in C[0,1]^*$. We compute

$$
\begin{aligned}
\widehat{A}^{-1}f(t) &= \frac{t}{6}(t+12), \quad \widehat{A}^{-1}g(t) = \frac{t^2}{2}, \quad \widehat{A}^{-1}q(t) = \frac{t(4-t)}{4},\\
\widehat{A}^{-2}f(t) &= \frac{t^2(t+18)}{18}, \quad \widehat{A}^{-2}g(t) = \frac{t^3}{6}, \quad \widehat{A}^{-2}q(t) = \frac{t^2(6-t)}{12},\\
F(\widehat{A}^{-1}g) &= \frac{1}{8}, \quad F(\widehat{A}^{-1}q) = \frac{13}{48}, \quad F(\widehat{A}^{-1}f) = \frac{17}{24}, \quad F(q) = \frac{1}{3},\ F(g) = \frac{1}{3},\\
F(f) &= \frac{10}{9}, \quad \Psi(\widehat{A}^{-1}q) = \frac{5}{12}, \quad \Psi(\widehat{A}^{-1}g) = \frac{1}{6}, \quad \Psi(\widehat{A}^{-1}f) = \frac{19}{18},\\
\Psi(f) &= \frac{13}{6}, \quad \Psi(q) = \frac{3}{4}, \quad \Psi(g) = \frac{1}{2}, \quad \Psi(\mathbf{z}) = 1, \quad F(\mathbf{z}) = 0.5.
\end{aligned}
$$

Substituting these values into (3.17), we get

$$
\det L = \begin{pmatrix} 0 & -\frac{1}{2} & 1-\frac{13}{48}-\frac{1}{2}(-1) & -\frac{1}{8}\\ 1 & -1 & -(-1)-\frac{5}{12} & -\frac{1}{6}\\ 0 & 1 & -\frac{3}{4} & -\frac{1}{2}\\ 0 & 0 & -\frac{1}{3} & 1-\frac{1}{3}\end{pmatrix} = \det\begin{pmatrix} 0 & -\frac{1}{2} & \frac{59}{48} & -\frac{1}{8}\\ 1 & -1 & \frac{7}{12} & -\frac{1}{6}\\ 0 & 1 & -\frac{3}{4} & -\frac{1}{2}\\ 0 & 0 & -\frac{1}{3} & \frac{2}{3}\end{pmatrix} = \frac{4}{9} \neq 0.
$$

Then, by Theorem 3.2, the problem (5.6) is correct. To find the unique solution of this problem we substitute the above values into (3.19) and get

$$
\begin{aligned}
u = B_1^{-1}f &= \frac{t^2(t+18)}{18} + \left(1, t, -t+\frac{t^2(6-t)}{12}, \frac{t^3}{6}\right)\begin{pmatrix} 0 & -\frac{1}{2} & \frac{59}{48} & -\frac{1}{8}\\ 1 & -1 & \frac{7}{12} & -\frac{1}{6}\\ 0 & 1 & -\frac{3}{4} & -\frac{1}{2}\\ 0 & 0 & -\frac{1}{3} & \frac{2}{3}\end{pmatrix}^{-1}\begin{pmatrix}17/24\\19/18\\13/6\\10/9\end{pmatrix}\\
&= \frac{1}{48}(16t^3+135t^2+144t+295).
\end{aligned}
$$

Now, to show the advantages of the decomposition method, we solve the previous problem by using Theorem 4.1.

If we compare (5.6) with (4.6), it is natural to take the operators $A, \widehat{A}$, the vectors $q,\ g,\ \mathbf{z}$, $F(Au)$, $F(A^2u)$, $\Phi(u)$, $\Phi(Au)$, $\Psi(Au)$, $\Psi(A^2u)$ and the matrices $N,\ D$ as in the extension method. Since $g = t \in D(A)$, $Ag - gF(Ag) = \frac{1}{2}(2-t) = q$, $\Phi(g) - N\Psi(Ag) = 0 - \int_0^1 1dx = -1 = D$, the conditions (4.3) are fulfilled and so the operator B_1 is quadratic. We compute $W = I - F(g) = \frac{2}{3} \neq 0$. Then $W^{-1} = \frac{3}{2}$ and, by Theorem 4.1, the problem (5.6) is correct. To find the solution of this problem by (4.9)-(4.10) we compute $\widehat{A}^{-1}f = \frac{1}{6}t(t+12)$, $\widehat{A}^{-1}g = \frac{t^2}{2}$, $\Psi(g) = \frac{1}{2}$, $F(f) = \frac{10}{9}$, $\Psi(f) = \frac{13}{6}$. Substituting these values into (4.10) we find

$$
Y = [\widehat{A}^{-1}g + \mathbf{z}N\Psi(g)]W^{-1} = \frac{3}{4}(t^2+1), \quad \tilde{f} = t^2 + 2t + 3.
$$

