ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2019, Volume 10, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

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EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 10, Number 2 (2019), 65 – 74

MAXIMAL REGULARITY ESTIMATES FOR HIGHER ORDER DIFFERENTIAL EQUATIONS WITH FLUCTUATING COEFFICIENTS

K.N. Ospanov, Zh.B. Yeskabylova, D.R. Beisenova

Communicated by M. Otelbaev

Key words: differential equation, oscillating coefficient, well-posedness, maximal regularity estimate.

AMS Mathematics Subject Classification: 34A30, 34B40, 34C11

Abstract. We give the well-posedness conditions in $L_2(-\infty,+\infty)$ for the following differential equation

$$-y''' + p(x)\,y' + q(x)\,y = f(x),$$

where p and q are continuously differentiable and continuous functions, respectively, and $f \in L_2(R)$. Moreover, we prove for the solution y of this equation the following maximal regularity estimate:

$$\|y'''\|_2 + \|py'\|_2 + \|qy\|_2 \le C\|f\|_2$$

(here $\|\cdot\|_2$ is the norm in $L_2(-\infty,+\infty)$). We assume that the intermediate coefficient p is fast oscillating and not controlled by the coefficient q. The sufficient conditions obtained by us are close to necessary ones. We give similar results for the fourth-order differential equation with singular intermediate coefficients.

DOI: https://doi.org/10.32523/2077-9879-2019-10-2-65-74

1 Introduction

In this paper we consider the solvability and smoothness properties for a solution of the following linear third-order differential equation

$$Ly = -y''' + p(x)\,y' + q(x)\,y = f(x) \tag{1.1}$$

where $x \in R = (-\infty,+\infty)$, p, q and f are real-valued functions and $f \in L_2(R)$. We assume that p is a continuously differentiable function and q is a continuous function. Important representatives of such equations are the stationary Korteweg-de Vries equation and its modifications, arising in the theory of distribution of long waves of small finite amplitudes (see, for example, [5] and the references therein), as well as the composite type equations used in the hydrodynamics and hydromechanics (see [6]). Moreover, the more general third-order equations are often reduced to form (1.1).

The smoothness problems for solutions of equation (1.1) are of great interest. The case of bounded domains and smooth coefficients is well understood and sufficiently well described in the literature. In the case of unbounded domains, although the solution of the odd-order equation (1.1) is smooth, it may not belong to any Sobolev space. Of course, this fact causes some difficulties for study of (1.1).

For such equations the study of separability or, in other words, the maximal regularity problem is important. We recall that if for some $C > 0$ the following inequality

$$\|y'''\|_2 + \|py'\|_2 + \|qy\|_2 \leq C(\|Ly\|_2 + \|y\|_2)$$

holds for every $y \in D(L)$, then the operator $Ly = -y''' + py' + qy$ (corresponding to equation (1.1)) is said to be separable in $L_2(R)$ [11, 18] (in this case, the authors [9] say that the operator L is L_2 - maximally regular). Here and in the sequel we denote by C, C_+, C_-, C_j ($j = 0, 1, 2, ...$) positive constants, which, in general, are different in different places. Solving of the maximal regularity problem allows us to show the optimal smoothness of a solution and its behavior at infinity. In addition, the maximal regularity makes it possible to apply the linearization method based on the fixed point theorems to the study of nonlinear differential equations [3, 10]. In the case $p = 0$ the maximal regularity problem for equation (1.1) and its quasilinear generalizations was investigated in [2, 3, 10, 11]. Moreover, in these works the existence and uniqueness of a solution was proved, and the important spectral and approximate properties of the resolvent L^{-1} were given. By perturbation theorems, these results easily can be extended to the case $p \neq 0$, when the growth of p at infinity is controlled by the coefficient q.

The main purpose of our paper is to find sufficient conditions for L_2-maximal regularity of the operator L, when the intermediate coefficient p changes independently (for example, it does not obey the potential q). For example, this is the case for a generalized stationary Korteweg-de Vries equation. We study also the correct solvability problem for (1.1). Along with the case of growing coefficient p, we will consider the case of the strongly oscillating p.

In Section 5 of the paper we consider the fourth-order differential equation with unbounded coefficients and give one result on the solvability and maximal regularity of the solution.

According to the methods of investigation, this work is close to [14, 15, 16, 17].

Definition 1. A function $y \in L_2(R)$ is called a solution of (1.1), if there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset C_0^{(3)}(R)$ (the set of thrice continuously differentiable functions with compact support) such that $\|y_n - y\|_2 \to 0$ and $\|Ly_n - f\|_2 \to 0$ as $n \to \infty$.

