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ON THE STRUCTURE OF FREE DUAL LEIBNIZ ALGEBRAS

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Abstract. It is proved that over a field of characteristic zero the free dual Leibniz algebras are the free associative-commutative algebras (without unity) with respect to the multiplication $a \circ b = ab + ba$ and their free generators are found. We construct the examples of subalgebras of two-generated free dual Leibniz algebra, that are free dual Leibniz algebras of countable rank.

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1 Introduction

Recall that an algebra g over an arbitrary field k equipped with a bilinear operation $[-,-]$ is called *Leibniz* if it satisfies the (right) Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

An algebra A over k is called *dual Leibniz* or *Zinbiel* (read Leibniz in reverse order) if it satisfies the identity

$$(xy)z = x(zy + yz). \tag{1.1}$$

Leibniz algebras form a Koszul operad in the sense of V. Ginzburg and M. Kapranov [4]. The notion of a dual Leibniz algebra defined by J.-L. Loday [6] is precisely the notion of a dual operad of Leibniz algebras in their sense. Moreover, any dual Leibniz algebra A with respect to the symmetrization $a \circ b = ab + ba$ is an associative and commutative algebra [6].

The emergence of notion of Lyndon-Shirshov words is closely related to the construction of bases of a free Lie algebras. In 1958 A.I. Shirshov [9] and R. Lyndon [2] constructed a basis of free Lie algebra that consists of regular (by Shirshov) or standard (by Lyndon) non-associative words. In the work [1] L.A. Bokut, Yu Chen called these words the Lyndon-Shirshov words. The Lyndon-Shirshov words have found numerous applications in the Lie superalgebras theory. For example, A.A. Mikhalev [7] and A.S. Shtern [10] showed that the basis of a free Lie superalgebra consists of non-associative Lyndon-Shirshov words and squares of non-associative odd Lyndon-Shirshov words.

J.-L. Loday [6] proved that the set of all non-associative words with left arranged parenthesis (left-normed words) forms a basis for a free dual Leibniz algebra. A. Dzhumadildaev and K. Tulenbaev [3] proved an analogue of Nagata-Higman's theorem [5] for dual Leibniz algebras (any dual Leibniz nil-algebra is nilpotent). They also proved that every finite-dimensional dual Leibniz algebra over an algebraically closed field is solvable. A. Naurabekova and U. Umirbaev [8] proved that in characteristic 0 any proper subvariety of the variety of dual Leibniz algebras is nilpotent and, as a consequence, the variety of dual Leibniz algebras is Spechtian and has base rank 1.

In this paper we study the structure of free dual Leibniz algebras.

The paper is organized as follows. In Section 2 we define a lexicographic order on the set of all left-normed words and we prove some combinatorial statements. In Section 3 we construct a basis of a free dual Leibniz algebra related to Lyndon-Shirshov words. Consequently, a free dual Leibniz algebra over a field of characteristic zero is a polynomial algebra without unity with respect to the multiplication $a \circ b = ab + ba$ and their free generators are left-normed Lyndon-Shirshov words. In Section 4 we construct examples of subalgebras of a two-generated free dual Leibniz algebra, that are free dual Leibniz algebras of countable rank.

2 The lexicographic order and minimal words

Throughout this paper we denote by k an arbitrary fixed field of characteristic 0. Let $A = DL\langle X\rangle$ be a free dual Leibniz algebra over k with a set of free generators $X = \{x_1, x_2, ..., x_n\}$. Let X^* be a set of all nonempty nonassociative words in the alphabet X with left arranged parenthesis. It was shown in [6] that X^* forms a linear basis of A.

Let $B = Ass\langle X\rangle$ be a free associative algebra over k with a set of free generators X. We denote by $\widehat{X}$ the set of all associative words (not including the empty word 1) in X. We order the set X assuming that $x_i > x_j$ if $i > j$. We define the lexicographic order $\geq$ on $\widehat{X}$, i.e. for any $u, v \in \widehat{X}$ we put $u > v$ if $u = zx_it_1$, $v = zx_jt_2$ and $x_i > x_j$, where $z, t_1, t_2 \in \left\{\widehat{X}, 1\right\}$, or if $v = ut$ for some $t \in \widehat{X}$.