Then $\widehat{A}^{-1}\tilde{f} = \frac{1}{3}t^3 + t^2 + 3t$, $F(\tilde{f}) = \frac{29}{12}$, $\Psi(\tilde{f}) = \frac{13}{3}$. Finally using (4.9) we get the unique solution of (5.6)

$$
u = B_1^{-1}f = \widehat{A}^{-1}\tilde{f} + YF(\tilde{f}) + zN\Psi(\tilde{f}) = \frac{1}{48}(16t^3+135t^2+144t+295).
$$

□

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References

[1] N. Apreutesei, A. Ducrot, V. Volpert, *Travelling waves for integro-differential equations in population dinamics* AIMS, Discrete Cont. Dyn. Syst. Ser.B. 11 (2009), no. 3, 541–561.

[2] M. Arisawa, *A remark on the definitions of viscosity solutions for the integro-differential equations with Lėvy operators.* J. Math. Pures Appl. 89 (2008), 567–574.

[3] G. Avalishvili, M. Avalishvili, D. Gordeziani *On a nonlocal problem with integral boundary conditions for a multidimensional elliptic equation.* Applied Mathemat. Letters. 24 (2004), no. 4, 566–571.

[4] M. Benchohra, S.K. Ntouyas *Existence results on the semiinfinite interval for first and second order integro-differential equations in Banach spaces with nonlocal conditions.* Acta Univ. Palacki. Olomuc, Fac. rer. nat. V Mathematica. 41 (2002), 13–19.

[5] A.V. Bitsatze, A.A. Samarskii, *On some simplest generalization of linear elliptic problems.* Dokl. Akad. Nauk SSSR. 185 (1969), 739–740 (in Russian).

[6] F. Bloom, *Ill posed problems for integro-differential equations in mechanics and electromagnetic theory.* SIAM. 1981.

[7] J.R. Cannon, *The solution of the heat equation subject to the specification of energy.* Quart. Appl. Math. 21 (1963), 155–160.

[8] J.M. Cushing, *Integro-differential equations and delay models in population dynamics.* Springer. 1977.

[9] A.A. Dezin, *Nonstandard problems.* Matematicheskie Zametki, 41 (1987), no. 3, 356–364.

[10] D.S. Dzhumabaev, *On one approach to solve the linear boundary value problems with Fredholm integro-differential equations.* Journal of Computational and Applied Mathematics, 294 (2016), 342–357.

[11] D.N. Georgiou, I.E. Kougias, *On fuzzy Fredholm and Voltera integral equations.* Journal of Fuzzy Mathematics, 9 (2001) , no. 4, 943–951.

[12] V.A. Il'in, E.I. Moiseev, *Two-dimensional nonlocal boundary value problem for Poissons operator in differential and difference variants.* Math. Model., 2 (1990), no. 8, 132–156 (in Russian).

[13] N.I. Ionkin, *Solution of one boundary value problem of heat conduction theory with a nonclassical boundary condition.* Differ. Uravn., 13 (1977), no. 2, 294–304. (in Russian).

[14] T.S. Kalmenov, N.E. Tokmaganbetov, *On a nonlocal boundary value problem for the multidimensional heat equation in a noncylindrical domain.* Siberian Mathematical Journal 54 (2013), no. 6, 1287–1293.

[15] L.I. Kamynin, *On a boundary problem in the theory of heat conduction with a nonclassical boundary condition.* Zh. Vychisl. Math. Fiz. 4 (1964), no. 6, 1006–1024. (in Russian).

[16] M. Kandemir, *Nonlocal boundary value problems with transmission conditions.* Gulf Journal of Mathematics, 3 (2015), Is. 1, no. 3, 1–17.

[17] B.K. Kokebaev, M. Otelbaev, A.N. Shynybekov, *About restrictions and extensions of operators.* D.A.N. SSSR, 271 (1983), no. 6, 1307–1310. (in Russian).

