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MINIMA OF FUNCTIONS ON (q_1, q_2)-QUASIMETRIC SPACES

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Abstract. Lower semicontinuous functions defined on a complete (q_1, q_2)-quasimetric spaces are considered. For these functions, minimum existence conditions are obtained.

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1 Introduction

Let X be a nonempty set, numbers $q_0 \geq 1$, $q_1 \geq 1$, $q_2 \geq 1$ be given.

Definition 1. A function $\rho : X \times X \to \mathbb{R}_+$ is called a (q_1, q_2)-quasimetric if the following properties hold:

- $\rho(x, y) = 0 \Leftrightarrow x = y \; \forall \; x, y \in X$ (the identity axiom);
- $\rho(x, z) \leq q_1\rho(x, y) + q_2\rho(y, z) \quad \forall x, y, z \in X$ (the (q_1, q_2)-generalized triangle inequality).

If ρ is a (q_1, q_2)-quasimetric, then the space (X, ρ) is called a (q_1, q_2)-quasimetric space. A (q_1, q_2)-quasimetric is called weakly symmetric if the following property holds:

- for every point $x \in X$, for every sequence $\{x_n\} \subset X$, if $\rho(x, x_i) \to 0$, then $\rho(x_i, x) \to 0$.

A (q_1, q_2)-quasimetric is called q_0-symmetric if the following property holds:

- $\rho(x, y) \leq q_0\rho(y, x) \; \forall \; x, y \in X$ (the q_0-symmetry axiom).

A $(1, 1)$-quasimetric space is called a quasimetric space. A quasimetric space is called a metric space if it is 1-symmetric.

The study of spaces endowed with a distance functions satisfying various properties goes back to M. Fréchet [12] and F. Hausdorff [14] and was continued in multiple papers (see, for example, [13, 15] as well as [6] and the references therein). The concept of (q_1, q_2)-quasimetric space was introduced and studied in the papers [4, 5]. In [6], a natural generalization of the (q_1, q_2)-quasimetric spaces – f-quasimetric spaces, were studied. In [4, 5], fixed point theorems and coincidence point theorems for mappings acting between (q_1, q_2)-quasimetric spaces were obtained. In this paper, we investigate the problem of the existence of points of minimum for functions defined on (q_1, q_2)-quasimetric spaces. This kind of propositions are used to derive sufficient conditions for existence of solutions to various abstract equations and inclusions (see, for example, Theorem 7 in [1]). The results on the existence of minima of specific functions and related results on solvability of certain types of equations and inclusions can be applied to various problems arising in optimization (see, for example, [2, 7]), set-valued analysis (see, for example, [17]) and control theory (see, for example, [11]). Moreover, they are closely related to covering mappings theory and coincidence points theorems, (see, for example, [3, 8, 9]).

Let us briefly describe the content of the paper. In Section 2, we recall some known topological properties of (q_1, q_2)-quasimetric spaces and propositions on the existence of minima for functions defined on metric and quasimetric spaces. In Section 3, we present a sufficient condition for the existence of points of minimum for functions defined on (q_1, q_2)-quasimetric spaces. In Section 4, we discuss the obtained result and compare it with some known analogous results.

2 Preliminaries

Let (X, ρ) be a (q_1, q_2)-quasimetric space. First, let us recall the topological properties of (q_1, q_2)-quasimetric spaces.

Given a point $x \in X$ and a number $r > 0$, denote

$$O(x, r) = \{y \in X : \rho(x, y) < r\}, \quad B(x, r) = \{y \in X : \rho(x, y) \leq r\}.$$

The set $O(x, r)$ is called an open ball centered at x with radius r, the set $B(x, r)$ is called a closed ball centered at x with radius r.

Define a topology τ on X as follows. The set $A \subset X$ is called open if for every point $a \in A$ there exists $r > 0$ such that $O_X(a, r) \subset A$. It is a straightforward task to ensure that the system τ of all such sets is a topology.

Note that unless the ball $O(x, r)$ is called "open", it may be not open in the sense of topology τ (see Example 3.4 in [5]). The same remark is valid for the closed ball.

