ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2019, Volume 10, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

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EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 10, Number 3 (2019), 08 – 19

ON STOLZ'S THEOREM AND ITS CONVERSION

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Communicated by T.V. Tararykova

Key words: monotone sequence, convex sequence, Stolz's theorem.

AMS Mathematics Subject Classification: 40D10, 40E10.

Abstract. The present paper extends and refines some results on the relative growth rate of real sequences. Various forms of Stolz's theorem and its conversion are considered.

DOI: https://doi.org/10.32523/2077-9879-2019-10-3-08-19

1 Introduction

The classical Stolz's theorem [10] establishes conditions on real sequences x_n and y_n when the existence of $\lim\limits_{n\to\infty} \dfrac{x_{n+1}-x_n}{y_{n+1}-y_n}$ implies the existence of $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n}$ and that this limit is equal to the first one.

We formulate Stolz's theorem in its general form, with the upper and lower limits instead of the conventional ones (see, for example, [5]).

Stolz's theorem. *Let x_n be a real sequence and y_n be a real strictly monotone sequence. If both sequences are infinitely small or if y_n is an infinitely large sequence and x_n is an arbitrary sequence, then*

$$\varliminf_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \leqslant \varliminf_{n\to\infty} \frac{x_n}{y_n}, \qquad \varlimsup_{n\to\infty} \frac{x_n}{y_n} \leqslant \varlimsup_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}. \tag{1}$$

The study of the cases when estimates (1) are equalities and derivation of reverse inequalities requires additional (Tauberian) conditions on the sequences under consideration.

2 Main results

We start with the monotone case (see, for example, [3]).

Theorem A. *Let x_n be a real sequence and y_n be a real strictly increasing sequence. If the sequence $\dfrac{x_{n+1}-x_n}{y_{n+1}-y_n}$ is increasing (decreasing), then so is the sequence $\dfrac{x_n}{y_n}$.*

In this result, one can assert that the limits of the above ratios are equal if both sequences x_n and y_n are either infinitely small or infinitely large.

This result cannot be completely reversed. However, the following result holds.

Theorem 2.1. *Let x_n be a real sequence and y_n be a real strictly monotone sequence.*

If the sequence $\dfrac{x_n}{y_n}$ is increasing, then

$$\varliminf_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \lim_{n\to\infty} \frac{x_n}{y_n}. \tag{1}$$

If the sequence $\dfrac{x_n}{y_n}$ *is decreasing, then*

$$\varlimsup_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \lim_{n\to\infty} \frac{x_n}{y_n}. \tag{2}$$

Proof. Let us prove the first assertion; the proof of the second one is similar. Given $n \in \mathbb{N}$, we have

$$\frac{x_{n+1}}{y_{n+1}} - \frac{x_n}{y_n} = \frac{(x_{n+1}-x_n)y_n - x_n(y_{n+1}-y_n)}{y_{n+1}y_n} = \frac{y_{n+1}-y_n}{y_{n+1}} \left(\frac{x_{n+1}-x_n}{y_{n+1}-y_n} - \frac{x_n}{y_n} \right) \geqslant 0.$$

Here, the first factor is positive for sufficiently large n if y_n tends to $+\infty$ or tends to $-\infty$. Hence, starting from some n, we have

$$\frac{x_{n+1}-x_n}{y_{n+1}-y_n} \geqslant \frac{x_n}{y_n}, \qquad n \geqslant n_0.$$

Taking the lower limits, we get

$$\varliminf_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \geqslant \varliminf_{n\to\infty} \frac{x_n}{y_n}.$$

A comparison with the first inequality in (1) proves (1). □

Remark 1. The conclusion of Theorem 2.1 remains valid also for infinitely small strictly monotone sequences y_n if one swaps equalities (1) and (2), with the other assumptions of the theorem being the same.

In what follows, we shall assume that reference sequences y_n, whose growth is compared with that of sequences x_n under study, are positive infinitely large sequences.

Let us now show that the conversion of Stolz's theorem, under which the inequality signs in (1) are reversed, holds for *rapidly growing sequences*. Following [2], a sequence y_n is called rapidly growing if

$$\frac{y_{n+1}}{y_n} \to +\infty, \qquad n \to \infty. \tag{3}$$

Theorem 2.2. *Let* x_n *be a positive sequence and* y_n *be a positive rapidly growing sequence. Then*

$$\varlimsup_{n\to\infty} \frac{x_n}{y_n} = \varlimsup_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}. \tag{4}$$

Moreover, if $\varlimsup\limits_{n\to\infty} \dfrac{x_n}{y_n} < +\infty$*, then in addition*

$$\varliminf_{n\to\infty} \frac{x_n}{y_n} = \varliminf_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}. \tag{5}$$