We can define a one-to-one correspondence between the sets X^* and $\widehat{X}$ matching every left-normed word $x_{i_1}(x_{i_2}(...(x_{i_{m-1}}x_{i_m})))$ the associative word $x_{i_1}x_{i_2}...x_{i_{m-1}}x_{i_m}$ obtained by removing all brackets. If w is a left-normed word, then the corresponding associative word we denote by $\widehat{w}$. If w is an associative word, then the corresponding left-normed word we denote by w^*. Now, we define an order on the set X^*. We say that $u > v$ for any $u, v \in X^*$ if $\widehat{u} > \widehat{v}$.

We denote by deg the standard degree function of algebras A and B, i.e deg $x_i = 1$ for any $x_i \in X$.

It is obvious that every element $f \in A$ is uniquely represented in the form

$$f = \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n,$$

where $w_i \in X^*$, $\alpha_i \in k$, $\alpha_i \neq 0$ for $i = 1, ..., n$ and $w_1 > w_2 > ... > w_n$. The word w_n is called *the minimal term* of f and denoted by $\overline{f}$.

Let C be an arbitrary dual Leibniz algebra and $c \in C$. We define c^i and $c^{\circ i}$ by induction in i as follows:

$$c^1 = c, c^{\circ 1} = c, c^{i+1} = cc^i, c^{\circ i+1} = c \circ c^{\circ i},$$

for all $i \geq 1$.

Lemma 2.1. *Let u, v be arbitrary elements in the set X^*. Then the following statements are true:*
(i) $uv \neq 0$ and uv are the linear combinations of elements of X^* with natural coefficients;
(ii) $\overline{uv} < u$;
(iii) $\overline{uv} \leq (\widehat{u}\widehat{v})^*$.

Proof. We prove the first statement by induction in deg u + deg v. If deg $u = 1$, then $uv \in X^*$. Assume that deg $u > 1$ and $u = x_iw$. It follows from identity (1.1) that

$$uv = (x_iw)v = x_i(wv + vw).$$

By induction, wv and vw are linear combinations of elements of X^* with natural coefficients. Consequently, $wv + vw$ and $uv = x_i(wv + vw)$ are also the linear combinations of elements of X^* with natural coefficients.

We prove the second statement of lemma by induction in deg u. If deg $u = 1$, then $\overline{uv} = uv < u$. Let $u = x_i u_1$. Then $uv = x_i(u_1v + vu_1)$. By induction, $\overline{u_1v} < u_1$. Obviously

$$\overline{u_1v + vu_1} = \overline{\overline{u_1v} + \overline{vu_1}} < u_1.$$

Consequently,

$$\overline{uv} = \overline{x_i(u_1v + vu_1)} = x_i\overline{u_1v + vu_1} < x_iu_1 = u.$$

We prove the third statement by induction in deg u. If deg $u = 1$, then $\overline{uv} = uv = (\widehat{u}\widehat{v})^*$. Assume that $u = x_iu'$, where $u' \in X^*$. Then $uv = x_i(u'v + vu')$. By induction, $\overline{u'v} \leq (\widehat{u'}\widehat{v})^*$. Consequently, $\overline{uv} = x_i\overline{u'v + vu'} \leq x_i(\widehat{u'}\widehat{v})^* = (\widehat{u}\widehat{v})^*$. □

Lemma 2.2. *Let $u, v, w \in X^*$. If $u < v$, then $\overline{wu} < \overline{wv}$ and $\overline{uw} < \overline{vw}$.*

Proof. We prove both inequalities of the lemma by induction in deg v + deg w. If deg $w = 1$, then obviously $\overline{wu} = wu < wv = \overline{wv}$. If $w = x_iw_1$, where $w_1 \in X^*$, then

$$wu = x_i(w_1u + uw_1), \quad wv = x_i(w_1v + vw_1).$$

By induction, $\overline{w_1u} < \overline{w_1v}$ and $\overline{uw_1} < \overline{vw_1}$. Then we have $\overline{w_1u + uw_1} < \overline{w_1v + vw_1}$ by Lemma 2.1 . Consequently, $\overline{wu} < \overline{wv}$.