The convergence of a sequence $\{x_n\} \subset X$ to a point $x \in X$ with respect to this topology is equivalent to the fulfilment of the equality $\lim\limits_{n\to\infty} \rho(x, x_n) = 0$. Moreover, it is obvious that $\lim\limits_{n\to\infty} \rho(x, x_n) = 0$ if and only if $\forall\, \varepsilon > 0\ \exists\, N : x_n \in O(x, \varepsilon)\ \forall\, n \geq N$.

Note that the first axiom of countability holds for τ (see Corollary 1.3 in [6]). Hence, a set $A \subset X$ is closed if and only if it coincides with the set of all limit points of A, and for every set $A \subset X$, its closure $\mathrm{cl}A$ coincides with the set of all limit points of A. The topology τ may not satisfy the T_2 axiom. Hence, the limit of a convergent sequence may be not unique.

A function $U : X \to \overline{\mathbb{R}}$ is called lower semicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$, inequality $f(x_0) \leq f(x) + \varepsilon$ holds for every point x in a neighbourhood of x_0. Hence, U is lower semicontinuous at a point $x_0 \in X$ if and only if

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 : \quad f(x_0) \leq f(x) + \varepsilon \quad \forall\, x \in O(x_0, \delta).$$

The function U is called lower semicontinuous if it is lower semicontinuous at every point $x \in X$. A function $U : X \to \overline{\mathbb{R}}$ is called proper if there exists $x \in X$ such that $U(x) < +\infty$.

Let us now recall metric properties of (q_1, q_2)-quasimetric space.

Definition 2. A sequence $\{x_i\} \subset X$ is called a Cauchy sequence if

$$\forall\, \varepsilon > 0 \quad \exists N \in \mathbb{N} : \quad \rho(x_i, x_{i+j}) < \varepsilon \quad \forall\, i, j \in \mathbb{N}, \quad i \geq N, \quad j \geq 1.$$

The space (X, ρ) is called complete if each Cauchy sequence in X has at least one limit.

Note that unlike metric spaces, a convergent sequence may be not fundamental in (q_1, q_2)-quasimetric spaces.

Let us recall some minimum existence theorems. Let (X, ρ) be a complete metric space, $U : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function bounded below by a given number γ, i.e. $U(x) \geq \gamma$ for all $x \in X$.

Theorem 2.1. (Theorem 3 in [1]) *Assume that a function U satisfies the Caristi-like condition with some $k > 0$, i.e.*

$$\forall\, x \in X : \quad \gamma < U(x) \quad \exists\, x' \in X \setminus \{x\} : \quad U(x') + k\rho(x, x') \le U(x). \tag{2.1}$$

Then for every point $x_0 \in X$ there exists a point $\bar{x} \in X$, such that the function U attains its minimum at $\bar{x}$ and

$$U(\bar{x}) = \gamma, \quad \rho(x_0, \bar{x}) \le \frac{U(x_0) - \gamma}{k}. \tag{2.2}$$

The term “Caristi-like condition” was introduced in [1]. In [10], a generalization of this concept was used to derive modifications of Ekeland’s variational principle.

Let us recall a similar result for quasimetrics from [16]. Let (X, ρ) be a complete metric space, $U : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function bounded below. Set

$$\gamma := \inf_{x \in X} U(x).$$

Theorem 2.2. (Theorem 2.5 in [16]) *Assume that a quasimetric ρ is lower semicontinuous in the second variable, i.e. for every $y \in X$ the function $x \mapsto \rho(y, x)$, $x \in X$, is lower semicontinuous. If*

$$\forall\, x \in X : \quad U(x) > \gamma \quad \exists x' \in X \setminus \{x\} : \quad U(x') + \rho(x, x') \le U(x),$$

then U has a point of minimum.

In the following section we derive an analogous proposition on the existence of minima for functions defined on (q_1, q_2)-quasimetric spaces.

3 Main result

Let $q_1 \ge 1$, $q_2 \ge 1$, $\gamma \in \mathbb{R}$, $k > 0$ be given, (X, ρ) be a complete (q_1, q_2)-quasimetric space, $U : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function such that

$$U(x) \ge \gamma \quad \forall\, x \in X.$$

If (X, ρ) is a compact space, then U attains its minimum. Below we provide sufficient conditions for the existence of minima of the function U without prior compactness assumption similar to those in Theorems 2.1 and 2.2.

