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BOUNDEDLY SOLVABLE NEUTRAL TYPE DELAY DIFFERENTIAL OPERATORS OF THE FIRST ORDER

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Key words: boundedly solvable operator, neutral type delay differential operator, spectrum.

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Abstract. In this work, by using methods of operator theory, all boundedly solvable extensions of the minimal operator generated by a linear neutral type delay differential-operator expression of the first order in a Hilbert space of vector-functions on a finite interval are described. Furthermore, the geometry of spectrum sets of these operators is studied.

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1 Introduction

It is known that many problems in chemistry, biology, medicine, economy, control theory, electrodynamics, ecology etc. are reduced the study of boundary value problems for functional differential equations for the first and second order in corresponding function spaces. In order words functional differential equations arise in many areas of science and technology. Problems in chemical reactions [3], [5], [17] and chemical kinetics [1], [6], [7], [8] etc. can be modelled by time-delay functional differential equations. The delay reaction-diffusion equation without diffusive term is expressed by following neutral type delay differential equations of the form

$$\frac{d}{dt}(x(t) + px(t-\tau)) + q(t)x(t-\sigma) = 0,\ t > 0$$

where: $p \in \mathbb{R},\ \tau > 0,\ \sigma \geq 0$, and $q(t) > 0$ for $t > 0$.

Note that asymptotic behaviour and oscillation of solutions of this equations have been investigated in detail by many mathematician (for example see [15] and references therein). In modelling of certain cell growth phenomena the important role is played by neutral type delay differential equations (see [2], and references therein). Value in the 21st century of these and connected problems is undisputed.

Let us recall that an operator $T : D(T) \subset H \to H$ in a Hilbert space H is called boundedly solvable, if T is one-to-one, $TD(T) = H$ and $T^{-1} : H \to H$ is bounded. In mathematical literature it is known that boundedly solvable extensions of any unbounded linear operator in a Hilbert space have been described by M.I. Vishik in [16]. Generalization of this results to nonlinear and complete additive Hausdorff topological spaces have been done by M.O. Otelbaev, B.K. Kokebaev and A.N. Shynybekov in their works [11], [12], [13], [14].

In this work, in Section 2 based on methods of operator theory all boundedly solvable extensions of the minimal operator generated by a linear neutral type delay differential-operator expression of the first order in a Hilbert space of vector-functions on a finite interval are described. Furthermore, in Section 3 the geometry of spectrum sets of these operators is studied. Finally, in Section 4 an example is given.

Everywhere in this paper by $\sigma(\,\cdot\,)$ and $\rho(\,\cdot\,)$ will be denoted spectrum and resolvent sets of an operator respectively.

2 Description of boundedly solvably extensions

We start with considering the following simplest scalar neutral type functional differential equation in nonhomogeneous form

$$a\dot{x}(t) + b\dot{x}(t-\tau) + cx(t) + dx(t-\tau) = h(t),\ t > 0$$

$$a \neq 0,\ b \neq 0,\ |c| + |d| > 0,\ a, b, c, d \in \mathbb{C},\ \tau > 0,$$
$$x(t) = \varphi(t) \in C^1[-\tau, 0],\ h : [0, \infty] \to \mathbb{C}.$$

Note that without loss of generality it can be assumed that $d = 0$. To verify it suffices to use the substitution $x(t) = exp\left(-\frac{d}{b}t\right)u(t),\ t > 0.$

Moreover the above boundary value condition can be choosen to be such that

$$\varphi(t) = 0,\ -\tau \leq t \leq 0.$$

To verify it suffices to use the following substitution

$$u(t) = \begin{cases} x(t) - \varphi(t), & \text{if } -\tau \leq t \leq 0, \\ x(t), & \text{if } t > 0. \end{cases}$$