Proof. We set $M = \varlimsup\limits_{n\to\infty} \dfrac{x_n}{y_n}$, $m = \varliminf\limits_{n\to\infty} \dfrac{x_n}{y_n}$. If $M = +\infty$, then equality (4) is true by Stolz's theorem. Assume now that $M < +\infty$. Setting $q_n := \dfrac{x_n}{y_n}$, we have $M = \varlimsup\limits_{n\to\infty} q_n$, $m = \varliminf\limits_{n\to\infty} q_n$. Transforming the difference

$$x_{n+1} - x_n = q_{n+1}y_{n+1} - q_n y_n = q_{n+1}(y_{n+1}-y_n) + y_n(q_{n+1}-q_n),$$

we get

$$\frac{x_{n+1}-x_n}{y_{n+1}-y_n} = q_{n+1} + \frac{y_n}{y_{n+1}} \, \frac{q_{n+1}-q_n}{1-y_n/y_{n+1}} =: q_{n+1} + r_n. \tag{6}$$

Let us estimate the remainder r_n in this equality. From the hypotheses of the theorem we see that, for a fixed $\varepsilon > 0$ and sufficiently large n,

$$|r_n| \leqslant \frac{y_n}{y_{n+1}} \frac{2(M+\varepsilon)}{1-\varepsilon} \to 0, \quad n \to \infty.$$

Hence (6) can be written as

$$\frac{x_{n+1}-x_n}{y_{n+1}-y_n} = q_{n+1} + o(1), \quad n \to \infty.$$

Taking the upper and lower limit in this formula proves equalities (4) and (5). The theorem is proved. □

Remark 2. This result also holds for negative sequences x_n and y_n. This can be easily checked by multiplying the terms of the sequence by -1. It should also be noted that such a conversion of Stolz's theorem for more slowly growing convex sequences is proved below in Theorem 2.7. Equality (4) was proved by a different method in [2, Propositions 5 and 6].

We also require some results from [2], which are given in an equivalent form convenient for our purposes.

Proposition 2.1. I. *Let a positive sequence y_n have Hadamard gaps; i.e.,*

$$p = p_y := \varliminf_{n\to\infty} \frac{y_{n+1}}{y_n} > 1.$$

Then, for any positive sequence x_n,

$$\varlimsup_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \leqslant \frac{p}{p-1} \varlimsup_{n\to\infty} \frac{x_n}{y_n}.$$

II. *Let y_n be an increasing positive sequence and $n = o(y_n)$ as $n \to \infty$. Then, for any convex positive sequence x_n,*

$$\varlimsup_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \leqslant b_y \varlimsup_{n\to\infty} \frac{x_n}{y_n}, \tag{7}$$

where

$$b_y := \varlimsup_{n\to\infty} \frac{1}{y_{n+1}-y_n} \inf_{m>n} \frac{y_m}{m-n}. \tag{8}$$

We recall that a sequence x_n is *convex* if the sequence $x_{n+1} - x_n$ is increasing; this is equivalent to the fact that the sequence $\dfrac{x_m - x_n}{m-n}$ is increasing in each of the indexes m, n, $m \neq n$, with the other index fixed.

Convex sequences play an important role in the study of quasi-analytic classes of functions, in the Denjoy–Carleman theory of infinitely differentiable functions [7], in the Roumieu–Komatsu theory of ultradistributions (see, for example, [1], [8], [6]).

The above results can be strengthened by considering not only upper, but also lower boundaries of the relative growth of sequences. Let us do this not only for increasing (infinitely large), but also for decreasing (infinitely small) convex sequences. We first require some preliminary "fundamental" uniform-type results, which have independent interest. We set, as usual, $a^+ = \max\{a, 0\}$.

Theorem 2.3. *Let x_n be a convex sequence and y_n be a positive strictly increasing sequence. Assume that the condition*

$$m \leqslant \frac{x_n}{y_n} \leqslant M, \qquad n \in \mathbb{N}, \tag{9}$$

is satisfied with nonnegative constants m, M, $m \leqslant M$. Then

$$M\, s_1^+(\theta) \leqslant \frac{x_{n+1} - x_n}{y_{n+1} - y_n} \leqslant M\, s_2(\theta), \qquad n \in \mathbb{N}, \tag{10}$$

where $\theta = \dfrac{m}{M}$ *and the quantities $s_1(\theta)$, $s_2(\theta)$ are defined as*

$$\begin{aligned} s_1(\theta) &= \inf_{n \geqslant 2} \frac{1}{y_n - y_{n-1}} \sup_{k<n} \frac{y_k - \theta y_n}{k-n}, \\ s_2(\theta) &= \sup_{n \geqslant 1} \frac{1}{y_{n+1} - y_n} \inf_{k>n} \frac{y_k - \theta y_n}{k-n}. \end{aligned} \tag{11}$$

Proof. Using the convexity of the sequence x_n and employing estimates (9), we can write, for arbitrary $n, k \in \mathbb{N}$, $k > n$,

$$x_{n+1} - x_n \leqslant \frac{x_k - x_n}{k-n} \leqslant \frac{M\, y_k - m\, y_n}{k-n} = M\, \frac{y_k - \theta y_n}{k-n}.$$

As a result, we have

$$x_{n+1} - x_n \leqslant M \inf_{k>n} \frac{y_k - \theta y_n}{k-n}, \qquad n \in \mathbb{N}. \tag{12}$$

Dividing both sides by $y_{n+1} - y_n > 0$ for all $n \in \mathbb{N}$, we have

$$\frac{x_{n+1} - x_n}{y_{n+1} - y_n} \leqslant M \frac{1}{y_{n+1} - y_n} \inf_{k>n} \frac{y_k - \theta y_n}{k-n} \leqslant M\, s_2(\theta).$$

This proves the upper estimate in (10).

