ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2019, Volume 10, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

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The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 515
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 10, Number 2 (2019), 75 – 83

COMMON FIXED POINT THEOREMS FOR TWO PAIRS OF SELF-MAPPINGS IN COMPLEX-VALUED METRIC SPACES

F. Rouzkard

Communicated by N.A. Bokayev

Key words: common fixed point, contractive type mapping, complex valued metric space, complex coefficient.

AMS Mathematics Subject Classification: 47H10, 54H25.

Abstract. In this paper, we consider complex-valued metric space and prove some coincidence point and common fixed point theorems involving two pairs of self-mappings satisfying the contraction condition with complex coefficients in these spaces. In this paper, we generalize, improve and simplify the proofs of some existing results.

DOI: https://doi.org/10.32523/2077-9879-2019-10-2-75-83

1 Introduction with preliminaries

The concept of complex-valued metric spaces was introduced and studied by Azam et al. [1] and Rouzkard et al. [5]. Naturally, this new idea can be utilized to define complex-valued normed spaces and complex-valued inner product spaces, which in turn, offer a wide scope for further investigations. Though complex-valued metric spaces form a special class of cone metric spaces, they are intended to define rational expressions which are not meaningful in cone metric spaces, and thus many results of analysis cannot be generalized to cone metric spaces. Indeed, the definition of a cone metric space is based on the underlying Banach space which is not a division ring. However, in complex-valued metric spaces we can study improvements of a lot of results of analysis involving divisions.

In this paper, we prove coincidence points and common fixed points theorems involving two pairs of weakly compatible mappings satisfying certain inequalities in a complex-valued metric space.

To begin with, we collect some definitions and basic facts on complex-valued metric spaces, which will be needed in the sequel.

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define the partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

We denote by $\mathbb{C}_+$ the following set

$$\mathbb{C}_+ = \{z \in \mathbb{C} : 0 \precsim z\}.$$

Definition 1. [5]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies the following conditions:
(d_1) $0 \precsim d(x, y)$ $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ $x, y \in X$;
(d_3) $d(x, y) \precsim d(x, z) + d(z, y)$ $x, y, z \in X$.
Then d is called a complex-valued metric on X, and (X, d) is called a complex-valued metric space.

Example 1. Let $X = X_1 \cup X_2$ where

$$X_1 = \{z \in \mathbb{C} : Re(z) \geq 0 \text{ and } Im(z) = 0\} \text{ and } X_2 = \{z \in \mathbb{C} : Re(z) = 0 \text{ and } Im(z) \geq 0\}.$$

Define $d : X \times X \to \mathbb{C}$, as follows

$$d(z_1, z_2) = \begin{cases} \frac{2}{3}|x_1 - x_2| + \frac{i}{2}|x_1 - x_2| & \text{if } z_1, z_2 \in X_1; \\ \frac{1}{2}|y_1 - y_2| + \frac{i}{3}|y_1 - y_2| & \text{if } z_1, z_2 \in X_2; \\ (\frac{2}{3}x_1 + \frac{1}{2}y_2) + i(\frac{1}{2}x_1 + \frac{1}{3}y_2) & \text{if } z_1 \in X_1, z_2 \in X_2; \\ (\frac{1}{2}y_1 + \frac{2}{3}x_2) + i(\frac{1}{3}y_1 + \frac{1}{2}x_2) & \text{if } z_1 \in X_2, z_2 \in X_1; \end{cases}$$

where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in X$. Then (X, d) is a complex-valued metric space.

Definition 2. Let (X, d) be a complex-valued metric space and $B \subseteq X$. We recall the following definitions:

(i) $b \in B$ is called an interior point of the set B whenever there is $0 \prec r \in \mathbb{C}$ such that

$$N(b, r) \subseteq B,$$

where $N(b, r) = \{y \in X : d(b, y) \prec r\}$.

(ii) A point $x \in X$ is called a limit point of B whenever for every $0 \prec r \in \mathbb{C}$,

$$N(x, r) \cap (B \backslash X) \neq \emptyset.$$

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.
The family $F = \{N(x, r) : x \in X, 0 \prec r\}$ is a sub-basis for a topology on X. We denote this complex topology by τ_c. Note that, the topology τ_c is Hausdorff.