We denote by L the closure in $L_2(R)$ of the differential expression $Ly = -y''' + p(x)\, y' + q(x)y$ defined on $C_0^{(3)}(R)$. By the definition of a closed operator, a function $y \in L_2(R)$ is a solution of equation (1.1) if and only if $y \in D(L)$ and $Ly = f$.

We present the main results. Let

$$\alpha_{g,\,h,\,\delta_+}(t) = \|g\|_{L_2(0,\,t)}\|1/h\|_{L_2(t-\delta_+,\,+\infty)} \;\; (t > 0),$$

$$\beta_{g,\,h,\,\delta_-}(\tau) = \|g\|_{L_2(\tau,\,0)}\|1/h\|_{L_2(-\infty,\,\tau+\delta_-)} \;\; (\tau < 0),$$

$$\gamma_{g,\,h,\,\delta_+,\,\delta_-} = \max\left(\sup_{t>0} \alpha_{g,\,h,\,\delta_+}(t),\; \sup_{\tau<0} \beta_{g,\,h,\,\delta_-}(\tau)\right).$$

Here g and $h \neq 0$ are given continuous functions, and $\delta_+ \geq 0$, $\delta_- \geq 0$. For a non-negative continuous function $v(x)$ we define

$$v^*(x) = \sup\left\{d : d^{-1} \geq \int\limits_{x-d/2}^{x+d/2} v(t)dt\right\}$$

and

$$v_n^*(x) = \inf\left\{d^{-1} : d^{-2n+1} \geq \int\limits_{x-d}^{x+d} v^2(t)dt\right\} \;\; (n = 1,\, 2),$$

where $x \in R$. For the first time such functions were introduced by M. Otelbaev in [1, 12, 18].
Theorem 1.1. *Let $p \geq 1$ be a continuously differentiable function, and q be a continuous function and assume that the following conditions are satisfied:*
(a) $\gamma_{1,\,(\sqrt{p})_1^*,\,0,\,0} < \infty$;
(b) for $a \geq 1$ and $b > 0$ the following estimates hold:

$$\frac{1}{a} \leq \frac{p^*(x)}{p^*(\eta)} \leq a \quad \forall \eta \in \left(x - \frac{b}{2}p^*(x),\, x + \frac{b}{2}p^*(x)\right),\ x \in R;$$

(c) $A(p, p^*) = \sup\limits_{x \in R} \left[\left(\int\limits_{x - \frac{p^*(x)}{4}}^{x + \frac{p^*(x)}{4}} p^2(t)dt \right)^{\frac{1}{2}} \cdot (p^*(x))^{\frac{3}{2}} \right] < \infty$;

(d) $\rho_{q,\,p} = \max \left[\sup\limits_{x>0} \|q\|_{L_2(0,\,x)} \|(p_2^*)^2\|_{L_2(\tau(x),\,+\infty)}, \sup\limits_{s<0} \|q\|_{L_2(s,\,0)} \|(p_2^*)^2\|_{L_2(-\infty,\,\eta(s))} \right] < \infty$, *where*

$$\tau\,(x) = \max\left(\frac{x}{4},\, x - 8\sup_{x>0}(p_2^*)^2(x)\right)$$

and

$$\eta\,(s) = \min\left(\frac{s}{4},\, s + 8\sup_{s<0}\,(p_2^*)^2\,(s)\right).$$

Then for any $f \in L_2(R)$ there exists a unique solution y of equation (1.1). *Moreover, there exists $C > 0$ such that for any such f and y the following estimate holds:*

$$\|y'''\|_2 + \|py'\|_2 + \|qy\|_2 \leq C\|f\|_2. \tag{1.2}$$

Remark 1. Conditions *(a)* and *(c)* of Theorem 1.1 are close to being necessary. In fact:

i) if $q = 0$ and condition (a) is not satisfied, then for p, satisfying the condition

$$\max\left(\sup_{\tau<0} \|p_1^*\|^2_{L_2(-\infty,\,\tau+\delta_-)} \|p_1^*\|^{-2}_{L_2(-\infty,\,\tau)}, \sup_{x>0} \|p_1^*\|^2_{L_2(x+\delta_+,+\infty)} \|p_1^*\|^{-2}_{L_2(x,+\infty)}\right) < +\infty,$$

equation (1.1) does not have a solution in $L_2(R)$, therefore, the assigned problem loses its meaning. This follows from Lemma 2.5;

ii) if $q = 0$, and for some $C > 0$ and for all f and y described above estimate (1.2) holds, then *(c)* holds. This fact follows from [12, Chapter 7, Theorem 3].