The proof of the lower estimate in (10) is similar. Given arbitrary $n, k \in \mathbb{N}$, $n \geqslant 2$, $k < n$, we have

$$x_n - x_{n-1} \geqslant \frac{x_k - x_n}{k-n} \geqslant \frac{M\, y_k - m\, y_n}{k-n} = M\, \frac{y_k - \theta y_n}{k-n}.$$

Therefore,

$$x_n - x_{n-1} \geqslant M \sup_{k<n} \frac{y_k - \theta y_n}{k-n}, \qquad n \in \mathbb{N}, \quad n \geqslant 2.$$

Dividing both sides by $y_n - y_{n-1} > 0$, this gives

$$\frac{x_n - x_{n-1}}{y_n - y_{n-1}} \geqslant M \frac{1}{y_n - y_{n-1}} \sup_{k<n} \frac{y_k - \theta y_n}{k-n} \geqslant M\, s_1(\theta), \quad n \geqslant 2,$$

implying the lower estimate in (10). The theorem is proved. □

Theorem 2.4. *Let x_n be a convex sequence and y_n be a positive strictly decreasing sequence. Also let condition* (9)

$$m \leqslant \frac{x_n}{y_n} \leqslant M, \qquad n \in \mathbb{N}$$

be satisfied with nonnegative constants m, M, $m \leqslant M$. Then

$$M\, p_1^+(\theta) \leqslant \frac{x_{n+1} - x_n}{y_{n+1} - y_n} \leqslant M\, p_2(\theta), \qquad n \in \mathbb{N},$$

where $\theta = \dfrac{m}{M}$ *and the quantities $p_1(\theta)$, $p_2(\theta)$ are defined as*

$$p_1(\theta) = \inf_{n\geqslant 1} \frac{1}{y_{n+1}-y_n} \inf_{k>n} \frac{y_k - \theta y_n}{k-n},$$
$$p_2(\theta) = \sup_{n\geqslant 2} \frac{1}{y_n-y_{n-1}} \sup_{k<n} \frac{y_k - \theta y_n}{k-n}.$$

The proof is nearly the same as that of the previous theorem and hence omitted.

Some efforts are required to change from uniform to asymptotic estimates. We define

$$\alpha_y = \overline{\lim_{n\to\infty}}\, n\left(\frac{y_{n+1}}{y_n}-1\right),$$
$$\underline{\alpha}_y = \underline{\lim_{n\to\infty}}\, n\left(\frac{y_{n+1}}{y_n}-1\right).$$

We first prove the following two results.

Theorem 2.5. *Let y_n be a strictly increasing positive sequence and x_n be a convex sequence satisfying the condition*

$$\frac{x_n}{n} \to +\infty, \quad n\to\infty. \tag{13}$$

Then

$$\underline{\lim_{n\to\infty}} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \geqslant \alpha_y^{-1} \underline{\lim_{n\to\infty}} \frac{x_n}{y_n}. \tag{14}$$

Proof. It is easily verified (geometrically clear) that a convex sequence x_n satisfying condition (13) obeys, starting with some number, the relation

$$x_{n+1}-x_n \geqslant \frac{x_n}{n},$$

from which we get

$$\frac{x_{n+1}-x_n}{y_{n+1}-y_n} \geqslant \frac{x_n}{y_n}\frac{y_n}{n(y_{n+1}-y_n)} = \frac{x_n}{y_n}\left[n\left(\frac{y_{n+1}}{y_n}-1\right)\right]^{-1}.$$

Now the required estimate (14) follows by taking lower limits. □

We next require the following estimates of the Euler–Mascheroni constant γ:

$$1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln(n+1) < \gamma < 1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n, \qquad n\in\mathbb{N}. \tag{15}$$

These inequalities follow from the fact that the sequence $a_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n$ decreases (to the limit γ), while the sequence $b_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln(n+1)$ increases and moreover, $0 < a_n - b_n < \frac{1}{n}$.

Theorem 2.6. *Let y_n be a strictly increasing sequence satisfying condition*

$$\frac{y_n}{n} \to +\infty. \tag{14'}$$

Then

$$b_y = \overline{\lim_{n\to\infty}} \frac{1}{y_{n+1}-y_n} \inf_{m>n} \frac{y_m}{m-n} \leqslant \begin{cases} 2^{\alpha_y}/\underline{\alpha}_y & \text{if } \alpha_y > 1, \\ 1/\underline{\alpha}_y & \text{if } \alpha_y = 1, \end{cases} \tag{16}$$

where b_y is defined in (8).