Consequently, it suffices to consider the following nonhomogeneous neutral type delay differential equation

$$a\dot{y}(t) + b\dot{y}(t-\tau) + cy(t) = h(t),\ t > 0$$

with the boundary condition

$$y(t) = 0,\ -\tau \leq t \leq 0.$$

This problem can be written in form

$$a\dot{y}(t) + bS_l^\tau \dot{y}(t) + cy(t) = h(t),\ t > 0$$

$$y(t) = 0,\ -\tau \leq t \leq 0,$$

where: $S_l^\tau : L^2(0,1) \to L^2(0,1)$, and

$$S_l^\tau f(t) = \begin{cases} f(t-\tau), & \text{if } \tau \leq t \leq 1, \\ 0, & \text{if } 0 \leq t < \tau, \end{cases} \quad f \in L^2(0,1).$$

In this work will be considered the following neutral type functional differential-operator expression of a form

$$l(u) = u'(t) + \sum_{k=1}^{n} A_k u'(t-\alpha_k) + \sum_{j=1}^{m} B_j u(t-\tau_j)$$

in the Hilbert space $L^2(H, (a,b))$ of vector-functions on a finite interval, where:
(1) for each $k = 1, 2, ..., n$

$$0 < \alpha_k < \infty$$

and for any $j = 1, 2, ..., m$

$$0 \leq \tau_j < \infty;$$

(2)H is a separable Hilbert space;

(3) for $k = 1, 2, ..., n,\ j = 1, 2, ..., m,\ A_k,\ B_j \in L(H)$ and $0 \in \rho\left(1 + \sum\limits_{k=1}^{n} A_k S_l^{\tau}\right)$,

$\sum\limits_{k=1}^{n} A_k \neq 0$ and $\sum\limits_{j=1}^{m} B_j \neq 0$.

The differential expression $l(\,\cdot\,)$ can be rewritten in the form

$$l(u) = Au'(t) + Bu(t),\ a < t < b,$$

where:

$$A : L^2(H, (a, b)) \to L^2(H, (a, b)),\ A = E + \sum_{k=1}^{n} A_k S_l^{\alpha_k},$$

$$B : L^2(H, (a, b)) \to L^2(H, (a, b)),\ B = \sum_{j=1}^{m} B_j S_l^{\tau_j},$$

and note that for $\gamma > 0,\ \gamma < b - a,\ S_l^{\gamma} : L^2(H, (a, b)) \to L^2(H, (a, b))$,

$$S_l^{\gamma} f(t) = \begin{cases} 0, & \text{if } a + \tau \leq t < b1, \\ f(t - \tau), & \text{if } a \leq t < a + \tau, \end{cases} \quad f \in L^2(H, (0, 1)).$$

Now assume that

$$m(v) = Av'(t),\ a < t < b.$$

In the standard way the minimal $M_0(L_0)$ and maximal $M(L)$ operators corresponding to the differential expression $m(\,\cdot\,)\,(l(\,\cdot\,))$ can be defined in $L^2(H, (a, b))$ (see [9]).

Along this work it will be assumed that $AB = BA$.

Now let $U(t, s),\ t, s \in [a, b]$ be the family of evolution operators corresponding to the homogeneous differential equation

$$\begin{cases} AU_t'(t, s)f + BU(t, s)f = 0,\ t, s \in (a, b), \\ U(s, s)f = f,\ f \in H \end{cases}$$

Note that if $\widetilde{L}$ is an extension of the minimal operator L_0, i.e. $L_0 \subset \widetilde{L} \subset L$, then $U^{-1} L_0 U = M_0,\ M_0 \subset U^{-1} \widetilde{L} U = \widetilde{M} \subset M,\ U^{-1} L U = M$.

Theorem 2.1. *Each boundedly solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L^2(H, (a, b))$ is generated by the differential-operator expression $l(\,\cdot\,)$ and boundary condition*

$$(W + E)u(a) = WU(a, b)u(b), \tag{2.1}$$

where $W \in L(H)$ and E is a identity operator in H. The operator W is determined uniquely by the extension $\widetilde{L}$, i.e $\widetilde{L} = L_W$.

Conversely, the restriction of the maximal operator L_0 to the manifold of vector-functions satisfying condition (2.1) for some bounded operator $W \in L(H)$ is a boundedly solvable extension of the minimal operator L_0 in $L^2(H, (a, b))$.