[18] M.A. Kraemer, L.V. Kalachev, *Analysis of a class of nonlinear integro-differential equations arising in a forestry application.* Q. Appl. Math. 61 (2003), no. 3, 513–535.

[19] M.G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian operators and its aplications.* Mat. Sb. 20 (1947), no. 3, 431–495.

[20] J. Medlock, M. Kot, *Spreading disease: integro-differential equations old and new.* Elsevier, Mathematical Biosciences. 184 (2003), 201-222.

[21] J.Von Neumann, *Allgemeine eigenwerttheorie hermitescher functional operatoren.* Math. Ann., Bd. 102, (1929-1930), 49–131.

[22] R. Oinarov, I.N. Parasidis, *Correct extensions of operators with finite defect in Banach spases.* Izv. Akad. Kaz. SSR. 5 (1988), 42–46 (in Russian).

[23] R. Oinarov, S.S. Sagintaeva, *Smooth expansions of minimal operator in Banach spase.* Izv, AN. Rep. Kaz., N.**5**, (1994), 43–48 (in Russian).

[24] I.N. Parassidis, P.C. Tsekrekos, *Correct selfadjoint and positive extensions of nondensely defined symmetric operators.* Abstract and Applied Analysis, Hindawi Publishing Corporation, (2005), no. 7, 767–790.

[25] I.N. Parasidis, P.C. Tsekrekos, *Correct and self-adjoint problems for quadratic operators.* Eurasian Mathematical Journal. 1 (2010), no. 2, 122–135.

[26] I.N. Parasidis, E. Providas, *Extension operator method for the exact solution of integro-differential equations.* In Pardalos P., Rassias T.(eds) Contributions in Mathematics and Engineering: In Honor of Constantin Caratheodory, Springer, Cham. (2016), 473–496.

[27] A.D. Polyanin, A.I. Zhurov, *Exact solutions to some classes of nonlinear integral, integro-functional and integro-differential equation.* Dokl. Math. 77 (2008), no. 2, 315–319.

[28] L.S. Pulkina, *A nonlocal problem with integral condition for a hyperbolic equation.* Differ. Equations. 40 (2004), no. 7, 15–27.

[29] M.A. Sadybekov, B.K. Turmetov, *On an analog of periodic boundary value problems for the poisson equation in the disk.* Differential Equations. 50 (2014), 264–268.

[30] A.A. Samarskii, *On certain problems of the modern theory of differential equations.* Differ. Uravn. 16 (1980), no. 11, 1221–1228.

[31] E.W. Sachs, A.K. Strauss, *Efficient solution of a partial integro-differential equation in finance.* Appl. Numer. Math. 58 (2008), 1687–1703.

[32] K. Schumacher, *Traveling front solutions for integro-differential equations.* I., J. Rene Angew. Math. 316 (1980), 54–70.

[33] G.A. Shishkin, *Linear Fredholm integro-differential equations.* Ulan-Ude, Buryat State University, (2007), UDK 517.948 Sh 655 (in Russian).

[34] E. Shivanian, *Analysis of meshless local radial point in terpolation(MLRPI) on a nonlinear partial integro-differential equation arising in population dynamics.* Engineering Analysis with Boundary Elements. 37 (2003), 1693–1702.

[35] J.D. Tamarkin, *The notion of the Green's function in the theory of integro-differential equations.* Trans. Amer. Math. Soc. 29 (1927), 755–800.

[36] A.S. Tersenov, *Ultraparabolic equations and unsteady heat transfer.* Journal of Evaluation Equations. 5 (2005), no. 2, 277–289.

[37] A.M. Wazwaz, *Linear and nonlinear integral equations, methods and applications.* Springer, Beijing. 2011.

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Events

5TH INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF MATHEMATICS AND COMPUTER SCIENCE: THEORY, METHODOLOGY, PRACTICE" (APRIL 18–20, 2019, YELETS, RUSSIA)

The XX century is marked by the enrichment of world science with outstanding achievements in the field of mathematics, solving many important problems that remain relevant in the modern world. Such problems include, in particular, the problems considered in fundamental works of academician S. Chaplygin. Based on his works new research paths were set, and serious applied problems were addressed in the fields of aerodynamics, gas dynamics, hydrodynamics, and mechanics. They were further intensively developed due to the achievements of contemporary information technology.

To commemorate his activities was organized a large-scale scientific event in the historic homeland of academician S. Chaplygin — the 5th international conference "Actual problems of mathematics and computer science: theory, methodology, practice", dedicated to the 150th anniversary of the birth of academician S. Chaplygin.