Example 1. Let us consider the following equation with unbounded and fast oscillating coefficients:

$$-y''' + (1 + 20e^{\sqrt{1+x^2}} \sin^2 e^{x^2})y' + x^{2n} \cos^2 5x\ y = f(x),\ f \in L_2(R), \tag{1.3}$$

It is easy to verify that all conditions of Theorem 1.1 are satisfied. Therefore, equation (1.3) for any $f \in L_2(R)$ has a unique solution $y \in L_2(R)$ and for some $C > 0$, for all such f and y holds the following maximal regularity estimate:

$$\|y'''\|_2 + \|(1 + 20e^{\sqrt{1+x^2}} \sin^2 e^{x^2})y'\|_2 + \|x^{2n} \cos^2 5x\ y\|_2 \leq C\|f\|_2.$$

2 Some integral inequalities

We denote by $\dot{C}^{(m)}\,[0,\,+\infty)$ (respectively $\dot{C}^{(m)}\,(-\infty,\,0]$) a set of the m times ($m \in N$) continuously differentiable on $[0, +\infty)$ (respectively $(-\infty,\,0]$) functions with compact support in $[0, +\infty)$ (respectively in $(-\infty,\,0]$). Lemmas 2.1 and 2.2 follow from [12, Chapter 3, Theorem 8] and [12, Chapter 3, Theorem 9], respectively, by the same way as in Lemma 2.1 [17]. Theorems 8 and 9 from [12] are proved using results of [7, 13].

Lemma 2.1. *Assume that $n = 1$ or $n = 2$, and $h \geq 0$ and g are continuous functions such that*

$$\alpha_{g,\,h_n^*,\,\delta_+} < \infty \tag{2.1}$$

for some $\delta_+ \geq 0$. Then for each $y \in \dot{C}^{(2)}\,[0,\,+\infty)$ the following estimate holds:

$$\left(\int\limits_0^{+\infty} g^2(t)y^2(t)dt\right)^{\frac{1}{2}} \leq C_+ \left(\int\limits_0^{+\infty} \left[\left(y^{(n+1)}\right)^2(t) + h^2(t){y'}^2(t)\right] dt\right)^{\frac{1}{2}}, \tag{2.2}$$

and $C_+ \leq C\,\alpha_{g,\,h_n^*,\,\delta_+}$. Conversely, if (2.2) holds with some constant C_+, then $\alpha_{g,\,h_n^*,\,0} < \infty$ and $C_+ \geq C_0\alpha_{g,\,h_n^*,\,0}$, where $C_0 > 0$ does not depend on g and h.

Lemma 2.2. *Assume that $n = 1$ or $n = 2$, and $h \geq 0$ and g are continuous functions such that for some $\delta_+ > 0$ condition (2.1) and*

$$\sup_{x>0} \int\limits_{x-\delta_+}^{+\infty} \left[h_n^*(t)\right]^{2n} dt \left(\int\limits_x^{+\infty} \left[h_n^*(\eta)\right]^{2n} d\eta\right)^{-1} < \infty. \tag{2.3}$$

are fulfilled. Then inequality (2.2) holds if and only if $\alpha_{g,\,h_n^*,\,0} < \infty$, and the minimal constant C_+ in (2.2) satisfies the following estimates:

$$C_2\alpha_{g,\,h_n^*,\,0} \leq C_+ \leq C_3\alpha_{g,\,h_n^*,\,0}, \tag{2.4}$$

where $C_2,\ C_3 > 0$ are independent of g and h.

Using Lemmas 2.1 and 2.2, we prove the following Lemmas 2.3 and 2.4, respectively.

Lemma 2.3. *Assume that $n = 1$ or $n = 2$, and $h \geq 0$ and g are continuous functions such that*

$$\beta_{g,\,h_n^*,\,\delta_-} < \infty \tag{2.5}$$

for some $\delta_- > 0$. Then for each $y \in \dot{C}^{(2)}\,(-\infty,\,0]$ the following estimate holds:

$$\left(\int\limits_{-\infty}^0 g^2(t)y^2(t)dt\right)^{\frac{1}{2}} \leq C_- \left(\int\limits_{-\infty}^0 \left[\left(y^{(n+1)}\right)^2(t) + h^2(t){y'}^2(t)\right]\right)^{\frac{1}{2}}, \tag{2.6}$$

where $C_- \leq \tilde{C}\,\beta_{g,\,h_n^*,\,\delta_-}$. Conversely, if (2.6) holds with some constant C_-, then $\beta_{g,\,h_n^*,\,0} < \infty$ and $C_- \geq C_1\beta_{g,\,h_n^*,\,0}$, where $C_1 > 0$ does not depend on g and h.