Proof. By definition of the characteristic α_y, for any $\alpha > \alpha_y$

$$\frac{y_{n+1}}{y_n} < 1 + \frac{\alpha}{n}$$

for all $n > n_0$. Hence, for the same n, we have

$$\frac{y_{n+s}}{y_n} = \frac{y_{n+s}}{y_{n+s-1}} \frac{y_{n+s-1}}{y_{n+s-2}} \dots \frac{y_{n+1}}{y_n} < \left(1 + \frac{\alpha}{n+s-1}\right)\left(1 + \frac{\alpha}{n+s-2}\right) \dots \left(1 + \frac{\alpha}{n}\right).$$

Taking logarithms and applying estimates (15), this gives

$$\ln \frac{y_{n+s}}{y_n} < \sum_{k=n}^{n+s-1} \ln\left(1 + \frac{\alpha}{k}\right) < \alpha \sum_{k=n}^{n+s-1} \frac{1}{k} = \alpha \left(\sum_{k=1}^{n+s-1} \frac{1}{k} - \sum_{k=1}^{n-1} \frac{1}{k}\right) < \alpha \ln \frac{n+s}{n-1},$$

and so

$$\frac{y_{n+s}}{y_n} < \left(\frac{n+s}{n-1}\right)^{\alpha}, \qquad n > n_0. \tag{17}$$

We fix $j \in \mathbb{N}$, $j \geqslant 2$, for the moment. Taking $s = nj - n$ in (17), we get

$$\frac{y_{nj}}{y_n} < \left(\frac{nj}{n-1}\right)^{\alpha} = j^{\alpha} \left(\frac{n}{n-1}\right)^{\alpha}, \qquad n > n_0.$$

Using this inequality, we estimate b_y as follows:

$$b_y = \overline{\lim_{n\to\infty}} \frac{1}{y_{n+1} - y_n} \inf_{m>n} \frac{y_m}{m-n} \leqslant \overline{\lim_{n\to\infty}} \frac{1}{y_{n+1} - y_n} \frac{y_{nj}}{nj-n} \leqslant$$
$$\leqslant \overline{\lim_{n\to\infty}} \frac{y_n}{n(y_{n+1} - y_n)} \frac{y_{nj}/y_n}{j-1} \leqslant \overline{\lim_{n\to\infty}} \frac{y_n}{n(y_{n+1} - y_n)} \left(\frac{n}{n-1}\right)^{\alpha} \frac{j^{\alpha}}{j-1} = \underline{\alpha}_y^{-1} \frac{j^{\alpha}}{j-1}.$$

Replacing α with α_y, this gives

$$b_y \leqslant \underline{\alpha}_y^{-1} \frac{j^{\alpha_y}}{j-1}. \tag{18}$$

Now the required estimates follow by putting $j = 2$ in (18) in the case $\alpha_y > 1$ or making $j \to \infty$ in the case $\alpha_y = 1$. □

The following conversion of Stolz's theorem is a direct corollary to Theorems 2.5 and 2.6.

Theorem 2.7. *Let x_n and y_n be convex sequences satisfying* (13), (14′) *respectively. Assume that $\alpha_y = 1$. Then*

$$\underline{\lim_{n\to\infty}} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \underline{\lim_{n\to\infty}} \frac{x_n}{y_n}, \qquad \overline{\lim_{n\to\infty}} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \overline{\lim_{n\to\infty}} \frac{x_n}{y_n}. \tag{19}$$

Proof. As was already noted, a convex sequence y_n satisfying condition (14′) has the property

$$y_{n+1} - y_n \geqslant \frac{y_n}{n}, \qquad n \geqslant n_0.$$

Hence $\underline{\alpha}_y = \alpha_y = 1$. By estimate (14) of Theorem 2.5 we get

$$\underline{\lim_{n\to\infty}} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} \geqslant \underline{\lim_{n\to\infty}} \frac{x_n}{y_n}.$$

This together with Stolz's theorem proves the first equality in (19). By Theorem 2.6 we have $b_y = 1$. But then the second equality in (19) is valid due to inequality (7) of assertion II of Proposition 2.1 (see also [2, Corollary 3]). □

Remark 3. In [9], the sequences satisfying the condition

$$\frac{y_{n+1}}{y_n} < 1 + \frac{\alpha}{n}, \qquad n > n(\alpha),$$

for some $\alpha > 0$ were called *quasi-monotone*. Such sequences were used in the study of trigonometric series. Any such sequence displays at most power-like growth and is characterized by the fact that, for some $\beta > 0$, the sequence $\frac{y_n}{n^\beta}$ is decreasing. Clearly, in Theorem 2.7 one speaks about sequences y_n subject to the condition of quasi-monotonicity with arbitrary $\alpha > 1$. If the reference sequence y_n is not rapidly growing (see Theorem 2.3), then a complete conversion of Stolz's theorem (in form (19)) is impossible. Nevertheless, one can give sharp estimates, which contain a variant of the reverse of Stolz's theorem.