Assume that $u = x_ju_1, v = x_kv_1$, where $u_1, v_1 \in X^* \cup \{1\}$. If $j < k$, then it is easy to see that $\overline{uw} < \overline{vw}$. Therefore, we assume that $j = k$. If $v_1 = 1$, then $u_1 \neq 1$ since $u < v$. By Lemma 2.1, $\overline{wu_1} < w$ and, consequently, $\overline{u_1w + wu_1} < w$. Then $\overline{uw} = x_j\overline{u_1w + wu_1} < x_jw = vw = \overline{vw}$. If $v_1 \neq 1$, then $u_1 \neq 1$ and

$$uw = x_j(u_1w + wu_1), \quad vw = x_j(v_1w + wv_1).$$

By induction, $\overline{u_1w} < \overline{v_1w}$ and $\overline{wu_1} < \overline{wv_1}$. Then, by Lemma 2.1, we have $\overline{u_1w + wu_1} < \overline{v_1w + wv_1}$ and, consequently, $\overline{uw} < \overline{vw}$. □

Corollary 2.1. *If $f, g \in A$, then $\overline{fg} = \overline{\overline{f}\,\overline{g}}$.*

Proof. Let

$$f = \alpha_1u_1 + \alpha_2u_2 + ... + \alpha_nu_n, \quad g = \beta_1v_1 + \beta_2v_2 + ... + \beta_mv_m,$$

where $u_i, v_j \in X^*$, $\alpha_i, \beta_j \in k$, $\alpha_i, \beta_j \neq 0$ for any i, j, $u_1 > u_2 > ... > u_n$ and $v_1 > v_2 > ... > v_m$. By Lemma 2.2,

$$\overline{u_nv_m} < \overline{u_iv_j}, \qquad (i, j) \neq (n, m).$$

Consequently, $\overline{fg} = \overline{u_nv_m}$. Since $\overline{f} = u_n$ and $\overline{g} = v_m$, it follows that $\overline{fg} = \overline{\overline{f}\,\overline{g}}$. □

Proposition 2.1. *Let $u = x_ju'$, $v = x_iv'$ be elements of X^* and $u < v$, where $j < i$ or $v' = 1$. Let $y_1, y_2, \ldots, y_k, \ldots$ be an arbitrary sequence of elements of X. Put $u_s = y_s(y_{s-1}\ldots(y_1u)\ldots)$ and $v_s = y_s(y_{s-1}\ldots(y_1v)\ldots)$ for any $s \geq 0$. Then the following inequalities are true:*

(i) $\overline{v_su_t} \leq \overline{u_sv_t}$ for any $s > t$;

(ii) $\overline{u_tv_s} \leq \overline{v_tu_s}$ for any $s \geq t$.

Proof. At first we consider inequality (ii) for $t = 0$. Then $u_tv_s = uv_s$, $v_tu_s = vu_s$. Note that uv_s is a linear combination of left-normed words beginning with x_j, and vu_s is a linear combination of left-normed words beginning with x_i. If $j < i$, then $\overline{uv_s} < \overline{vu_s}$. If $j = i$, then $v' = 1$, $uv_s = x_i(u'v_s + v_su')$ and $vu_s = x_iu_s$. Note that $\widehat{v_s}\widehat{u'} = \widehat{u_s}$. By Lemma 2.1, $\overline{v_su'} \leq u_s$. Hence

$$\overline{u'v_s + v_su'} \leq u_s \text{ and } \overline{uv_s} = x_i\overline{u'v_s + v_su'} \leq x_iu_s = vu_s = \overline{vu_s}.$$

Now we prove both statements of lemma together by induction on $s+t$. If $s+t=0$, then $s=t=0$ and should consider only inequality (ii) for $t=0$.