Lemma 2.4. *Assume that $n = 1$ or $n = 2$, and $h \geq 0$ and g are continuous functions such that for some $\delta_- > 0$ the conditions (2.5) and*

$$\sup_{\tau<0} \int\limits_{-\infty}^{\tau+\delta_-} \left[h_n^*(t)\right]^{2n} dt \left(\int\limits_{-\infty}^{\tau} \left[h_n^*(\eta)\right]^{2n} d\eta\right)^{-1} < \infty. \tag{2.7}$$

are fulfilled. Then inequality (2.6) holds if and only if $\beta_{g,\,h_n^*,\,0} < \infty$, and the minimal constant C_- in (2.6) satisfies the following estimates:

$$C_4\beta_{g,\,h_n^*,\,0} \leq C_- \leq C_5\beta_{g,\,h_n^*,\,0}, \tag{2.8}$$

where $C_4,\ C_5 > 0$ *are independent of* g *and* h.

Lemma 2.5. *Assume that* $n = 1$ *or* $n = 2$, *and* $h \geq 0$ *and* g *are continuous functions such that for some* $\delta_+ > 0$ *and* $\delta_- > 0$ *conditions* (2.1), (2.3), (2.5) *and* (2.7) *are fulfilled. Then for* $y \in C_0^{(2)}(R)$ *the inequality*

$$\left(\int_{-\infty}^{+\infty} g^2(t)y^2(t)dt\right)^{1/2} \leq C\left(\int_{-\infty}^{+\infty}\left[\left(y^{(n+1)}\right)^2(t)+h^2(t)y'^2(t)\right]dt\right)^{1/2} \quad (n=1,\,2) \tag{2.9}$$

holds if and only if $\gamma_{g,\,h_n^*,\,0,\,0} < \infty$, *and the minimal constant* C *in* (2.9) *satisfies the following estimates:*

$$C_6\gamma_{g,\,h_n^*,\,0,\,0} \leq C \leq C_7\gamma_{g,\,h_n^*,\,0,\,0}, \tag{2.10}$$

where $C_6,\ C_7 > 0$ *are independent of* g *and* h.

Proof. By Lemmas 2.2 and 2.4 and estimates (2.2) and (2.6), for $y \in C_0^{(n+1)}(R)$ we have

$$\|g(t)\,y(t)\|_2 \leq C_1(\delta)\,\beta_{g,\,h_n^*,\,0}\left(\left\|y^{(n+1)}\right\|^2_{L_2(-\infty,\,0)}+\|h(t)\,y'\|^2_{L_2(-\infty,\,0)}\right)^{1/2}$$

$$+C(\delta)\,\alpha_{g\,,\,h_n^*,\,0}\left(\left\|y^{(n+1)}\right\|^2_{L_2(0,\,+\infty)}+\|h(t)\,y'\|^2_{L_2(0,\,+\infty)}\right)^{1/2}$$

$$\leq \tilde{C}(\delta)\,\gamma_{g,\,h_n^*,\,0,\,0}\left(\left\|y^{(n+1)}\right\|^2_2+\|h(t)\,y'\|^2_2\right)^{1/2}.$$

Then, assuming $C = \tilde{C}(\delta)\,\gamma_{1,\,h_n^*,\,0,\,0}$, we obtain (2.9). Inequalities (2.4) and (2.8) imply estimates (2.10). □

3 On the two-term differential equation

We denote by l the closure in the $L_2(R)$ of the differential expression $l_0y = -y''' + p(x)\,y'$ defined on the set $C_0^{(3)}(R)$. We consider the following degenerate differential equation

$$ly = -y''' + p(x)\,y' = f. \tag{3.1}$$

Definition 2. A function $y \in L_2(R)$ is called a solution of (3.1), if there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset C_0^{(3)}(R)$ such that $\|y_n - y\|_2 \to 0$ and $\|l_0y_n - f\|_2 \to 0$ as $n \to \infty$.

It is clear that a function $y \in L_2(R)$ is a solution of equation (3.1) if and only if $y \in D(l)$ and $ly = f$.

