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FINITE GROUPS WITH GIVEN SYSTEMS OF PROPERMUTABLE SUBGROUPS

A.A. Trofimuk

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Keywords: supersoluble group, propermutable subgroup, saturated formation, factorized groups, Sylow and maximal subgroups.

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Abstract. Let H be a subgroup of a finite group G. Then we say that H is propermutable in G provided G has a subgroup B such that $G = N_G(H)B$ and H permutes with all subgroups of B. In this paper, we present new properties of propermutable subgroups. Also we provide new information on the structure of a group with propermutable Sylow (Hall, maximal) subgroups and a group $G = AB$ with propermutable subgroups A and B.

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1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [4]. The notation $Y \leq X$ ($Y < X$) means that Y is a subgroup (proper subgroup) of a group X.

A subgroup H of G is called *seminormal* in G if there exists a subgroup B such that $G = HB$ and HX is a subgroup of G for each subgroup X of B. The groups with given systems of seminormal subgroups were investigated in works of many authors, see, for example, the references in [13].

Following [16] a subgroup H is called *propermutable* in G if G has a subgroup B such that $G = N_G(H)B$ and H permutes with all subgroups of B. The groups with some propermutable subgroups were investigated in [1, 16, 17].

Obviously, if a subgroup H is seminormal in G, then H is propermutable in G. The opposite is not always true. For example, in the group

$$G = \langle a, b, c \mid |a| = |b| = 3, |c| = 2, ab = ba, ac = ca, b^c = b^{-1}\rangle \simeq Z_3 \times S_3$$

([6], IdGroup=[18,3]), the subgroup $A = \langle c\rangle$ is propermutable in G, since $N_G(A) = \langle ac\rangle$ and $B = \langle b\rangle$, but A is not seminormal in G.

In this paper, we present new properties of propermutable subgroups. Also we provide new information on the structure of a group with propermutable Sylow (Hall, maximal) subgroups and a group $G = AB$ with propermutable subgroups A and B.

2 Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel. A group whose chief factors have prime orders is called *supersoluble*. Recall that a *p-closed* group is a group with a normal Sylow p-subgroup and a *p-nilpotent* group is a group with a normal Hall p'-subgroup.

Denote by G', $Z(G)$, $F(G)$ and $\Phi(G)$ the derived subgroup, centre, Fitting and Frattini subgroups of G, respectively, and by $\mathrm{O}_p(G)$ the largest normal p-subgroup of G. Denote by $\pi(G)$ the set of all prime divisors of order of G. We use E_{p^t} to denote an elementary abelian group of order p^t and Z_m to denote a cyclic group of order m. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$.

The monographs [5, 10] contain the necessary information of the theory of formations.

A class group $\mathfrak{F}$ is called a formation if the following statements is true:

(1) if $G \in \mathfrak{F}$ and $N \lhd G$, then $G/N \in \mathfrak{F}$.

(2) if $G/N_1 \in \mathfrak{F}$ and $G/N_2 \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$.

A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. The formations of all supersoluble, nilpotent and abelian groups are denoted by $\mathfrak{U}$, $\mathfrak{N}$ and $\mathfrak{A}$, respectively. Let $\mathfrak{F}$ be a formation. Recall that the $\mathfrak{F}$-residual of G is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$ and is denoted by $G^{\mathfrak{F}}$.

Recall that a group G is said to be *siding* if every subgroup of the derived subgroup G' is normal in G, see [14, Definition 2.1]. It is clear that if G is a siding group, then G is supersoluble, every subgroup and quotient subgroup of G is a siding group. Metacyclic groups and soluble T-groups (groups in which every subnormal subgroup is normal) are siding groups. The group $G = (Z_6 \times Z_2) \rtimes Z_2$ ([6], IdGroup(G)=[24,8]) is a siding group but is neither a metacyclic nor a T-group.