The Ivan Bunin Yelets State University (Russia), the Samarkand State University (Uzbekistan), the Higher School of Insurance and Finance (Bulgaria),the Khachatur Abovyan Armenian State Pedagogical University (Armenia), and the Scientific and Methodological Council on Mathematics of the Ministry of Science and Higher Education of Russia held through April 18–20, 2019 the 5th International Conference "Actual problems of mathematics and computer science: theory, methodology, practice" dedicated to the 150th anniversary of academician S. Chaplygin.

The conference marked the three major milestones related to the development of mathematical science in the Lipetsk region and in the oldest university center in the region — the Ivan Bunin Yelets State University.

1. April 2019 is the 150th anniversary of the birth of S. Chaplygin (1869–1942) — a well-known Russian scientist, academician of the Academy of Sciences of the USSR. S. Chaplygin is an outstanding representative of the Lipetsk region, whose surname is immortalized in the name of the city Chaplygin (previously Ranenburg) in the Lipetsk region.

2. 2019 year is the 80th anniversary of the foundation of the Faculty of Physics and Mathematics. It is the oldest faculty of the Ivan Bunin Yelets State University, where students of the scientific school of academician N. Zhukovsky were taught, whose famous representative was academician S. Chaplygin.

3. In October 2019 there will be 10 years since the organization of the Lipetsk Branch of the Scientific and Methodological Council for Mathematics of the Ministry of Science and Higher Education of Russia on the basis of the Ivan Bunin Yelets State University.

The main goals of the conference were the creation of conditions for international scientific communication of representatives of fundamental and applied areas in the field of mathematics, understanding the importance of scientific works of S. Chaplygin, the actualization of his scientific achievements, taking into account the rapid development of information technologies and their adaptation to modern mathematical education.

The plenary session of the conference was opened by the Rector of the Ivan Bunin Yelets State University Professor E. Gerasimova and continued by the President of the International Academy of the History of Science Professor S. Demidov (Moscow, Russia), who presented the talk "Pure and Applied Mathematics at the M.V. Lomonosov Moscow State University in the first half of the twentieth century: N. Luzin and S. Chaplygin".

Professor A. Soleev (Samarkand, Uzbekistan) devoted his talk to basic ideas and general provisions of the Power Geometry. Professor A. Soldatov (Moscow, Russia) focused on the consideration of the Dirichlet problem for equations of mixed type. In her talk Professor G. Zhukova (Moscow, Russia) discussed the dependence of solutions to singularly perturbed linear differential systems on a small parameter. The talk of Professors O. Masina (Yelets, Russia) and O. Druzhinina (Moscow, Russia) was devoted to the analysis of the known and developed by the authors approaches to the study of the stability of intelligent control systems. The talks of Professors V. Tikhomirov (Moscow, Russia), T. Sergeeva (Moscow, Russia) and E. Smirnov (Yaroslavl, Russia) addressed the issues of improving mathematical education, introducing novelty into the teaching process while maintaining the best traditions of high-quality teaching mathematics, laid by S. Chaplygin in his productive teaching activities.

The relevance of the event was noted in the talks of Professors A. Abylkasymova (Alma-Ata, Kazakhstan), A. Borovskikh (Moscow, Russia), S. Grozdev (Sofia, Bulgaria), M. Mkrtchyan (Yerevan, Armenia) and other scientists. At the end of the plenary session, talks were presented by the authors of this communication on the history of the Scientific and Methodological Council on Mathematics of the Ministry of Science and Higher Education of Russia and its contribution to the development of mathematics and its applications in Russian education, as well as on the activities of the Lipetsk Branch of the Scientific and Methodological Council.

The following sections were working at the conference: “Modern Directions in Mathematics”, “Applied problems of mathematics”, “Computer modeling, information technologies and systems”, “New theories, models and technologies of teaching mathematics and computer science at schools and universities”, “Actualization of the problems of the history of mathematics and mathematical education in modern conditions”.

At the conference there were more than 250 participants, including leading foreign specialists from Armenia, Bulgaria, Uzbekistan, Kazakhstan, well-known scientists from more than twenty regions of Russia, as well as young researchers. Overall, it was a successful conference, which helped to increase the scientific and innovative activity of the region, stimulated the participants to develop mathematics, information technologies and mathematical education.

S. Dvoryatkina, S. Rozanova, S. Shcherbatykh