Proof. First, all boundedly solvable extensions $\widetilde{M}$ of the minimal operator M_0 in $L^2(H,(a,b))$ in terms of boundary values will be described. Consider the following so-called Cauchy extension M_c,

$$M_c u = Au'(t),$$

$$M_c : D(M_c) = \{u \in W_2^1 H(a,b) : u(a) = 0\} \subset L^2(H,(a,b)) \to L^2(H,(a,b))$$

of the minimal operator M_0. It is clear that M_c is a boundedly solvable extension of M_0 and

$$M_c^{-1} := L^2(H,(a,b)) \longrightarrow L^2(H,(a,b)),\ M_c^{-1}f(t) = \int_a^t A^{-1}f(x)dx, \quad f \in L^2(H,(a,b)).$$

Now assume that $\widetilde{M}$ is a boundedly solvable extension of the minimal operator M_0 in $L^2(H,(a,b))$. In this case it is known that the domain of $\widetilde{M}$ can be written as direct sum of the form

$$D(\widetilde{M}) = D(M_0) \oplus (M_c^{-1} + K)V,$$

where $V = KerM = H, K \in L(H)$ (see [16]). Therefore for each $u \in D(\widetilde{M})$

$$u(t) = u_0(t) + M_c^{-1}f + Kf, \quad u_0 \in D(M_0), \quad f \in H.$$

That is,

$$u(t) = u_0(t) + (t-a)A^{-1}f + Kf, \quad u_0 \in D(M_0), \quad f \in H.$$

Hence

$$u(a) = Kf, \quad u(b) = (b-a)A^{-1}f + Kf = \left((b-a)A^{-1} + K\right) f$$

and from these relations it follows that

$$\frac{1}{b-a}Au(a) = \frac{AK}{b-a}f,\ \frac{1}{b-a}Au(b) = \left(E + \frac{AK}{b-a}\right) f.$$

The last relations can also be written in following form

$$\frac{1}{b-a}Au(a) = Tf,\ \frac{1}{b-a}Au(b) = (E+T)f$$

where: $T = \dfrac{AK}{b-a}$.

Consequently,

$$\frac{1}{b-a}(E+T)Au(a) = \frac{1}{b-a}TAu(b),$$

i.e.

$$(E+T)Au(a) = TAu(b).$$

Then

$$(A+S)u(a) = Su(b),\ S = TA,$$

hence

$$(E + A^{-1}S)u(a) = A^{-1}Su(b).$$

The last relation can be expressed in the form

$$(E+W)u(a) = Wu(b), \tag{2.2}$$

where: $W = A^{-1}S,\ W \in L(H)$.

The uniqueness of the operator $W \in L(H)$ is clear from the work [16]. Therefore $\widetilde{M} = M_W$. This completes the proof of the first assertion.

Conversely, if M_W is a operator generated by the differential expression $m(\cdot)$ and boundary condition (2.2), then M_W is boundedly invertible and

$$M_W^{-1} := L^2(H,(a,b)) \longrightarrow L^2(H,(a,b)),$$

$$M_W^{-1}f(t) = \int_a^t A^{-1}f(x)dx + W\int_a^b A^{-1}f(x)dx, \quad f \in L^2(H,(a,b)).$$

Consequently, all boundedly solvable extensions of the minimal operator M_0 in $L^2(H,(0,1))$ are generated by the differential expression $m(\cdot)$ and boundary condition (2.2) with any linear bounded operator W.

Now consider the general case. For this in $L^2(H,(a,b))$ we introduce the operator of the form

$$U : L^2(H,(a,b)) \to L^2(H,(a,b)),$$

$$(Uz)(t) := U(t,a)z(t), z \in L^2(H,(a,b)).$$

The properties of the family of evolution operators $U(t,s), t,s \in [a,b]$ imply that the operator U is linear continuous boundedly solvable and such that

$$(U^{-1}z)(t) = U(a,t)z(t).$$

On the other hand from the relations

$$U^{-1}L_0U = M_0, \quad U^{-1}\widetilde{L}U = \widetilde{M}, \quad U^{-1}LU = M$$

it follows that the operator U is one-to-one between of sets of boundedly solvable extensions of the minimal operators L_0 and M_0 in $L^2(H,(a,b))$.