Definition 3. Let (X, d) be a complex-valued metric space and $\{x_n\}_{n\geq 1}$ be a sequence in X and $x \in X$. We say that

(i) the sequence$\{x_n\}_{n\geq 1}$ converges to x if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$,

(ii) the sequence$\{x_n\}_{n\geq 1}$ is a Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and $m \in \mathbb{N}$, $d(x_n, x_{n+m}) \prec c$,

(iii) the metric space (X, d) is a complete complex-valued metric space if every Cauchy sequence is convergent.

Definition 4. [2] Two families of self-mappings $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ $(i.e\ \{T_i\}, \{S_i\} : X \to X)$ are said to be pairwise commuting if:

(i) $T_iT_j = T_jT_i$, $i, j \in \{1, 2, ...m\}$.

(ii) $S_iS_j = S_jS_i$, $i, j \in \{1, 2, ...n\}$.

(iii) $T_iS_j = S_jT_i$, $i \in \{1, 2, ...m\}$, $j \in \{1, 2, ...n\}$.

Definition 5. [3] Let S and I be self-mappings of a set X. If $w = Sx = Ix$ for some $x \in X$, then x is called a point of coincidence of S and I, and w is called a point of coincidence of S and I.

Definition 6. [4] Let S and T be two self-mappings defined on a set X. S and T are said to be weakly compatible if they commute at their coincidence points.

In [1], Azam et al. established the following two lemmas.

Lemma 1.1. *[1] Let (X, d) be a complex-valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.2. *[1] Let (X, d) be a complex-valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $m, n \to \infty$.*

2 Main result

In this section, we give the unique point of coincidence and unique common fixed point theorems in complex-valued metric spaces.

Our first main result is the following theorem.

Theorem 2.1. *Let S, T, I and J be self-mappings defined on a complex-valued metric space (X, d) satisfying $TX \subseteq IX, SX \subseteq JX$ and*

$$\lambda d(Sx, Ty) \precsim Ad(Ix, Jy) + B\frac{d(Jy, Sx)d(Ix, Ty)}{1 + d(Ix, Jy) + d(Ix, Sx) + d(Jy, Ty)} \tag{2.1}$$

for all $x, y \in X$, where $\lambda, A, B \in \mathbb{C}_+$ and $0 \prec A + B \prec \lambda$. If one of SX, TX, IX or JX is a complete subspace of X, then

(a) both pairs $\{S, I\}$ and $\{T, J\}$ have a unique point of coincidence in X,

(b) if both pairs $\{S, I\}$ and $\{T, J\}$ are weakly compatible, then S, T, I and J have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $SX \subseteq JX$, we find a point x_1 in X such that $Sx_0 = Jx_1$. Also, since $TX \subseteq IX$, we choose a point x_2 with $Tx_1 = Ix_2$. Thus, in general, for the point x_{2n-2} one can find a point x_{2n-1} such that $Sx_{2n-2} = Jx_{2n-1}$ and then a point x_{2n} with $Tx_{2n-1} = Ix_{2n}$ for $n = 1, 2, \ldots$. Repeating such arguments one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$y_{2n-1} = Sx_{2n-2} = Jx_{2n-1}, \ \ y_{2n} = Tx_{2n-1} = Ix_{2n}, \quad n = 1, 2, \ldots$$

Using inequality (2.1), we have

$$\begin{aligned} \lambda d(Sx_{2n}, Tx_{2n+1}) \ &\precsim \ Ad(Ix_{2n}, Jx_{2n+1}) \\ &\quad + B\frac{d(Jx_{2n+1}, Sx_{2n})d(Ix_{2n}, Tx_{2n+1})}{1 + d(Ix_{2n}, Jx_{2n+1}) + d(Ix_{2n}, Sx_{2n}) + d(Jx_{2n+1}, Tx_{2n+1})} \end{aligned}$$

or

$$\begin{aligned} \lambda d(y_{2n+1}, y_{2n+2}) \ &\precsim \ Ad(y_{2n}, y_{2n+1}) \\ &\quad + B\frac{d(y_{2n+1}, y_{2n+1})d(y_{2n}, y_{2n+2})}{1 + d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}, \end{aligned}$$

so that

$$|\lambda||d(y_{2n+1}, y_{2n+2})| \leq |A||d(y_{2n}, y_{2n+1})|,$$

therefore

$$|d(y_{2n+1}, y_{2n+2})| \leq |\frac{A}{\lambda}||d(y_{2n}, y_{2n+1})|.$$

We rewrite this in the form

$$|d(y_{2n+1}, y_{2n+2})| \leq h_1|d(y_{2n}, y_{2n+1})|, \tag{2.2}$$

where $h_1 = |\frac{A}{\lambda}|$.