Suppose that $s+t>0$. Let $s>t$. Since $v_s = y_s v_{s-1}$, $u_s = y_s u_{s-1}$ we have

$$v_s u_t = y_s(v_{s-1}u_t + u_t v_{s-1}), \quad u_s v_t = y_s(u_{s-1}v_t + v_t u_{s-1}).$$

If $s-1>t$, then, by induction, we have

$$\overline{v_{s-1}u_t} \leq \overline{u_{s-1}v_t} \text{and} \overline{u_t v_{s-1}} \leq \overline{v_t u_{s-1}}.$$

If $s-1=t$, then

$$v_{s-1}u_t + u_t v_{s-1} = u_{s-1}v_t + v_t u_{s-1}.$$

In both cases we obtain (i).

Suppose that $s \geq t$. If $t \geq 1$, then

$$u_t v_s = y_t(u_{t-1}v_s + v_s u_{t-1}), \quad v_t u_s = y_t(v_{t-1}u_s + u_s v_{t-1}).$$

In this case $s > t-1$ and, by induction, $\overline{u_{t-1}v_s} \leq \overline{v_{t-1}u_s}$ and $\overline{v_s u_{t-1}} \leq \overline{u_s v_{t-1}}$. Consequently, $\overline{u_t v_s} \leq \overline{v_t u_s}$. □

Corollary 2.2. *Let w_1, w_2 be arbitrary elements of X^*. If $w_1 \leq w_2$, then $\overline{w_1 w_2} \leq \overline{w_2 w_1}$.*

Proof. We can assume that $w_1 < w_2$. Then there exists an obvious representation

$$w_1 = u_s, w_2 = v_s$$

for some sequence of elements $y_1, y_2, \ldots, y_s \in X^*$ and $u, v \in X^*$ satisfying the statement of Proposition 2.1. By inequality (ii) of Proposition 2.1, we have $\overline{w_1 w_2} = \overline{u_s v_s} \leq \overline{v_s u_s} = \overline{w_2 w_1}$. □

3 The Lyndon-Shirshov words

Recall that an associative word $u \in \widehat{X}$ is called *Lyndon-Shirshov* (see, for example [1]), if for any representation $u = u_1 u_2$, where $u_1, u_2 \in \widehat{X}$, we have $u > u_2 u_1$. The following lemma is proved in [1, Lemma 2.13].

Lemma 3.1. *[1] For any $u \in \widehat{X}$ there exists a unique decomposition $u = u_1 u_2 \ldots u_k$, where u_i is the associative Lyndon-Shirshov word for any $1 \leq i \leq k$ and $u_1 \leq u_2 \leq \ldots \leq u_k$.*

A left-normed word $v \in X^*$ is called *a left-normed Lyndon-Shirshov word*, if $\widehat{v}$ is an associative Lyndon-Shirshov word.

Proposition 3.1. *Let $u, v \in X^*$ and $u_2^* \leq v$ for any decomposition $\widehat{u} = u_1 u_2$ with $u_2 \neq 1$. Then $\overline{uv} = (\widehat{u}\widehat{v})^*$.*

Proof. It is obvious that the statement of the proposition is true for deg $u = 1$. It is clear that if $u = x_i u'$, then the left-normed words u', v satisfy the statement of the proposition. Consequently, $\overline{u'v} = (\widehat{u'}\widehat{v})^*$. Since $u' \leq v$, it follows that $\overline{u'v} \leq \overline{vu'}$ by Corollary 2.2. By Lemma 2.1, $\overline{u'v + vu'} = (\widehat{u'}\widehat{v})^*$. Consequently, $\overline{uv} = x_i \overline{u'v + vu'} = x_i(\widehat{u'}\widehat{v})^* = (\widehat{u}\widehat{v})^*$. □