An extension $\widetilde{L}$ of the minimal operator L_0 is boundedly solvable in $L^2(H,(a,b))$ if and only if the operator $\widetilde{M} = U^{-1}\widetilde{L}U$ is an extension of the minimal M_0 in $L^2(H,(a,b))$. Then $u \in D(\widetilde{L})$ if and only if

$$(W+E)U(a,a)u(a) = WU(a,b)u(b),$$

that is,

$$(W+E)u(a) = WU(a,b)u(b).$$

This proves the validity of the claims in theorem. □

Corollary 2.1. *The resolvent operator $R_\lambda(L_W), \lambda \in \rho(L_W)$ of any boundedly solvable operator L_W of the minimal operator L_0, generated by the differential expression $l(\cdot)$ with the boundary condition*

$$(W+E)u(a) = WU(a,b)u(b), W \in L(H)$$

is of the form $R_\lambda(K_W) : L^2(H,(a,b)) \to L^2(H,(a,b))$, where

$$\begin{aligned} R_\lambda(L_W)f(t) &= U(t,a)\left(E - W\left(e^{\lambda A^{-1}(b-a)} - E\right)\right)^{-1} W\int_a^b e^{\lambda A^{-1}(b-s)}A^{-1}f(s)ds \\ &+ \int_a^t e^{\lambda A^{-1}(t-s)}A^{-1}U(a,s)f(s)ds, \ f \in L^2(H,(a,b)). \end{aligned}$$

3 Spectrum of boundedly solvable extensions

In this section the structure of the spectrums of boundedly solvable extensions of the minimal operator L_0 will be investigated.

The validity of following assertion is clear.

Theorem 3.1. *If $\widetilde{L}$ is a boundedly solvable extension of the minimal operator L_0 and $\widetilde{M} = U^{-1}\widetilde{L}U$ is a boundedly solvable extension of the minimal operator M_0, then for the spectrums of these extensions the equality $\sigma(\widetilde{L}) = \sigma(\widetilde{M})$ holds.*

For the spectrums of boundedly solvable extensions of L_0 the following proposition is valid.

Theorem 3.2. *If L_W is a boundedly solvable extension of the minimal operator L_0 in the space $L^2(H,(a,b))$, then the spectrum of L_W has the form*

$$\sigma(L_W) = \{\alpha\nu : \alpha \in \sigma(A),\ \nu \in \Omega\},$$

$$\Omega := \left\{\lambda \in \mathbb{C} : \lambda = ln|\frac{\mu+1}{\mu}| + iarg\left(\frac{\mu+1}{\mu}\right) + 2p\pi i,\ \mu \in \sigma(W) \setminus \{0,1\},\ p \in \mathbb{Z}\right\}.$$

Proof. First, the spectrum of a boundedly solvable extension $M_W = U^{-1}L_W U$ of the minimal operator M_0 in $L^2(H,(a,b))$ will be investigated. For this we consider the following spectral problem

$$M_W u = \lambda u + f, \lambda \in \mathbb{C},\ f \in L^2(H,(a,b)).$$

Then it is clear that $\lambda \neq 0$ and

$$Au' = \lambda u + f,\ \lambda \in \mathbb{C} \setminus \{0\},\ f \in L^2(H,(a,b))$$

$$(W+E)u(a) = Wu(b), W \in L(H).$$

The above problem can be formulated the following form

$$(A \otimes D_W)\, u = \lambda u + f,$$

where $D_W : D(L_W) \subset L^2(a,b) \to L^2(a,b)$,

$$D(L_W) = \left\{u \in W_2^1(a,b) : (W+E)u(a) = Wu(b)\right\}.$$

It is known that the operator D_W is a boundedly solvable in $L^2(a,b)$ and the spectrum has the following structure

$$\sigma(D_W) = \left\{\lambda \in \mathbb{C} : \lambda = \ln|\frac{\mu+1}{\mu}| + i\arg(\frac{\mu+1}{\mu}) + 2p\pi i, \mu \in \sigma(K)\backslash\{0,-1\},\ p \in \mathbb{Z}\right\}$$

([10]). The problem considered above is equivalent to the following spectral problem

$$(A^{-1} \otimes D_W^{-1})u = \frac{1}{\lambda}u - \frac{1}{\lambda}(A^{-1} \otimes D_W^{-1})f,\ \lambda \in \mathbb{C} \setminus \{0\},\ f \in L^2(H,(a,b))$$

for the operator $A^{-1} \otimes D_W^{-1}$ in $L^2(H,(a,b))$.