Lemma 2.1. ([7, VI.9]) (1) *The class $\mathfrak{U}$ is a hereditary saturated formation.*

(2) *Every minimal normal subgroup of a supersoluble group has prime order.*

(3) *Let N be a normal subgroup of G and assume that G/N is supersoluble. If N is either cyclic or $N \leq Z(G)$, or $N \leq \Phi(G)$, then G is supersoluble.*

(4) *Each supersoluble group has an Sylow tower of supersoluble type.*

(5) *The derived subgroup of a supersoluble group is nilpotent.*

(6) *A group G is supersoluble if and only if every maximal subgroup of G has prime index.*

If H is a subgroup of G, then $H_G = \bigcap_{x\in G} H^x$ is called *the core* of H in G. If a group G contains a maximal subgroup M with trivial core, then G is said to be *primitive* and M is its *primitivator*. A simple check proves the following lemma.

Lemma 2.2. *Let $\mathfrak{F}$ be a saturated formation and G be a group. Assume that $G \notin \mathfrak{F}$, but $G/N \in \mathfrak{F}$ for all non-trivial normal subgroups N of G. Then G is a primitive group.*

Lemma 2.3. ([7, II.3.2]) *Let G be a soluble primitive group and M be a primitivator of G. Then the following statements hold:*

(1) $\Phi(G) = 1$;

(2) *$F(G) = C_G(F(G)) = O_p(G)$ and $F(G)$ is an elementary abelian subgroup of order p^n for some prime p and some positive integer n;*

(3) *G contains a unique minimal normal subgroup N and, moreover, $N = F(G)$;*

(4) *$G = F(G) \rtimes M$ and $O_p(M) = 1$;*

Lemma 2.4. ([10, Lemma 5.8, Lemma 5.11]) *Let $\mathfrak{F}$ and $\mathfrak{H}$ be non-empty formations, K be normal in G. Then:*

(1) $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$;

(2) $G^{\mathfrak{F}\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$;

(3) *if $\mathfrak{H} \subseteq \mathfrak{F}$, then $G^{\mathfrak{F}} \leq G^{\mathfrak{H}}$;*

3 Finite groups with propermutable Sylow, Hall and maximal subgroups

Recall that $A^G = \langle A^g \mid g \in G\rangle$ is the smallest normal subgroup of G containing A.

Basic properties of propermutable subgroups are given in [16]. Some of them are presented in the following lemma.

Lemma 3.1. ([16]) *Let A and B be subgroups of G and let N be a normal subgroup of G.*

(1) *If A is propermutable in G, then AN/N is propermutable in G/N.*

(2) *If $AB = BA$ and $G = N_G(A)B$, then $A^G = A(A^G \cap B)$.*

It is clear that the following lemma is true.

Lemma 3.2. *Let A be a subgroup of G. If A is propermutable in G, then A is seminormal in A^G. In particular, if $A^G = G$, then A is seminormal in G.*

Lemma 3.3. 1. *Let A be a subgroup of G. If A is propermutable in G, then A^G is soluble in each of the following cases:*

(1.1) *A is 2-nilpotent;*

(1.2) *A is soluble and $3 \notin \pi(A)$.*

2. *Let p be the smallest prime divisor of the order of G. If A is propermutable in G and p does not divide the order of A, then p does not divide the order of A^G.*

3. *Let A be propermutable in a soluble group G and let r be the largest in $\pi(G)$. If A is r-closed, then A_r is subnormal in G.*

Proof. 1. Let us prove both assertions 1 and 2 at once. By Lemma 3.2, A is seminormal in A^G. Then by [8, Lemmas 10–11], A^{A^G} is either soluble or a p'-group. Since A^{A^G} is subnormal in G, it follows that by [10, Theorem 5.31], $(A^{A^G})^G = A^G$ is either soluble or a p'-group.

3. By Lemma 3.2, A is seminormal in A^G. Then A_r is subnormal in A^G by [13, Lemma 1.8]. Hence, A_r is subnormal in G. □

The following theorem generalizes some results of the papers [8, 9, 13].