We require some additional facts. Given a strictly increasing sequence y_n and a number $\theta \in [0,\, 1]$, we set

$$\begin{aligned} \tilde{s}_1(\theta) &= \varliminf_{l\to\infty} \inf_{n\geqslant l+1} \frac{1}{y_n - y_{n-1}} \sup_{l\leqslant k<n} \frac{y_k - \theta y_n}{k-n}, \\ \tilde{s}_2(\theta) &= \varlimsup_{n\to\infty} \frac{1}{y_{n+1} - y_n} \inf_{k>n} \frac{y_k - \theta y_n}{k-n}. \end{aligned} \tag{20}$$

$$\begin{aligned} s_{1,l}(\theta) &= \inf_{n\geqslant l+1} \frac{1}{y_n - y_{n-1}} \sup_{l\leqslant k<n} \frac{y_k - \theta\, y_n}{k-n}, \\ s_{2,l}(\theta) &= \sup_{n\geqslant l} \frac{1}{y_{n+1} - y_n} \inf_{k>n} \frac{y_k - \theta\, y_n}{k-n}. \end{aligned} \tag{21}$$

It is clear that $\tilde{s}_1(\theta) = \varliminf\limits_{l\to\infty} s_{1,l}(\theta)$, $\tilde{s}_2(\theta) = \lim\limits_{l\to\infty} s_{2,l}(\theta)$, and moreover, for $0 \leqslant \theta_1 < \theta < 1$, we have

$$s_{1,l}(\theta_1) \leqslant s_{1,l}(\theta), \qquad s_{2,l}(\theta_1) \geqslant s_{2,l}(\theta).$$

Let us prove the reverse inequalities.

Theorem 2.8. *Let y_n be an increasing sequence and let $0 \leqslant \theta_1 < \theta < 1$ and $l \in \mathbb{N}$. Then*

$$s_{2,l}(\theta_1) \leqslant \frac{1-\theta_1}{1-\theta}\, s_{2,l}(\theta). \tag{22}$$

Proof. For $\theta > \theta_1$ and $k > n$, we have

$$\begin{gathered} y_k > y_n, \quad (\theta - \theta_1) y_k > (\theta - \theta_1) y_n, \\ [(1-\theta_1) - (1-\theta)] y_k > [(1-\theta_1)\theta - (1-\theta)\theta_1] y_n, \\ (1-\theta_1)(y_k - \theta y_n) > (1-\theta)(y_k - \theta_1\, y_n), \quad \frac{y_k - \theta_1\, y_n}{k-n} < \frac{1-\theta_1}{1-\theta}\, \frac{y_k - \theta y_n}{k-n}, \\ \frac{1}{y_{n+1} - y_n} \inf_{k>n} \frac{y_k - \theta_1\, y_n}{k-n} \leqslant \frac{1-\theta_1}{1-\theta}\, \frac{1}{y_{n+1} - y_n} \inf_{k>n} \frac{y_k - \theta y_n}{k-n}. \end{gathered}$$

Taking the supremum over $n \geqslant l$ we get estimate (22). □

Before proceeding with the proof of the next estimate, we mention the following fact. If a strictly increasing sequence y_n satisfies for some $l \in \mathbb{N}$ and $\theta \in (0,\, 1)$ the condition

$$y_{l+1}/y_l \leqslant \theta^{-1}, \tag{23}$$

then, for the same l,

$$s_{1,l}(\theta) \leqslant \frac{\theta\, y_{l+1} - y_l}{y_{l+1} - y_l} \leqslant 0.$$

This inequality can be easily proved by putting $n = l+1$ in the first inequality in (21). So, if condition (23) holds for some sequence $l = l_k \to \infty$, then $\tilde{s}_1(\theta) \leqslant 0$. However, as we shall see later (see estimate (27) of Theorem 2.10), meaningful estimates with a reference sequence y_n appear in the case when $\tilde{s}_1(\theta) > 0$. Hence it is natural to assume that y_n has Hadamard gaps; i.e.,

$$p_y := \varliminf_{n\to\infty} \frac{y_n}{y_{n-1}} > 1. \tag{24}$$

In [4] this quantity is called the sparsity index of a sequence y_n.

Theorem 2.9. *Let y_n be a positive sequence with Hadamard gaps; i.e., $p_y > 1$. Then for arbitrary θ, θ_1, $0 \leqslant \theta_1 < \theta < 1$ and sufficiently large $l \in \mathbb{N}$,*

$$s_{1,l}(\theta_1) \geqslant \frac{\theta_1}{\theta}\, s_{1,l}(\theta) - (\theta - \theta_1)\,\frac{p_y}{p_y - 1}. \tag{25}$$

Proof. The following relations are straightforward:

$$\sup_{l\leqslant k<n} \frac{y_k - \theta_l\, y_n}{k-n} = \sup_{l\leqslant k<n} \left(\frac{y_k - \theta y_n}{k-n} + \frac{(\theta-\theta_l) y_n}{k-n} \right) \geqslant$$
$$\geqslant \sup_{l\leqslant k<n} \frac{y_k - \theta y_n}{k-n} + \inf_{l\leqslant k<n} \frac{(\theta-\theta_l) y_n}{k-n} = \sup_{l\leqslant k<n} \frac{y_k - \theta y_n}{k-n} - (\theta-\theta_l) y_n.$$