In this case

$$\sigma\left((A \otimes D_W)^{-1}\right) = \left\{\lambda^{-1} : \lambda \in \sigma((A \otimes D_W)\right\},$$

that is,

$$\sigma(A^{-1} \otimes D_W^{-1}) = \left\{\lambda^{-1} : \lambda \in \sigma((A \otimes D_W)\right\}.$$

From this and [4] we have

$$\sigma\left((A \otimes D_W)^{-1}\right) = \left\{\alpha\nu : \alpha \in \sigma(A^{-1}),\ \nu \in \sigma(D_W^{-1})\right\}.$$

Consequently,

$$\sigma(A \otimes D_W) = \{\alpha\nu : \alpha \in \sigma(A), \nu \in \sigma(D_W)\},$$

that is,

$$\sigma(M_W) = \{\alpha\nu : \alpha \in \sigma(A), \nu \in \sigma(D_W)\}.$$

Then by Theorem 3.1 the validity of the assertion in last theorem it is clear. □

4 Application

Consider the following reaction-diffusion equation of neutral type without diffusive term of the form

$$\frac{d}{dt}(x(t) + px(t-\tau)) + qx(t-\sigma) = f(t),\ t \in [0, T],$$

where $p, q \in \mathbb{R};\ 0 < \tau, \sigma < T < \infty$, with the initial condition

$$x(t) = 0,\ t \in [-max(\tau, \sigma), 0]$$

(see [15]).

It is clear that if $|p| < 1$, then $0 \in \rho\,(E + pS_l^\tau)$, $E + pS_l^\tau : L^2(0,1) \to L^2(0,1)$.

The last initial value problem can be considered as the Cauchy problem of the form

$$\begin{cases} L_c x(t) = f(t),\ 0 < t < T, \\ x(0) = 0 \end{cases}$$

where $L_c x(t) = Ax'(t) + Bx(t),\ A = E + pS_l^\tau,\ B = qS_l^\sigma$.

In this case by Corollary 2.2 this problem has a unique L^2-solution in $[0, T]$ and it can be represented in the following form

$$x(t) = L_c^{-1} f(t) = U(t,0) \int_0^t A^{-1} U(0,s) f(s) ds,\ f \in L^2(0,T).$$

Here $U(t,s),\ t, s \in [0,T]$ is the family of evolution operators corresponding to the problem

$$\begin{cases} AU_t'(t,s)f + BU(t,s)f = 0, \\ U(s,s)f = f,\ f \in \mathbb{R}. \end{cases}$$

Note that the Cauchy operator L_c is a boundedly solvable extension of the corresponding minimal operator in $L^2(0,T)$. To verify this it suffices to take $W = 0$. For $W = 0$

$$\sigma(D_W^{-1}) = \{0\}.$$

Then

$$\sigma(A^{-1} \otimes D_W^{-1}) = \{0\}.$$

Consequently, by Theorem 3.2 $\sigma(L_c) = \varnothing$.

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Events

5TH INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF MATHEMATICS AND COMPUTER SCIENCE: THEORY, METHODOLOGY, PRACTICE" (APRIL 18–20, 2019, YELETS, RUSSIA)

The XX century is marked by the enrichment of world science with outstanding achievements in the field of mathematics, solving many important problems that remain relevant in the modern world. Such problems include, in particular, the problems considered in fundamental works of academician S. Chaplygin. Based on his works new research paths were set, and serious applied problems were addressed in the fields of aerodynamics, gas dynamics, hydrodynamics, and mechanics. They were further intensively developed due to the achievements of contemporary information technology.

To commemorate his activities was organized a large-scale scientific event in the historic homeland of academician S. Chaplygin — the 5th international conference "Actual problems of mathematics and computer science: theory, methodology, practice", dedicated to the 150th anniversary of the birth of academician S. Chaplygin.