Definition 3. We say that a function U satisfies the Caristi-like condition with the constant $k > 0$ if

$$\forall\, x \in X : \quad \gamma < U(x) \quad \exists\, x' \in X : \quad q_2 U(x') + k q_1 \rho(x, x') \le U(x) + (q_2 - 1)\gamma. \tag{3.1}$$

Obviously, if (X, ρ) is a metric space, then we can take $q_1 = q_2 = 1$. In this case, conditions (3.1) and (2.1) coincide. The same remark is valid if (X, ρ) is a quasimetric space. Taking $q_1 = q_2 = 1$ and $k = 1$, we observe that condition (3.1) coincides with the sufficient condition for minima existence in Theorem 2.2.

Theorem 3.1. *Assume that $q_2 > 1$ and a function $U : X \to \mathbb{R}_+$ satisfies the Caristi-like condition with the constant $k > 0$. Then for every point $x_0 \in X$, there exists a point $\bar{x} \in X$, such that the function U attains its minimum at $\bar{x}$ and*

$$U(\bar{x}) = \gamma, \qquad \bar{x} \in \mathrm{cl}B\left(x_0, \frac{U(x_0) - \gamma}{k}\right). \tag{3.2}$$

Proof. Without loss of generality we assume that $\gamma = 0$ and $k = 1$.

Take an arbitrary point $x_0 \in X$ such that $U(x_0) < +\infty$. Let us show that there exists a sequence $\{x_n\} \subset X$ such that

$$U(x_i) \leq \frac{U(x_0)}{q_2^i} \quad \forall\, i = 1, 2, ..., \tag{3.3}$$

$$B(x_i, U(x_i)) \subset B(x_{i-1}, U(x_{i-1})) \quad \forall\, i = 1, 2, ... \tag{3.4}$$

Applying the Caristi-like condition at the point $x = x_0$ we obtain a point $x_1 \in X$ such that

$$q_2 U(x_1) + q_1 \rho(x_0, x_1) \leq U(x_0).$$

This inequality implies that

$$U(x_1) \leq \frac{U(x_0)}{q_2}$$

and for every $x \in B(x_1, U(x_1))$ the relation

$$\rho(x_0, x) \leq q_1 \rho(x_0, x_1) + q_2 \rho(x_1, x) \leq (U(x_0) - q_2 U(x_1)) + q_2 U(x_1) = U(x_0)$$

holds. Hence, conditions (3.3) and (3.4) hold for $i = 1$.

Assume now that for some number $k \in \mathbb{N}$ conditions (3.3) and (3.4) hold for $i = k$. Applying the Caristi-like condition at the point $x = x_k$ we obtain a point $x_{k+1} \in X$ such that

$$q_2 U(x_{k+1}) + q_1 \rho(x_k, x_{k+1}) \leq U(x_k). \tag{3.5}$$

Inequalities (3.5) and (3.3) imply that

$$U(x_{k+1}) \leq \frac{U(x_k)}{q_2} \leq \frac{U(x_0)}{q_2^{k+1}}.$$

Inequality (3.5) implies that for every $x \in B(x_{k+1}, U(x_{k+1}))$ the relation

$$\rho(x_k, x) \leq q_1 \rho(x_k, x_{k+1}) + q_2 \rho(x_{k+1}, x) \leq (U(x_k) - q_2 U(x_{k+1})) + q_2 U(x_{k+1}) = U(x_k)$$

holds. Hence, conditions (3.3) and (3.4) hold for $i = k + 1$. The inductive construction of a sequence $\{x_i\}$ satisfying (3.3) and (3.4) is complete.

Since $q_2 > 1$, inequality (3.3) implies that $U(x_i) \to 0$ as $i \to \infty$. Inclusions (3.4) imply that

$$B(x_{i+j}, U(x_{i+j})) \subset B(x_i, U(x_i)) \quad \forall\, i, j \in \mathbb{N}.$$

Therefore,

$$x_{i+j} \in B(x_i, U(x_i)) \quad \forall\, i, j \in \mathbb{N}.$$

Since $U(x_i) \to 0$ as $i \to \infty$, the sequence $\{x_i\}$ is a Cauchy one. Hence, the completeness of (X, ρ) implies that there exists $\bar{x} \in X$ such that $x_i \to \bar{x}$ as $i \to \infty$.