Let us divide by $y_n - y_{n-1} > 0$. Taking $p' \in (1, p_y)$ and assuming that l is so large that the condition $\dfrac{y_n}{y_{n-1}} > p'$ is satisfied for for $n > l$, we get

$$\frac{1}{y_n - y_{n-1}} \sup_{l\leqslant k<n} \frac{y_k - \theta_l\, y_n}{k-n} \geqslant \frac{1}{y_n - y_{n-1}} \sup_{l\leqslant k<n} \frac{y_k - \theta y_n}{k-n} - \frac{(\theta-\theta_l) y_n}{y_n - y_{n-1}} \geqslant$$
$$\geqslant \frac{1}{y_n - y_{n-1}} \sup_{l\leqslant k<n} \frac{y_k - \theta y_n}{k-n} - (\theta-\theta_l)\,\frac{p'}{p'-1}.$$

Taking the infimum over $n > l$, this gives

$$s_{1,l}(\theta_l) \geqslant s_{1,l}(\theta) - (\theta - \theta_l)\,\frac{p'}{p'-1}.$$

The proof is complete by replacing p' with p_y. $\square$

We can now give a reverse of Stolz's theorem.

Theorem 2.10. *Let x_n be a positive convex sequence and y_n be a positive strictly increasing sequence. Assume that*

$$m = \varliminf_{n\to\infty} \frac{x_n}{y_n}, \qquad M = \varlimsup_{n\to\infty} \frac{x_n}{y_n},$$

where $0 \leqslant m \leqslant M < +\infty$. Then

$$\varlimsup_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} \leqslant M\, \tilde{s}_2(\theta). \tag{26}$$

If, in addition, y_n has Hadamard gaps; i.e., $\varliminf\limits_{n\to\infty} \dfrac{y_{n+1}}{y_n} = p_y > 1$, *then*

$$M\,\tilde{s}_1^{+}(\theta) \leqslant \varliminf_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}. \tag{27}$$

Here $\theta = \dfrac{m}{M}$ *and* $\tilde{s}_1(\theta)$, $\tilde{s}_2(\theta)$ *are given by formulas* (20).

Proof. Given $l \in \mathbb{N}$, we set $m_l = \inf\limits_{n\geqslant l} \dfrac{x_n}{y_n}$, $\quad M_l = \sup\limits_{n\geqslant l} \dfrac{x_n}{y_n}$. It is clear that

$$m_l \leqslant \frac{x_n}{y_n} \leqslant M_l, \qquad n \geqslant l.$$

By Theorem 2.3 we have

$$M_l\, s_{1,l}(\theta_l) \leqslant \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \leqslant M_l\, s_{2,l}(\theta_l), \qquad n \geqslant l, \tag{28}$$

where $\theta_l = \dfrac{m_l}{M_l}$ and $s_{1,l}(\theta_l)$, $s_{2,l}(\theta_l)$ are given by formulas (21). From Theorem 2.8 and the right-hand side of (28) we get

$$\sup_{n\geqslant l} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \leqslant M_l\, s_{2,l}(\theta_l) \leqslant M_l\, \frac{1-\theta_l}{1-\theta}\, s_{2,l}(\theta).$$

Taking into account that

$$\lim_{l\to\infty} m_l = m, \qquad \lim_{l\to\infty} M_l = M, \qquad \lim_{l\to\infty} \theta_l = \theta, \qquad \lim_{l\to\infty} s_{2,l}(\theta) = \tilde{s}_2(\theta),$$

we arrive at inequality (26).

Using the left-hand side of (28) and Theorem 2.9, we get

$$\inf_{n>l} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \geqslant M_l\, s_{1,l}(\theta_l) \geqslant M_l \left(s_{1,l}(\theta) - (\theta-\theta_l)\, \frac{p_y}{p_y-1} \right).$$

Inequality (27) is proved by taking the lower limit as $l \to \infty$; here we again used the fact that $M_l \to M$ and $\theta_l \to \theta$. □

As a corollary, we have the following refinement of Proposition 2.1.

Proposition 2.2. *Assume that a positive sequence y_n has Hadamard gaps, and* $p = \varliminf\limits_{n\to\infty} \dfrac{y_{n+1}}{y_n} > 1$. *Let x_n be an arbitrary increasing convex sequence with*

$$\varliminf_{n\to\infty} \frac{x_n}{y_n} = m, \qquad \varlimsup_{n\to\infty} \frac{x_n}{y_n} = M,$$

where $0 \leqslant m \leqslant M < +\infty$. *Then*

$$\frac{mp-M}{p-1} \leqslant \varliminf_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}, \qquad \varlimsup_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} \leqslant \frac{Mp-m}{p-1}. \tag{29}$$