Theorem 3.1. 1. *Let H be a Hall π-subgroup of G. Suppose that H is propermutable in G. Then G is π-soluble in each of the following cases:*

(1.1) *H is 2-nilpotent;*

(1.2) *H is soluble and $3 \notin \pi$.*

2. *Let P be a Sylow p-subgroup of G. If P is propermutable in G, then G is p-soluble.*

3. *Let p be the largest prime in $\pi(G)$ and let P be a Sylow p-subgroup in G. If P is propermutable in G, then P is normal in G.*

4. *If all Sylow subgroups in G are propermutable, then G is supersoluble.*

5. *If all maximal subgroups in G are propermutable, then G is supersoluble.*

Proof. 1. By Lemma 3.3 (1), H^G is soluble. Since G/H^G is π'-group, it follows that G is π-soluble.

2. Since P is 2-nilpotent then from Step 1, G is p-soluble.

3. By Lemma 3.2, P is seminormal in P^G. Then P is normal in P^G by [13, Lemma 1.8]. Hence, P is normal in G, because P is subnormal in G.

4. Assume that the statement is not true and let G be a counterexample of minimal order. Let N be an arbitrary nontrivial normal subgroup in G and let S/N be a Sylow s-subgroup of G/N. Then $S/N = S_1N/N$, where S_1 is a Sylow s-subgroup of G. Since S_1 is propermutable in G, we have by Lemma 3.1 (1), S/N is propermutable in G/N. Thus, the condition of the lemma holds for the quotient group and by induction, G/N is supersoluble and G is primitive by Lemma 2.2.

From Step 2 it follows that G is p-soluble for every $p \in \pi(G)$. Hence, G is soluble. By Lemma 2.3, G has a unique minimal normal subgroup N, $N = F(G) = O_p(G) = C_G(N)$, N is an elementary abelian subgroup of order p^n and $G = N \rtimes M$, where M is a maximal subgroup of G with trivial core. From Step 3 follows that p is the largest prime in $\pi(G)$ and $N = P$, where P is a Sylow p-subgroup of G. It is clear that M is a Hall p'-subgroup of G.

Let $N_1 \leq N = P$ such that $|N_1| = p$, and Q is a Sylow q-subgroups of M. Since Q is propermutable in G, we have $G = N_G(Q)Y$ and QX is a subgroup of G for every subgroup X of Y. By Lemma 3.1 (2), $Q^G = Q(Q^G \cap Y)$. Because $N \leq Q^G$, it follows that $N \leq Q^G \cap Y \leq Y$ and $QN_1 \leq G$ by Lemma 3.2. Since G is p-closed, $Q \leq N_G(N_1)$. Hence, $M \leq N_G(N_1)$ and N_1 is normal in $G = NM$. Then $N_1 = N$ and by Lemma 2.1 (3), G is supersoluble, a contradiction.

5. Let M be a maximal subgroup of G. By Lemma 3.2, M is seminormal in M^G. Since M is maximal in G, we have either $M^G = M$ or $M^G = G$. If $M^G = M$, then M is normal in G and $|G : M|$ is prime. If $M^G = G$, then M is seminormal in G. By [13, Lemma 1.4], $|G : M|$ is prime. By Lemma 2.1 (6), G is supersoluble. □

4 Finite factorizable groups with propermutable factors

Theorem 4.1. *Assume that A and B are propermutable subgroups of a group G and $G = AB$. Then the following statements hold.*

1. *Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$. If $A, B \in \mathfrak{F}$ and the derived subgroup G' is nilpotent, then $G \in \mathfrak{F}$.*
2. *If A and B are supersoluble, then $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$.*
3. *If A and B have Sylow towers of supersoluble type, then G has a Sylow tower of supersoluble type.*
4. *If A is nilpotent and B is supersoluble, then G is supersoluble.*
5. *If A is supersoluble and B is a normal siding subgroup of G, then G is supersoluble.*

Proof. 1. Assume that the claim is false and let G be a minimal counterexample. If N is a nontrivial normal subgroup of G, then the subgroups AN/N and BN/N are propermutable in G/N by Lemma 3.1 (1) and belong to $\mathfrak{F}$. Since

$$(G/N)' = G'N/N \simeq G'/G' \cap N,$$

it follows that the derived subgroup $(G/N)'$ is nilpotent. Consequently, G/N satisfies the hypothesis of the theorem and by induction, $G/N \in \mathfrak{F}$. Then G is primitive by Lemma 2.2. Since G is soluble, therefore we apply Lemma 2.3. We save to G the notation of this lemma, in particular, $N = G'$ and G/N is abelian.