So, U is lower semicontinuous, $x_i \to \bar{x}$ as $i \to \infty$, $U(x_i) \to 0$ as $i \to \infty$ and $U(x) \geq 0$ for every $x \in X$. Therefore, $U(\bar{x}) = 0$. Moreover, $\bar{x} \in \mathrm{cl}B(x_0, U(x_0))$, since by virtue of (3.4) we have $x_i \in B(x_0, U(x_0))$ for every $i \in \mathbb{N}$. $\square$

4 Discussion of main results

First, let us discuss the proposition of Theorem 3.1.

Inclusion in (3.2) cannot be replaced by the inclusion

$$\bar{x} \in B\left(x_0, \frac{U(x_0) - \gamma}{k}\right). \tag{4.1}$$

Consider an appropriate example.

Example 1. Let $X = \mathbb{R}$, $q \geq 1$,

$$\rho(x, y) = \begin{cases} |x - y|, & \text{if } (x - y) \in \mathbb{Q}; \\ q|x - y|, & \text{if } (x - y) \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

It is a straightforward task to ensure that ρ is (q, q)-quasimetric, the space (X, ρ) is complete, the topology of (X, ρ) coincides with the standard topology of the real line.

Consider the function

$$U : X \to \overline{\mathbb{R}}, \quad U(x) := |x|, \quad x \in X.$$

It satisfies the Caristi-like condition with $k = 1$ and $\gamma = 0$. The only point of minimum for U is the point $\bar{x} = 0$. However, if $q_0 > 1$, then inclusion (4.1) fails to hold for $x_0 := \sqrt{2}$, since

$$\rho(x_0, \bar{x}) = q_0\sqrt{2} > \sqrt{2} = \frac{U(x_0) - \gamma}{k}.$$

If the space (X, ρ) is weakly symmetric, then inclusion in (3.2) implies that

$$\rho(x_0, \bar{x}) \leq q_1 \frac{U(x_0) - \gamma}{k}.$$

This fact is a corollary of the following assertion.

Proposition 4.1. *If a (q_1, q_2)-quasimetric space (X, ρ) is weakly symmetric, then*

$$\mathrm{cl} B(x_0, r) \subset B(x_0, q_1 r) \quad \forall\, x_0 \in X, \quad \forall\, r > 0.$$

Proof. Take an arbitrary sequence $\{x_j\} \subset B(x_0, r)$ convergent to a point $\bar{x}$ as $j \to \infty$. We have

$$\rho(x_0, \bar{x}) \leq q_1 \rho(x_0, x_j) + q_2 \rho(x_j, \bar{x}) \leq q_1 r + q_2 \rho(x_j, \bar{x}) \quad \forall\, j.$$

Since $x_j \to \bar{x}$ as $j \to \infty$ and the space (X, ρ) is weakly symmetric, we have $\rho(x_j, \bar{x}) \to 0$ as $j \to \infty$. Therefore, passing to the limit as $j \to \infty$ in the inequality above, we obtain $\rho(x_0, \bar{x}) \leq q_1 r$. □

Assume now that the function $x \mapsto \rho(y, x)$, $x \in X$, is lower semicontinuous for every $y \in X$. In this case, the inclusion in (3.2) is equivalent to inclusion (4.1). This fact is a corollary of the following assertion.

Proposition 4.2. *Let (X, ρ) be a (q_1, q_2)-quasimetric space. If the function $x \mapsto \rho(y, x)$, $x \in X$, is lower semicontinuous for every $y \in X$, then*

$$\mathrm{cl} B(x_0, r) = B(x_0, r) \quad \forall\, x_0 \in X, \quad \forall\, r > 0.$$

Proof. Take an arbitrary point $x_0 \in X$, a number $r > 0$ and a sequence $\{x_j\} \subset B(x_0, r)$ convergent to a point $\bar{x}$ as $j \to \infty$. Since the function $x \mapsto \rho(x_0, x)$ is lower semincontinuous and $\rho(x_0, x_j) \leq r$, we have $\rho(x_0, \bar{x}) \leq r$. □

Let us now show that the assumption $q_2 > 1$ is essential in Theorem 3.1.