Since $\lambda, A \in \mathbb{C}_+$ and $0 \prec A \prec \lambda$ then $h_1 = |\frac{A}{\lambda}| < 1$.

Again, using inequality (2.1),

$$\begin{aligned} \lambda d(Sx_{2n}, Tx_{2n-1}) \ &\precsim \ Ad(Ix_{2n}, Jx_{2n-1}) \\ &\quad + B\frac{d(Jx_{2n-1}, Sx_{2n})d(Ix_{2n}, Tx_{2n-1})}{1 + d(Ix_{2n}, Jx_{2n-1}) + d(Ix_{2n}, Sx_{2n}) + d(Jx_{2n-1}, Tx_{2n-1})}, \end{aligned}$$

or

$$\begin{aligned} \lambda d(y_{2n+1}, y_{2n}) \ &\precsim \ Ad(y_{2n}, y_{2n-1}) \\ &\quad + B\frac{d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})}{1 + d(y_{2n}, y_{2n-1}) + d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})}, \end{aligned}$$

therefore,

$$|d(y_{2n+1}, y_{2n})| \leq h_1|d(y_{2n}, y_{2n-1})|, \tag{2.3}$$

where $h_1 = |\frac{A}{\lambda}|$.

Combining (2.2) and (2.3), we have

$$|d(y_{2n+1}, y_{2n+2})| \leq h|d(y_{2n}, y_{2n-1})|,$$

where $h = h_1^2$.

Continuing this process, we get

$$|d(y_{2n+1}, y_{2n+2})| \leq h_1|d(y_1, y_2)|. \tag{2.4}$$

By using inequality (2.1), we have

$$|d(y_{2n+3}, y_{2n+2})| \leq |\frac{A}{\lambda}||d(y_{2n+2}, y_{2n+1})| = h_1|d(y_{2n+2}, y_{2n+1})|. \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$|d(y_{2n+2}, y_{2n+3})| \leq h_1^{2n+1}|d(y_1, y_2)|. \tag{2.6}$$

From (2.4) and (2.6), we get

$$|d(y_n, y_{n+1})| \leq \frac{\max\{1, h_1\}}{h_1^2} h_1^n|d(y_1, y_2)|, \text{for } \ n = 2, 3, \ldots$$

Since $0 < h_1 < 1$, for $m, n(m > n)$, we have

$$|d(y_n, y_m)| \ \leq \ [\frac{{h_1}^n}{{h_1}^2(1-h_1)}]\max\{1, h_1\}|d(y_1, y_2)| \to 0 \quad \text{as} \quad m, n \to \infty.$$

In view of Lemma 1.2, the sequence $\{y_n\}$ is a Cauchy sequence in (X, d). Now suppose IX is a complete subspace of X, then the subsequence $y_{2n} = Tx_{2n-1} = Ix_{2n}$ converges to some u in IX. That is,

$$y_{2n} = Ix_{2n} = Tx_{2n-1} \to u \quad \text{as} \quad n \to \infty. \tag{2.7}$$

As $\{y_n\}$ is a Cauchy sequence which contains a convergent subsequence $\{y_{2n}\}$, we can find $v \in X$ such that

$$Iv = u. \tag{2.8}$$

We claim that $Sv = u$. Using inequalities (2.1) and (2.8), we have

$$\begin{aligned} \lambda d(Sv, y_{2n}) = \lambda d(Sv, Tx_{2n-1}) \ \precsim \ & Ad(Iv, Jx_{2n-1}) \\ & + B\frac{d(Jx_{2n-1}, Sv)d(Iv, Tx_{2n-1})}{1 + d(Iv, Jx_{2n-1}) + d(Iv, Sv) + d(Jx_{2n-1}, Tx_{2n-1})} \\ & = Ad(u, y_{2n-1}) + B\frac{d(y_{2n-1}, Sv)d(u, y_{2n})}{1 + d(u, y_{2n-1}) + d(u, Sv) + d(y_{2n-1}, y_{2n})}. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, using (2.7), we have

$$\lambda d(Sv, u) \precsim 0.$$

since $0 \prec \lambda$, this implies that $d(Sv, u) = 0$, that is,

$$Sv = u. \tag{2.9}$$

Now, combining (2.8) and (2.9), we have

$$Iv = Sv = u,$$

that is, u is a point of coincidence of I and S.