Lemma 3.2. *Let $v_1, \ldots, v_m$ be left-normed Lyndon-Shirshov words and $v_1 \leq v_2 \leq \ldots \leq v_m$. Then*

$$\overline{v_1 \circ v_2 \circ \ldots \circ v_m} = (\widehat{v_1}\widehat{v_2} \ldots \widehat{v_m})^*.$$

Proof. We prove the lemma by induction on m. We put $u = (\widehat{v_1}\widehat{v_2}\ldots\widehat{v_{m-1}})^*$. Suppose that the statement of lemma holds for $m-1$, i.e. $\overline{v_1 \circ v_2 \circ \ldots \circ v_{m-1}} = u$. Since $v_1 \leq v_m$, it follows that $u \leq v_m$. Let u' be an arbitrary proper nontrivial right tail of u. Then $\widehat{u'} = t\widehat{v_{i+1}}\ldots\widehat{v_{m-1}}$, where t is a right proper divisor of $\widehat{v_i}$. If $t=1$, then we have $u' \leq v_m$ since $v_{i+1} \leq v_m$.

If $t \neq 1$, then t is a right proper nontrivial divisor of an associative Lyndon-Shirshov word v_i. It is well known [1, Lemma 2.10] that any nontrivial proper tail of an associative Lyndon-Shirshov word is strictly less than the word itself. Consequently, $t < \widehat{v_i} \leq \widehat{v_m}$ and $u' \leq v_m$. So the couple u, v_m satisfies the statement of Proposition 3.1. Consequently, $\overline{uv_m} = (\widehat{u}\widehat{v_m})^*$. By Corollary 2.2, we have $\overline{uv_m} \leq \overline{v_m u}$. By Lemma 2.1 and Corollary 2.1, we obtain

$$\overline{v_1 \circ v_2 \circ \ldots \circ v_m} = \overline{(v_1 \circ \ldots \circ v_{m-1})v_m + v_m(v_1 \circ \ldots \circ v_{m-1})} = (\widehat{u}\widehat{v_m})^* = (\widehat{v_1}\ldots\widehat{v_m})^*.$$

□

Theorem 3.1. *Let $A = DL\langle X\rangle$ be a free dual Leibniz algebra over a field of characteristic 0 freely generated by $X = \{x_1, x_2, ..., x_n\}$. The elements*

$$u_1 \circ u_2 \circ \ldots \circ u_m, \tag{3.1}$$

where $u_1, u_2, ..., u_m$ are left-normed Lyndon-Shirshov words in the alphabet X and $u_1 \leq u_2 \leq ... \leq u_m$, form a basis of A.

Proof. By Lemma 3.2, we have

$$\overline{u_1 \circ u_2 \circ \ldots \circ u_m} = (\widehat{u_1}\widehat{u_2}\ldots\widehat{u_m})^*.$$

Furthermore, the correspondence

$$u_1 \circ u_2 \circ \ldots \circ u_m \mapsto \widehat{u_1}\widehat{u_2}\ldots\widehat{u_m}$$

is a one to one correspondence between the set of elements of form (3.1) and $\widehat{X}$. Hence the basicity of a set of elements of form (3.1) is equivalent to the basicity of the set X^*. □

Corollary 3.1. *Let Z be a set of left-normed Lyndon-Shirshov words in the alphabet X. Then a free dual Leibniz algebra $A = DL\langle X\rangle$ over a field of characteristic 0 is a polynomial algebra (without identity) in the variables Z with respect to the multiplication $x \circ y = xy + yx$.*

4 Free subalgebras

Let $R = DL\langle x, y\rangle$ be a two-generated free dual Leibniz algebra over a field k of characteristic 0. Assuming that $x < y$, we introduce the lexicographic orders on the sets of all associative words and all left normed words in the alphabet x, y. Put $y_n = yx^n$ for any $n \geq 0$ and $z_m = \underbrace{x(...(x}_{m} y^m))$ for any $m > 0$.