Example 2. Let $X = \{(n_1, n_2) : n_1, n_2 \in \mathbb{N}\}$. Define the function $\rho : X \times X \to \mathbb{R}_+$ as follows:

$$\rho((n_1,n_2),(m_1,m_2)) = \begin{cases} \dfrac{1}{m_2}, & \text{if} \quad n_1 > m_1; \\ 1, & \text{if} \quad n_1 < m_1; \\ 1, & \text{if} \quad n_1 = m_1, n_2 > m_2; \\ \dfrac{m_2 - n_2}{n_2 m_2} \cdot \dfrac{1}{n_1(n_1+1)}, & \text{if} \quad n_1 = m_1, n_2 < m_2; \\ 0, & \text{if} \quad n_1 = m_1, n_2 = m_2. \end{cases}$$

I. Let us show that ρ is a quasimetric. Obviously it suffices to verify the triangle inequality

$$\rho(a,c) \le \rho(a,b) + \rho(b,c) \quad \forall\, a = (n_1,n_2), \quad b = (m_1,m_2), \quad c = (k_1,k_2). \tag{4.2}$$

If at least two out of three points a, b or c coincide, or $\rho(a,b) = 1$, or $\rho(b,c) = 1$, then the triangle inequality holds. Thus, we assume that a, b and c are pairwise distinct, $\rho(a,b) < 1$ and $\rho(b,c) < 1$. Then the definition of ρ implies that one of the following relations holds:

(i) $n_1 > m_1$ and $m_1 > k_1$;
(ii) $n_1 > m_1$, $m_1 = k_1$ and $m_2 < k_2$;
(iii) $n_1 = m_1$, $n_2 < m_2$ and $m_1 > k_1$;
(iv) $n_1 = m_1$, $n_2 < m_2$, $m_1 = k_1$ and $m_2 < k_2$.

If (i) holds, then $n_1 > m_1$ and $n_1 > k_1$. Hence, $\rho(a,c) = \rho(b,c) = \dfrac{1}{k_2}$, which implies (4.2). If (ii) holds, then $n_1 > k_1$ and $m_2 < k_2$. Thus, $\rho(a,c) = \dfrac{1}{k_2} < \dfrac{1}{m_2} = \rho(a,b)$, which implies (4.2). If (iii) holds, then $n_1 > k_1$ and $m_1 > k_1$. Thus, $\rho(a,c) = \rho(b,c) = \dfrac{1}{k_2}$, which implies (4.2). If (iv) holds, then $n_1 = k_1$ and $n_2 < k_2$. Hence,

$$\rho(a,c) = \frac{k_2 - n_2}{k_2 n_2} \frac{1}{n_1(n_1+1)} = \left(\frac{m_2 - n_2}{m_2 n_2} + \frac{k_2 - m_2}{k_2 m_2} \right) \frac{1}{n_1(n_1+1)} = \rho(a,b) + \rho(b,c),$$

which completes the proof of inequality (4.2).

II. Let us show that (X, ρ) is complete. Let $\{x_i\} \subset X$, $x_i = (n_1^i, n_2^i)$, $i = 1, 2, \ldots$ be a Cauchy sequence. Then for $\varepsilon = 1$ there exists $N \in \mathbb{N}$ such that $\rho(x_i, x_j) < 1\, \forall j > i > N$. Therefore, $n_1^i \ge n_1^{i+1}\, \forall\, i > N$ and hence for $i > N$ the sequence $\{n_1^i\}$ decreases. Therefore, the sequence $\{n_1^i\}$ is stationary, i.e. there exist numbers N and n_1 such that $n_1^i = n_1\, \forall i \ge N$.

The sequence $\{n_2^i\}$ increases when i is sufficiently large, i.e. there exists a number N such that if $j \ge i \ge N$, then $n_2^i \le n_2^{i+1}$. Indeed, since the sequence $\{n_1^i\}$ is stationary and $\rho(x_i, x_j) < 1$ for sufficiently large i and j such that $j \ge i$, it follows that $n_2^i \le n_2^j$ by definition of ρ.

Let us show that the sequence $\{x_i\}$ converges. Obviously, if $\{x_i\}$ is stationary, then it converges. Assume that $\{x_i\}$ is not stationary. Since $\{n_1^i\}$ is stationary and $\{n_2^i\}$ increases for sufficiently large i, then $n_2^i \to \infty$ as $i \to \infty$. Therefore, for an arbitrary point $(m_1, m_2) \in X$ such that $m_1 > n_1$, for sufficiently large i we have

$$\rho((m_1,m_2),(n_1^i,n_2^i)) = \frac{1}{n_2^i} \to 0 \quad \text{as} \quad i \to \infty.$$

Therefore, $x_i \to (m_1, m_2)$ as $i \to \infty$. Hence, (X, ρ) is complete.