The Ivan Bunin Yelets State University (Russia), the Samarkand State University (Uzbekistan), the Higher School of Insurance and Finance (Bulgaria),the Khachatur Abovyan Armenian State Pedagogical University (Armenia), and the Scientific and Methodological Council on Mathematics of the Ministry of Science and Higher Education of Russia held through April 18–20, 2019 the 5th International Conference "Actual problems of mathematics and computer science: theory, methodology, practice" dedicated to the 150th anniversary of academician S. Chaplygin.

The conference marked the three major milestones related to the development of mathematical science in the Lipetsk region and in the oldest university center in the region — the Ivan Bunin Yelets State University.

1. April 2019 is the 150th anniversary of the birth of S. Chaplygin (1869–1942) — a well-known Russian scientist, academician of the Academy of Sciences of the USSR. S. Chaplygin is an outstanding representative of the Lipetsk region, whose surname is immortalized in the name of the city Chaplygin (previously Ranenburg) in the Lipetsk region.

2. 2019 year is the 80th anniversary of the foundation of the Faculty of Physics and Mathematics. It is the oldest faculty of the Ivan Bunin Yelets State University, where students of the scientific school of academician N. Zhukovsky were taught, whose famous representative was academician S. Chaplygin.

3. In October 2019 there will be 10 years since the organization of the Lipetsk Branch of the Scientific and Methodological Council for Mathematics of the Ministry of Science and Higher Education of Russia on the basis of the Ivan Bunin Yelets State University.

The main goals of the conference were the creation of conditions for international scientific communication of representatives of fundamental and applied areas in the field of mathematics, understanding the importance of scientific works of S. Chaplygin, the actualization of his scientific achievements, taking into account the rapid development of information technologies and their adaptation to modern mathematical education.

The plenary session of the conference was opened by the Rector of the Ivan Bunin Yelets State University Professor E. Gerasimova and continued by the President of the International Academy of the History of Science Professor S. Demidov (Moscow, Russia), who presented the talk "Pure and Applied Mathematics at the M.V. Lomonosov Moscow State University in the first half of the twentieth century: N. Luzin and S. Chaplygin".

Professor A. Soleev (Samarkand, Uzbekistan) devoted his talk to basic ideas and general provisions of the Power Geometry. Professor A. Soldatov (Moscow, Russia) focused on the consideration of the Dirichlet problem for equations of mixed type. In her talk Professor G. Zhukova (Moscow, Russia) discussed the dependence of solutions to singularly perturbed linear differential systems on a small parameter. The talk of Professors O. Masina (Yelets, Russia) and O. Druzhinina (Moscow, Russia) was devoted to the analysis of the known and developed by the authors approaches to the study of the stability of intelligent control systems. The talks of Professors V. Tikhomirov (Moscow, Russia), T. Sergeeva (Moscow, Russia) and E. Smirnov (Yaroslavl, Russia) addressed the issues of improving mathematical education, introducing novelty into the teaching process while maintaining the best traditions of high-quality teaching mathematics, laid by S. Chaplygin in his productive teaching activities.

The relevance of the event was noted in the talks of Professors A. Abylkasymova (Alma-Ata, Kazakhstan), A. Borovskikh (Moscow, Russia), S. Grozdev (Sofia, Bulgaria), M. Mkrtchyan (Yerevan, Armenia) and other scientists. At the end of the plenary session, talks were presented by the authors of this communication on the history of the Scientific and Methodological Council on Mathematics of the Ministry of Science and Higher Education of Russia and its contribution to the development of mathematics and its applications in Russian education, as well as on the activities of the Lipetsk Branch of the Scientific and Methodological Council.

The following sections were working at the conference: "Modern Directions in Mathematics", "Applied problems of mathematics", "Computer modeling, information technologies and systems", "New theories, models and technologies of teaching mathematics and computer science at schools and universities", "Actualization of the problems of the history of mathematics and mathematical education in modern conditions".

At the conference there were more than 250 participants, including leading foreign specialists from Armenia, Bulgaria, Uzbekistan, Kazakhstan, well-known scientists from more than twenty regions of Russia, as well as young researchers. Overall, it was a successful conference, which helped to increase the scientific and innovative activity of the region, stimulated the participants to develop mathematics, information technologies and mathematical education.

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