Since $u = Sv \in SX \subseteq JX$, there exists $w \in X$ such that

$$u = Jw. \tag{2.10}$$

We claim that $Tw = u$. Using inequality (2.1), we have

$$\lambda d(u, Tw) = \lambda d(Sv, Tw) \precsim Ad(Iv, Jw) + B\frac{d(Jw, Sv)d(Iv, Tw)}{1 + d(Iv, Jw) + d(Iv, Sv) + d(Jw, Tw)},$$

or

$$\lambda d(u, Tw) \precsim 0,$$

which, by using $0 \prec \lambda$, implies that $d(u, Tw) = 0$, that is

$$u = Tw. \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$u = Jw = Tw,$$

that is, u is a point of coincidence of J and T.

Now, suppose that u' is another point of coincidence of I and S, that is,

$$u' = Iv' = Sv',$$

for some $v' \in X$. Using inequality (2.1), we have

$$\lambda d(u', u) = \lambda d(Sv', Tw) \quad \precsim \quad Ad(Iv', Jw) + B\frac{d(Jw, Sv')d(Iv', Tw)}{1 + d(Iv', Jw) + d(Iv', Sv') + d(Jw, Tw)},$$

which implies (by using $0 \prec \lambda$) that $d(u', u) = 0$, that is, $u' = u$. Therefore, we proved that u is a unique point of coincidence of $\{I, S\}$ and $\{J, T\}$.

Now, we prove that S, T, I and J have a unique common fixed point.

Since $\{I, S\}$ and $\{J, T\}$ are weakly compatible, and $u = Iv = Sv = Jw = Tw$, we can write

$$Su = S(Iv) = I(Sv) = Iu = w_1 \quad \text{(say)}$$

and

$$Tu = T(Jw) = J(Tw) = Ju = w_2 \quad \text{(say)}.$$

By using inequality (2.1), we get

$$\lambda d(w_1, w_2) = \lambda d(Su, Tu) \quad \precsim \quad Ad(Iu, Ju) + B\frac{d(Ju, Su)d(Iu, Tu)}{1 + d(Iu, Ju) + d(Iu, Su) + d(Ju, Tu)},$$

which implies (by using $0 \prec \lambda$) that $w_1 = w_2$, that is,

$$Su = Iu = Tu = Ju,$$

which by using inequality (2.1) implies that

$$\lambda d(Sv,Tu) \precsim Ad(Iv,Ju)+B\frac{d(Ju,Sv)d(Iv,Tu)}{1+d(Iv,Ju)+d(Iv,Sv)+d(Ju,Tu)}.$$

Hence, we deduce (by using $0 \prec \lambda$) that $Sv = Tu$, that is, $u = Tu$. This implies that

$$u = Su = Iu = Tu = Ju.$$

So, u is a unique common fixed point of S, I, J and T. The proofs for the cases in which SX, JX and TX are complete are similar, and are omitted. □

Putting $I = J = I_X$, where I_X is the identity mapping from X into X in Theorem 2.1, we get the following corollary.

Corollary 2.1. *Let S, T be self-mappings defined on a complex-valued metric space (X, d) satisfying*

$$\lambda d(Sx,Ty) \precsim Ad(x,y)+B\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)+d(x,Sx)+d(y,Ty)} \tag{2.12}$$

for all $x, y \in X$, where $\lambda, A, B \in \mathbb{C}_+$ and $0 \prec A + B \prec \lambda$. If one of SX or TX is a complete subspace of X, then S and T have a unique common fixed point in X.