III. Consider the function $U : X \to \mathbb{R}$,

$$U(n_1, n_2) = \frac{1}{n_1+1} + \frac{1}{n_1(n_1+1)n_2}, \quad (n_1, n_2) \in X. \tag{4.3}$$

It is obvious that $\inf\limits_{x\in X} U(x) = 0$ and U does not attain its minimum. The Caristi-like condition with $k = 1$ and $\gamma = 0$ holds for the function U, since for every $x = (n_1, n_2) \in X$, the inequality in (3.1) holds with $x' = (n_1, n_2 + 1)$.

IV. Let us show that every point $\hat{x} \in X$ is a point of strict local minimum of the function U, i.e.

$$\forall\, \hat{x} \in X \quad \exists\, r > 0 : \quad U(x) > U(\hat{x}) \quad \forall\, x \in B(\hat{x}, r) \setminus \{\hat{x}\}. \tag{4.4}$$

Take an arbitrary point $\hat{x} = (n_1, n_2)$. Let us show that $U(x) > U(\hat{x})$ for any $x \in B(\hat{x}, r)$, $x \neq \hat{x}$, where $r \in \left(0, \dfrac{1}{n_1 n_2(1+n_1)(1+n_2)}\right)$.

Since $\rho(\hat{x}, x) \le r < 1$, the definition of ρ implies that either $n_1 > m_1$ or $n_1 = m_1$ and $n_2 < m_2$. The second case is impossible, since otherwise

$$\rho(\hat{x}, x) = \rho((n_1, n_2), (m_1, m_2)) = \frac{m_2 - n_2}{n_2 m_2} \cdot \frac{1}{n_1(n_1+1)} =$$

$$= \left(\frac{1}{n_2} - \frac{1}{m_2}\right)\frac{1}{n_1(1+n_1)} \ge \left(\frac{1}{n_2} - \frac{1}{n_2+1}\right)\frac{1}{n_1(1+n_1)} = \frac{1}{n_1 n_2(1+n_1)(1+n_2)} > r$$

which contradicts the choice of r. Hence, $n_1 > m_1$. Therefore,

$$U(\hat{x}) = \frac{1}{1+n_1} + \frac{1}{n_1(n_1+1)n_2} \le \frac{1}{1+n_1} + \frac{1}{n_1(n_1+1)} =$$

$$= \frac{1}{n_1} \le \frac{1}{1+m_1} < \frac{1}{1+m_1} + \frac{1}{m_1(m_1+1)m_2} = U(x).$$

V. Let us summarize the above. In **I** and **II**, it is shown that (X, ρ) is a complete $(1,1)$-quasimetric space. In **III**, it is shown that the function U satisfies the Caristi-like condition with $k = 1$ and $\gamma = 0$. It follows from **IV** that U is lower semicontinuous. Hence, all the assumptions of Theorem 3.1 hold except the assumption $q_2 > 1$.

Note that Example 2 also shows that the assumption of lower semicontinuity of quasimetric ρ in the second argument is essential in Theorem 2.2. In this example all the assumptions of Theorem 2.2 hold except for the lower semicontinuity of ρ in the second argument. Indeed, the function $\rho((2,2), \cdot)$ in Example 2 is not lower semicontinuous, since $(1,n) \to (2,1)$, $\rho((2,2),(1,n)) \to 0$ as $n \to \infty$ and $\rho((2,2),(2,1)) > 0$.

Let us now compare Theorem 3.1 with the analogous results from [1] and [16]. It is obvious that Theorem 2.1 (Theorem 3 in [1]) does not follow from Theorem 3.1 because of the assumption $q_2 > 1$ in Theorem 3.1 which never holds in metric space. Conversely, Theorem 3.1 does not follow from Theorem 2.1, since Theorem 2.1 is not applicable to (q_1, q_2)-quasimetric spaces when $q_2 > 1$. The same remark is valid for Theorem 2.2 (Theorem 2.5 in [16]).

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