Lemma 4.1. *In the algebra R for any non-negative integers $t_1, t_2, ..., t_m$ the following equality is true*

$$\overline{y_{t_1}(...(y_{t_{m-1}}y_{t_m})...)} = (\widehat{y_{t_1}}\widehat{y_{t_2}}...\widehat{y_{t_m}})^*.$$

Proof. We prove the lemma by induction in m. Suppose that the statement of the lemma holds for $m-1$. For convenience denote $(\widehat{y_{t_2}}...\widehat{y_{t_m}})^*$ by w. If $t_1 = 0$, then

$$\overline{y_{t_1}w} = (\widehat{y_{t_1}}\widehat{y_{t_2}}...\widehat{y_{t_m}})^*.$$

If $t_1 \neq 0$, then

$$\overline{y_{t_1}w} = \overline{y(x^{t_1}w + wx^{t_1})} = y\overline{x^{t_1}w + wx^{t_1}}.$$

It is obvious that $\overline{x^{t_1}w} < \overline{wx^{t_1}}$, by Proposition 3.1 and by the induction hypothesis,

$$\overline{x^{t_1}w + wx^{t_1}} = \overline{x^{t_1}w} = (\widehat{x^{t_1}}\widehat{w})^*.$$

Consequently, $\overline{y_{t_1}w} = y(\widehat{x^{t_1}}\widehat{w})^* = (\widehat{y_{t_1}}\widehat{w})^* = (\widehat{y_{t_1}}\widehat{y_{t_2}}...\widehat{y_{t_m}})^*$. $\square$

Lemma 4.2. *In the algebra R for any non-negative integers $t_1, t_2, ..., t_m$ the following equality is true*

$$\overline{(yy_{t_1})(...((yy_{t_{m-1}})(yy_{t_m}))...)} = (y\widehat{y_{t_1}}y\widehat{y_{t_2}}...y\widehat{y_{t_m}})^*.$$

Proof. We prove the lemma by induction in m. Suppose that the statement of the lemma holds for $m-1$. For convenience denote $(y\widehat{y_{t_2}}...y\widehat{y_{t_m}})^*$ by w. Then

$$\overline{(yy_{t_1})w} = \overline{y(y_{t_1}w + wy_{t_1})} = y\overline{y_{t_1}w + wy_{t_1}}.$$

It is obvious that $\overline{y_{t_1}w} < \overline{wy_{t_1}}$, by Proposition 3.1 and by the induction hypothesis,

$$\overline{y_{t_1}w + wy_{t_1}} = \overline{y_{t_1}w} = (\widehat{y_{t_1}}\widehat{w})^*.$$

Consequently, $\overline{(yy_{t_1})w} = y(\widehat{y_{t_1}}\widehat{w})^* = (y\widehat{y_{t_1}}y\widehat{y_{t_2}}...y\widehat{y_{t_m}})^*$. $\square$

Lemma 4.3. *In the algebra R for any non-negative integers $t_1, t_2, ..., t_m$ the following equality is true*

$$\overline{(yz_{t_1})(...((yz_{t_{m-1}})(yz_{t_m}))...)} = (y\widehat{x^{t_1}}y\widehat{x^{t_2}}...y\widehat{x^{t_m}}\widehat{y^{t_1+t_2+...+t_m}})^*.$$

Proof. We prove the lemma by induction in m. Suppose that the statement of the lemma holds for $m-1$. For convenience denote $(y\widehat{x^{t_2}}...y\widehat{x^{t_m}}\widehat{y^{t_2+...+t_m}})^*$ by w. Then

$$\overline{(yz_{t_1})w} = \overline{y(z_{t_1}w + wz_{t_1})} = y\overline{z_{t_1}w + wz_{t_1}}.$$