Proof. Let us estimate the quantities $\tilde{s}_1(\theta)$, $\tilde{s}_2(\theta)$ given in formula (20), where $\theta = \dfrac{m}{M}$. Replacing the supremum in the first of the expressions in (20) with the value of the sequence for $k = n-1$, we get

$$\tilde{s}_1(\theta) \geqslant \varliminf_{l\to\infty} \inf_{n\geqslant l+1} \frac{1}{y_n - y_{n-1}} \, \frac{y_{n-1}-\theta y_n}{n-1-n} = \varliminf_{l\to\infty} \inf_{n\geqslant l+1} \frac{\theta y_n - y_{n-1}}{y_n - y_{n-1}} =$$

$$= \varliminf_{l\to\infty} \inf_{n\geqslant l+1} \frac{\theta \dfrac{y_n}{y_{n-1}} - 1}{\dfrac{y_n}{y_{n-1}} - 1} \geqslant \frac{\theta p - 1}{p-1}.$$

Here we used the fact that the function $x \mapsto \dfrac{\theta x - 1}{x-1}$ is increasing on $(1, +\infty)$.

To estimate $\tilde{s}_2(\theta)$, we replace the infimum in the second expression in (20) with the value of the sequence for $k = n+1$. We get

$$\tilde{s}_2(\theta) \leqslant \varlimsup_{n\to\infty} \frac{1}{y_{n+1} - y_n} \frac{y_{n+1} - \theta y_n}{n+1-n} = \varlimsup_{n\to\infty} \frac{\dfrac{y_{n+1}}{y_n} - \theta}{\dfrac{y_{n+1}}{y_n} - 1} \leqslant \frac{p-\theta}{p-1}.$$

At the last step, we used the fact that the function $x \mapsto \dfrac{x-\theta}{x-1}$ is decreasing on the interval $(1, +\infty)$. To complete the proof it suffices to employ Theorem 10. □

The estimates of Theorem 2.10 (and Proposition 2.2, which is a corollary to Theorem 2.10) are sharp — it is easily seen that they are attained on the sequence $y_n = e^{\rho n}$, $\rho \geqslant 1$.

As was already noted, the lower estimate in (29) is meaningful if (in contrast to condition (23)), the sequence y_n increases faster than some geometric progression; i.e., condition (24) is satisfied with $p = \varliminf_{n\to\infty} \dfrac{y_{n+1}}{y_n} > 1$. In some cases, one can "suppress" the growth of sequences and obtained a meaningful answer in the spirit of Stolz's theorem.

We first recall that a positive sequence a_n is called *logarithmically convex* if $\dfrac{a_{n+1}}{a_n}$ increases with increasing n.

Theorem 2.11. *Let x_n and y_n be logarithmically convex sequences satisfying for some $p > 1$ the conditions*

$$\frac{x_n}{p^n n} \to +\infty, \qquad \frac{y_n}{p^n n} \to +\infty, \qquad \lim_{n\to\infty} n\left(\frac{y_{n+1}}{p y_n} - 1\right) = 1.$$

Then

$$\varliminf_{n\to\infty} \frac{x_{n+1} - p x_n}{y_{n+1} - p y_n} = \varliminf_{n\to\infty} \frac{x_n}{y_n}, \qquad \varlimsup_{n\to\infty} \frac{x_{n+1} - p x_n}{y_{n+1} - p y_n} = \varlimsup_{n\to\infty} \frac{x_n}{y_n}. \tag{30}$$

Proof. Indeed, it suffices to apply Theorem 2.7 to the sequences $X_n = x_n p^{-n}$, $Y_n = y_n p^{-n}$. Let us check that these sequences are convex (the remaining conditions of this theorem clearly hold). We have

$$X_{n+1} - 2X_n + X_{n-1} = \frac{x_{n+1} - 2x_n p + x_{n-1} p^2}{p^{n+1}} \geqslant \frac{x_n}{p^{n+1}} \left(\frac{x_{n+1}}{x_n} - \frac{x_n}{x_{n-1}}\right) \geqslant 0.$$

Here we used the fact that the minimum of this quadratic is attained for $p = \dfrac{x_n}{x_{n-1}}$. It remains to observe that

$$\frac{X_{n+1} - X_n}{Y_{n+1} - Y_n} = \frac{x_{n+1} - p x_n}{y_{n+1} - p y_n}, \qquad \frac{X_n}{Y_n} = \frac{x_n}{y_n}.$$

□

Note that the hypotheses of Theorem 3.1 are satisfied, for example, for sequences of the form $y_n = p^n n \ln^s n$, $y_n = p^n n \ln^s n \ln^t(\ln n)$, where $p > 1$, $s > 0$, $t \in \mathbb{R}$, and, more generally, $y_n = p^n n\, l(n)$ where $l(x)$ slowly tends to $+\infty$.

Acknowledgments

The author is grateful to the journal editorial staff and an anonymous reviewer, whose comments significantly improved the text of the article. This work was supported by the Russian Foundation for Basic Research, project no. 18-01-00236.