If $A^G = G$ and $B^G = G$, then by Lemma 3.2, the subgroups A and B are seminormal in G. By [15, Corollary 3.1 (2)], $G \in \mathfrak{F}$. Suppose that $A^G < G$. Since AN is normal in G, we have $A^G \leq AN$. On the other hand, $AN \leq A^G$, because N is the unique minimal normal subgroup of G. Hence, $AN = A^G$. By Lemma 3.2, A is seminormal in A^G, hence $A^G \in \mathfrak{F}$ by induction. If $B^G < G$, then by analogy, $B^G \in \mathfrak{F}$ and $G = AB = A^G B^G \in \mathfrak{F}$ by [15, Corollary 3.1 (2)].

If $B^G = G$, then by Lemma 3.2, B is seminormal in G. Then $G = AB = A^G B \in \mathfrak{F}$ by [15, Corollary 3.1 (2)].

2. Let $H = (G')^{\mathfrak{N}}$. Then the derived subgroup $(G/H)' = G'H/H = G'/H$ is nilpotent. From Step 1 it follows that G/H is supersoluble. Therefore, $G^{\mathfrak{U}} \leq H$. Because $\mathfrak{U} \subseteq \mathfrak{N}\mathfrak{A}$, we have $G^{(\mathfrak{N}\mathfrak{A})} = (G^{\mathfrak{A}})^{\mathfrak{N}} = (G')^{\mathfrak{N}} = H \leq G^{\mathfrak{U}}$. Hence, $G^{\mathfrak{U}} = H$.

3. We proceed by induction on $|G|$. Since A is 2-nilpotent, it follows that by Lemma 3.3 (1), A^G is soluble and $G = A^G B$ is soluble. Let $r \in \pi(G)$ and let r be the largest. It is clear that a Sylow r-subgroup A_r is normal in A. By Lemma 3.3 (3), A_r is subnormal in G. Similarly, a Sylow r-subgroup

B_r of B is subnormal in G. Since $R = A_rB_r$ is a Sylow subgroup of G, we have G is r-closed. The subgroups $AR/R \simeq A/A \cap R$ and $BR/R \simeq B/B \cap R$ are propermutable in $G/R = (AR/R)(BR/R)$ and have Sylow towers of supersoluble type. By induction, G/R has an Sylow tower of supersoluble type, hence G has an Sylow tower of supersoluble type.

4. Assume that the claim is false and let G be a minimal counterexample. If N is a nontrivial normal subgroup of G, then the subgroups AN/N and BN/N are propermutable in G/N by Lemma 3.1 (1), $AN/N \simeq A/A \cap N$ is nilpotent and $BN/N \simeq B/B \cap N$ is supersoluble. Then by induction, $G/N = (AN/N)(BN/N)$ is supersoluble and G is primitive by Lemma 2.2. By Lemma 2.1 (4) and from Step 3, G has an Sylow tower of supersoluble type and therefore we apply Lemma 2.3. We save to G the notation of this lemma, in particular, $N = G_p$ is the Sylow p-subgroup for the largest $p \in \pi(G)$. Since $G = AB$, it follows that $N = A_pB_p$, where A_p and B_p are Sylow p-subgroups of A and B respectively, see [7, VI.4.6]. Since A is propermutable in G, $G = N_G(A)Y$ and $AX \leq G$ for all subgroups X of Y.