It is obvious that

$$\overline{z_{t_1}w + wz_{t_1}} = \underbrace{x(...(x}_{t_1}(\overline{y^{t_1}w + wy^{t_1}}))...)$$

and $\overline{wy^{t_1}} < \overline{y^{t_1}w}$. By Proposition 3.1 and by the induction hypothesis,

$$\overline{y^{t_1}w + wy^{t_1}} = \overline{wy^{t_1}} = (\widehat{w}\widehat{y^{t_1}})^*.$$

Consequently,

$$\overline{(yz_{t_1})w} = y(\underbrace{x(...(x}_{t_1}(\widehat{w}\widehat{y^{t_1}})^*)...) = (y\widehat{x^{t_1}}y\widehat{x^{t_2}}...y\widehat{x^{t_m}}\widehat{y^{t_1+t_2+...+t_m}})^*.$$

$\square$

Denote by R_1, R_2, R_3 subalgebras of R generated by the sets $Y_1 = \{y_0, y_1, \ldots, y_n, \ldots\}$, $Y_2 = \{yy_1, yy_2, \ldots, yy_n, \ldots\}$, $Y_3 = \{yz_1, yz_2, ..., yz_n, ...\}$ respectively.

Theorem 4.1. *The algebras R_1, R_2, R_3 are free dual Leibniz algebras with sets of free generators Y_1, Y_2, Y_3 respectively.*

Proof. Let $D = DL\langle x_0, x_1, ..., x_n, ...\rangle$ be a free dual Leibniz algebra over a field k. It is clear that the mappings $\varphi_1 : D \to R_1$, $\varphi_2 : D \to R_2$, $\varphi_3 : D \to R_3$, given by $x_i \longmapsto y_i$, $x_i \longmapsto yy_i$, $x_i \longmapsto yz_i$ respectively, are homomorphisms.

By Lemma 4.1, Lemma 4.2, Lemma 4.3, from the inequality $x_{i_1}(...(x_{i_{t-1}}x_{i_t})) \neq x_{j_1}(...(x_{j_{m-1}}x_{j_m}))$ follows the inequalities

$$\overline{y_{i_1}(...(y_{i_{t-1}}y_{i_t}))} \neq \overline{y_{j_1}(...(y_{j_{m-1}}y_{j_m}))},$$
$$\overline{y_{i_1}(...(y_{i_{t-1}}y_{i_t}))} \neq \overline{y_{j_1}(...(y_{j_{m-1}}y_{j_m}))},$$
$$\overline{(yy_{i_1})(...((yy_{i_{t-1}})(yy_{i_t}))...)} \neq \overline{(yy_{j_1})(...((yy_{j_{m-1}})(yy_{j_m}))...)}.$$

Hence, the kernels of the homomorphisms φ_1, φ_2, φ_3 are equal to zero. □

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Events

5TH INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF MATHEMATICS AND COMPUTER SCIENCE: THEORY, METHODOLOGY, PRACTICE" (APRIL 18–20, 2019, YELETS, RUSSIA)

The XX century is marked by the enrichment of world science with outstanding achievements in the field of mathematics, solving many important problems that remain relevant in the modern world. Such problems include, in particular, the problems considered in fundamental works of academician S. Chaplygin. Based on his works new research paths were set, and serious applied problems were addressed in the fields of aerodynamics, gas dynamics, hydrodynamics, and mechanics. They were further intensively developed due to the achievements of contemporary information technology.

To commemorate his activities was organized a large-scale scientific event in the historic homeland of academician S. Chaplygin — the 5th international conference "Actual problems of mathematics and computer science: theory, methodology, practice", dedicated to the 150th anniversary of the birth of academician S. Chaplygin.

The Ivan Bunin Yelets State University (Russia), the Samarkand State University (Uzbekistan), the Higher School of Insurance and Finance (Bulgaria),the Khachatur Abovyan Armenian State Pedagogical University (Armenia), and the Scientific and Methodological Council on Mathematics of the Ministry of Science and Higher Education of Russia held through April 18–20, 2019 the 5th International Conference "Actual problems of mathematics and computer science: theory, methodology, practice" dedicated to the 150th anniversary of academician S. Chaplygin.