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Received: 28.12.2018

Events

5TH INTERNATIONAL CONFERENCE "ACTUAL PROBLEMS OF MATHEMATICS AND COMPUTER SCIENCE: THEORY, METHODOLOGY, PRACTICE" (APRIL 18–20, 2019, YELETS, RUSSIA)

The XX century is marked by the enrichment of world science with outstanding achievements in the field of mathematics, solving many important problems that remain relevant in the modern world. Such problems include, in particular, the problems considered in fundamental works of academician S. Chaplygin. Based on his works new research paths were set, and serious applied problems were addressed in the fields of aerodynamics, gas dynamics, hydrodynamics, and mechanics. They were further intensively developed due to the achievements of contemporary information technology.

To commemorate his activities was organized a large-scale scientific event in the historic homeland of academician S. Chaplygin — the 5th international conference "Actual problems of mathematics and computer science: theory, methodology, practice", dedicated to the 150th anniversary of the birth of academician S. Chaplygin.

The Ivan Bunin Yelets State University (Russia), the Samarkand State University (Uzbekistan), the Higher School of Insurance and Finance (Bulgaria),the Khachatur Abovyan Armenian State Pedagogical University (Armenia), and the Scientific and Methodological Council on Mathematics of the Ministry of Science and Higher Education of Russia held through April 18–20, 2019 the 5th International Conference "Actual problems of mathematics and computer science: theory, methodology, practice" dedicated to the 150th anniversary of academician S. Chaplygin.

The conference marked the three major milestones related to the development of mathematical science in the Lipetsk region and in the oldest university center in the region — the Ivan Bunin Yelets State University.

1. April 2019 is the 150th anniversary of the birth of S. Chaplygin (1869–1942) — a well-known Russian scientist, academician of the Academy of Sciences of the USSR. S. Chaplygin is an outstanding representative of the Lipetsk region, whose surname is immortalized in the name of the city Chaplygin (previously Ranenburg) in the Lipetsk region.

2. 2019 year is the 80th anniversary of the foundation of the Faculty of Physics and Mathematics. It is the oldest faculty of the Ivan Bunin Yelets State University, where students of the scientific school of academician N. Zhukovsky were taught, whose famous representative was academician S. Chaplygin.

3. In October 2019 there will be 10 years since the organization of the Lipetsk Branch of the Scientific and Methodological Council for Mathematics of the Ministry of Science and Higher Education of Russia on the basis of the Ivan Bunin Yelets State University.

The main goals of the conference were the creation of conditions for international scientific communication of representatives of fundamental and applied areas in the field of mathematics, understanding the importance of scientific works of S. Chaplygin, the actualization of his scientific achievements, taking into account the rapid development of information technologies and their adaptation to modern mathematical education.

The plenary session of the conference was opened by the Rector of the Ivan Bunin Yelets State University Professor E. Gerasimova and continued by the President of the International Academy of the History of Science Professor S. Demidov (Moscow, Russia), who presented the talk "Pure and Applied Mathematics at the M.V. Lomonosov Moscow State University in the first half of the twentieth century: N. Luzin and S. Chaplygin".

Professor A. Soleev (Samarkand, Uzbekistan) devoted his talk to basic ideas and general provisions of the Power Geometry. Professor A. Soldatov (Moscow, Russia) focused on the consideration of the Dirichlet problem for equations of mixed type. In her talk Professor G. Zhukova (Moscow, Russia) discussed the dependence of solutions to singularly perturbed linear differential systems on a small parameter. The talk of Professors O. Masina (Yelets, Russia) and O. Druzhinina (Moscow, Russia) was devoted to the analysis of the known and developed by the authors approaches to the study of the stability of intelligent control systems. The talks of Professors V. Tikhomirov (Moscow, Russia), T. Sergeeva (Moscow, Russia) and E. Smirnov (Yaroslavl, Russia) addressed the issues of improving mathematical education, introducing novelty into the teaching process while maintaining the best traditions of high-quality teaching mathematics, laid by S. Chaplygin in his productive teaching activities.

The relevance of the event was noted in the talks of Professors A. Abylkasymova (Alma-Ata, Kazakhstan), A. Borovskikh (Moscow, Russia), S. Grozdev (Sofia, Bulgaria), M. Mkrtchyan (Yerevan, Armenia) and other scientists. At the end of the plenary session, talks were presented by the authors of this communication on the history of the Scientific and Methodological Council on Mathematics of the Ministry of Science and Higher Education of Russia and its contribution to the development of mathematics and its applications in Russian education, as well as on the activities of the Lipetsk Branch of the Scientific and Methodological Council.

The following sections were working at the conference: “Modern Directions in Mathematics”, “Applied problems of mathematics”, “Computer modeling, information technologies and systems”, “New theories, models and technologies of teaching mathematics and computer science at schools and universities”, “Actualization of the problems of the history of mathematics and mathematical education in modern conditions”.

At the conference there were more than 250 participants, including leading foreign specialists from Armenia, Bulgaria, Uzbekistan, Kazakhstan, well-known scientists from more than twenty regions of Russia, as well as young researchers. Overall, it was a successful conference, which helped to increase the scientific and innovative activity of the region, stimulated the participants to develop mathematics, information technologies and mathematical education.

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