Corollary 2.2. *Let $\{T_i\}_1^m$,$\{J_i\}_1^p$ and $\{S_i\}_1^l$,$\{I_i\}_1^n$ be two finite pairwise commuting families of self-mappings defined on a complex-valued metric space (X, d) such that the mappings S,T,I and J (with $T = T_1T_2...T_m$, $J = J_1J_2...J_p$, $I = I_1I_2...I_n$ and $S = S_1S_2...S_l$) satisfy $TX \subset IX$, $SX \subset JX$ and inequality (2.1). If one of TX, SX, IX or JX is a complete subspace of X, then the component maps of the two families $\{T_i\}_1^m$,$\{J_i\}_1^p$ and $\{S_i\}_1^l$,$\{I_i\}_1^n$ have a unique common fixed point.*

Proof. Appealing to the componentwise commutativity of various pairs, one immediately concludes that $SI = IS$ and $TJ = JT$ and, hence, obviously both pairs (S, I) and (T, J) are weakly compatible. Note that all conditions of Theorem (2.1) (for mappings S, T, I and J) are satisfied ensuring the existence of a unique common fixed point u in X, i.e. $Su = Tu = Iu = Ju = u$. We are required to show that u is a common fixed point of all the component maps of the families. For this, consider

$$\begin{aligned} S(S_ku) &= ((S_1S_2...S_l)S_k)u = (S_1S_2...S_{l-1})((S_lS_k)u) \\ &= (S_1...S_{l-2})(S_{l-1}S_k(S_lu)) = (S_1...S_{l-2})(S_kS_{l-1}(S_lu)) = ... \\ &= S_1S_k(S_2S_3S_4...S_lu) = S_kS_1(S_2S_3S_4...S_lu) = S_k(Su) = S_ku. \end{aligned}$$

Similarly one can show that

$$\begin{gathered} T_ku = T_kJu = JT_ku, T_ku = T_kTu = TT_ku, \\ J_ku = TJ_ku = JJ_ku, S_ku = IS_ku = SS_ku, \\ I_ku = II_ku = SI_ku, T_ku = TT_ku = JT_ku, \end{gathered}$$

which implies that (for every k) S_ku, T_ku, I_ku and J_ku are other fixed points of S, T, I and J.

By using the uniqueness of a common fixed point for S, T, I and J, we can write $S_ku = T_ku = I_ku = J_ku = u$ (for every k) which shows that u is a common fixed point of the families $\{T_i\}_1^m$, $\{S_i\}_1^l$, $\{I_i\}_1^p$ and $\{J_i\}_1^n$. This completes the proof of the theorem. □

Theorem 2.2. *Let S, T, I and J be self-mappings defined on complex-valued metric space (X, d) satisfying $TX \subseteq IX, SX \subseteq JX$ and*

$$\lambda d(Sx, Ty) \precsim A \frac{d(Ix, Sx)d(Jy, Ty)}{1 + d(Ix, Jy) + d(Ix, Sx) + d(Jy, Ty)} \tag{2.13}$$

for all $x, y \in X$, where $\lambda, A \in \mathbb{C}_+$ and $0 \prec A \prec \lambda$. If one of SX, TX, IX or JX is a complete subspace of X, then

(a) both pairs $\{S, I\}$ and $\{T, J\}$ have a unique point of coincidence in X,

(b) if both pairs $\{S, I\}$ and $\{T, J\}$ are weakly compatible, then S, T, I and J have a unique common fixed point in X.

Proof. The proof of this theorem is identical to that of Theorem 2.1. □

Corollary 2.3. *Let $\{T_i\}_1^m, \{J_i\}_1^p$ and $\{S_i\}_1^l, \{I_i\}_1^n$ be two pairwise commuting families of self-mappings defined on a complex-valued metric space (X, d) such that the mappings S, T, I and J (with $T = T_1T_2...T_m$, $J = J_1J_2...J_p$, $I = I_1I_2...I_n$ and $S = S_1S_2...S_l$) satisfy $TX \subset IX$, $SX \subset JX$ and inequality* (2.13). *If one of TX, SX, IX or JX is a complete subspace of X, then the component maps of the two families $\{T_i\}_1^m, \{J_i\}_1^p$ and $\{S_i\}_1^l, \{I_i\}_1^n$ have a unique common fixed point.*

Proof. The proof of this Corollary is identical to that of Corollary 2.2. □

Acknowledgments

The author is very thankful for many helpful suggestions and corrections made by the referees who reviewed this paper.

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Fayyaz Rouzkard
Department of Mathematics
Farhangian University
Tehran, Iran.
E-mails: fayyazrouzkard@gmail.com and fayyaz_rouzkard@yahoo.com

Received: 08.07.2017
Revised: 24.07.2018