The conference marked the three major milestones related to the development of mathematical science in the Lipetsk region and in the oldest university center in the region — the Ivan Bunin Yelets State University.

1. April 2019 is the 150th anniversary of the birth of S. Chaplygin (1869–1942) — a well-known Russian scientist, academician of the Academy of Sciences of the USSR. S. Chaplygin is an outstanding representative of the Lipetsk region, whose surname is immortalized in the name of the city Chaplygin (previously Ranenburg) in the Lipetsk region.

2. 2019 year is the 80th anniversary of the foundation of the Faculty of Physics and Mathematics. It is the oldest faculty of the Ivan Bunin Yelets State University, where students of the scientific school of academician N. Zhukovsky were taught, whose famous representative was academician S. Chaplygin.

3. In October 2019 there will be 10 years since the organization of the Lipetsk Branch of the Scientific and Methodological Council for Mathematics of the Ministry of Science and Higher Education of Russia on the basis of the Ivan Bunin Yelets State University.

The main goals of the conference were the creation of conditions for international scientific communication of representatives of fundamental and applied areas in the field of mathematics, understanding the importance of scientific works of S. Chaplygin, the actualization of his scientific achievements, taking into account the rapid development of information technologies and their adaptation to modern mathematical education.

The plenary session of the conference was opened by the Rector of the Ivan Bunin Yelets State University Professor E. Gerasimova and continued by the President of the International Academy of the History of Science Professor S. Demidov (Moscow, Russia), who presented the talk "Pure and Applied Mathematics at the M.V. Lomonosov Moscow State University in the first half of the twentieth century: N. Luzin and S. Chaplygin".

Professor A. Soleev (Samarkand, Uzbekistan) devoted his talk to basic ideas and general provisions of the Power Geometry. Professor A. Soldatov (Moscow, Russia) focused on the consideration of the Dirichlet problem for equations of mixed type. In her talk Professor G. Zhukova (Moscow, Russia) discussed the dependence of solutions to singularly perturbed linear differential systems on a small parameter. The talk of Professors O. Masina (Yelets, Russia) and O. Druzhinina (Moscow, Russia) was devoted to the analysis of the known and developed by the authors approaches to the study of the stability of intelligent control systems. The talks of Professors V. Tikhomirov (Moscow, Russia), T. Sergeeva (Moscow, Russia) and E. Smirnov (Yaroslavl, Russia) addressed the issues of improving mathematical education, introducing novelty into the teaching process while maintaining the best traditions of high-quality teaching mathematics, laid by S. Chaplygin in his productive teaching activities.

The relevance of the event was noted in the talks of Professors A. Abylkasymova (Alma-Ata, Kazakhstan), A. Borovskikh (Moscow, Russia), S. Grozdev (Sofia, Bulgaria), M. Mkrtchyan (Yerevan, Armenia) and other scientists. At the end of the plenary session, talks were presented by the authors of this communication on the history of the Scientific and Methodological Council on Mathematics of the Ministry of Science and Higher Education of Russia and its contribution to the development of mathematics and its applications in Russian education, as well as on the activities of the Lipetsk Branch of the Scientific and Methodological Council.

The following sections were working at the conference: “Modern Directions in Mathematics”, “Applied problems of mathematics”, “Computer modeling, information technologies and systems”, “New theories, models and technologies of teaching mathematics and computer science at schools and universities”, “Actualization of the problems of the history of mathematics and mathematical education in modern conditions”.

At the conference there were more than 250 participants, including leading foreign specialists from Armenia, Bulgaria, Uzbekistan, Kazakhstan, well-known scientists from more than twenty regions of Russia, as well as young researchers. Overall, it was a successful conference, which helped to increase the scientific and innovative activity of the region, stimulated the participants to develop mathematics, information technologies and mathematical education.

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