Suppose that $A_p = 1$. Then $N = B_p \leq B$. We choose a minimal normal subgroup N_1 of B such that $N_1 \leq N$. Since B is supersoluble, we have $|N_1| = p$ by Lemma 2.1 (2). By Lemma 3.1 (2), $A^G = A(A^G \cap Y)$. Since $A_p = 1$ and $N \leq A^G$, we have $N_1 \leq N \leq Y$ and there exists a subgroup $AN_1 = N_1 \rtimes A$ by Lemma 3.2. Hence, N_1 is normal in G. Therefore, $N_1 = N$ and by Lemma 2.1 (3), G is supersoluble, a contradiction. Thus, the assumption $A_p = 1$ is false and $A_p \neq 1$.

Assume that $B_p = 1$. Hence, $N = A_p \leq A$ and $N = A$ by Lemma 2.3 (2). Then $B \cap N = 1$ and B is maximal in G. By Lemma 3.2, B is seminormal in B^G. Since B is maximal in G, we have either $B^G = B$ or $B^G = G$. If $B^G = B$, then B is normal in G and $|G : B|$ is prime. If $B^G = G$, then B is seminormal in G. By [13, Lemma 1.4], $|G : B|$ is prime. Hence, $|N| = p$ and by Lemma 2.1 (3), G is supersoluble, a contradiction. Thus, the assumption $B_p = 1$ is false and $B_p \neq 1$.

Let Y_1 be a Hall p'-subgroup of Y. Then AY_1 is a subgroup of G and $Y_1 \leq N_G(A_p)$, because A_p is normal in AY_1. Since N is abelian, a Sylow p-subgroup Y_p of Y centralizes A_p. Because A_p is characteristic in A and A is normal in $N_G(A)$, we have A_p is normal in $N_G(A)$. Hence, A_p is normal in $G = N_G(A)Y = N_G(A)Y_pY_1$ and $A_p = N$. Because A is nilpotent and by Lemma 2.3 (2), it follows that $A = N$. Since B is supersoluble, we have B_p is normal in B. In this case, B_p is normal in $N = A$ and therefore is normal in G. Thus $B_p = N$ and $G = AB = NB = B$ is supersoluble, a contradiction.

5. If $A^G = G$, then by Lemma 3.2, A is seminormal in G. Then G is supersoluble by [13, Corollary 2.2]. Hence, $A^G < G$. By Dedekind's identity, $A^G = A(A^G \cap B)$. Since A is seminormal in A^G by Lemma 3.2 and $A^G \cap B$ is a normal siding subgroup of A^G, it follows that A^G is supersoluble by indution. Then by [13, Corollary 2.2], $G = A^GB$ is supersoluble.

□

In monograph [4, p. 149], it is presented the following definition: two subgroups A and B of a group G are said to be *mutually permutable* if $UB = BU$ and $AV = VA$ for all $U \leq A$ and $V \leq B$.

Since every normal subgroup and every subgroup of prime index are seminormal and therefore are propermutable in a group, the following corollary holds.

Corollary 4.1. *Let A and B be supersoluble subgroups of G and $G = AB$.*

1. *Suppose that A is nilpotent. Then G is supersoluble in each of the following cases:*

(1.1) *A and B are mutually permutable, see* [2, Theorem 3.2]*;*

(1.2) *A and B are seminormal in G, see* [13, Theorem 2.1]*;*

(1.3) *the indices of A and B in G are prime, see* [12, Theorem A]*;*

2. *If G' is nilpotent, then G is supersoluble in each of the following cases:*

(2.1) *A and B are normal in G, see* [3]*;*

(2.2) *A and B are mutually permutable, see* [2, Theorem 3.8]*;*

(2.3) *A and B are seminormal in G, see* [13, Theorem 2.2]*;*
(2.4) *the indices of A and B in G are prime, see* [12, Corollary 3.6].
3. *If B is normal and siding, then G is supersoluble in each of the following cases:*
(3.1) *A is normal in G and B is a soluble* T*-group, see* [11, Theorem 3]*;*
(3.2) *A is seminormal in G, see* [13, Corollary 2.2]*;*
(3.3) *the indices of A and B in G are prime, see* [12, Theorem B]*;*

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