Lemma 3.1. *Let* p *be such that*

$$p \in C_{loc}^{(1)}(R)\,,\, p \geq 1, \tag{3.2}$$

and

$$\gamma_{1,\,(\sqrt{p})_1^*,\,\delta_+,\,\delta_-} < \infty \tag{3.3}$$

for some $\delta_+ > 0$ *and* $\delta_- > 0$. *Then for each* $f \in L_2(R)$ *there exists a unique solution* y *of equation* (3.1) *and there exists* $C > 0$ *such that for any such* p, δ_+, δ_-, f *and* y *the following estimate holds:*

$$\|y''\|_2 + \|\sqrt{p}y'\|_2 + \|y\|_2 \leq \sqrt{3\left(1 + C^2\gamma^2_{1,\,(\sqrt{p})_1^*,\,\delta_+,\,\delta_-}\right)}\,\|f\|_2\,. \tag{3.4}$$

Proof. Assuming $g=1$ and $h=\sqrt{p}$ in Lemmas 2.1 and 2.3, for $y\in C_0^{(3)}(R)$ we have

$$\|y\|_2\le C_-\left(\int_{-\infty}^{0}\left[y''^2+p(t)y'^2\right]dt\right)^{\frac{1}{2}}+C_+\left(\int_{0}^{+\infty}\left[y''^2+p(t)y'^2\right]dt\right)^{\frac{1}{2}}$$

$$\le C\,\gamma_{1,(\sqrt{p})_1^*,\delta_+,\delta_-}\left[\|y''\|^2+\|p(t)y'\|^2\right]^{\frac{1}{2}}. \tag{3.5}$$

Further

$$(ly,y')=(-y'''+p(x)\,y',y')=(-y''',y')+(p(x)\,y',y')$$

$$=-\int_R y'''\bar{y}'dx+\int_R p\,|y'|^2\,dx=\int_R y''\bar{y}''dx+\int_R p\,|y'|^2\,dx=|y''|_2^2+\|\sqrt{p}y'\|_2^2. \tag{3.6}$$

Taking into account condition (3.2), by the Hölder inequality, we obtain

$$|(ly,y')|\le\left\|\frac{1}{\sqrt{p}}ly\right\|_2\|\sqrt{p}y'\|_2.$$

This inequality and (3.6) imply that $\|y''\|_2^2+\left\|\sqrt{p}y'\right\|_2^2\le\|ly\|_2^2$. By (3.5), we obtain estimate (3.4).

We show, that estimate (3.4) is also valid for a solution of equation (3.1). Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $C_0^{(3)}(R)$ such that $\|y_n-y\|_2\to 0$, $\|ly_n-f\|_2\to 0$ $(n\to\infty)$. By (3.4),

$$\|y_n''\|_2^2+\|\sqrt{p}y_n'\|_2^2+\|y_n\|_2^2\le C_1\,\|ly_n\|_2^2. \tag{3.7}$$

Moreover, for any n, $m\in N$ we have

$$\|y_n''-y_m''\|_2^2+\|\sqrt{p}\,(y_n'-y_m')\|_2^2+\|y_n-y_m\|_2^2\le C_1\,\|ly_n-ly_m\|_2^2. \tag{3.8}$$

We denote by $\dot{W}_{2,\sqrt{p}}^2(R)$ the completion of $C_0^{(2)}(R)$ with respect to the norm $\|y\|_W=\left(\|y''\|_2^2+\left\|\sqrt{p}\,y'\right\|_2^2+\|y\|_2^2\right)^{1/2}$. According to (3.8), $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\dot{W}_{2,\sqrt{p}}^2(R)$. Then there exists an element z such that $\|y_n-z\|_W\to 0$ $(n\to\infty)$. By Definition 2 z is a solution of (3.1). For $n\to\infty$ from inequality (3.7) we obtain

$$\|z\|_W\le C\,\|f\|_2. \tag{3.9}$$

By (3.9), a solution of equation (3.1) is unique: $z=y$.

Now, we show that for each $f\in L_2(R)$ there exists a solution to equation (3.1). By Definition 2, it suffices to prove that $R(l)=L_2(R)$. Assume the contrary, let $R(l)\ne L_2(R)$. Then there exists the non-zero element $z\in L_2(R)$ such that $z\perp R(l)$. So, $(ly,z)=0$ for any $y\in C_0^{(3)}(R)$. On the other hand

$$(ly,\,z)=\int_R y\left(z'''-[p(x)\,z]'\right)dx,\quad y\in C_0^{(3)}(R).$$

So we obtain

$$z''-pz=C_1. \tag{3.10}$$

By (3.10), we have that z is a twice continuously differentiable function.

The following two cases are possible.

1. $C_1 \neq 0$. Without loss of generality, we can assume that $C_1 = 1$. So,

$$z'' - p(x)z = 1, \ x \in R. \tag{3.11}$$

The solution z of this equation can be written as

$$z\left(x\right) = C_2 z_1\left(x\right) + C_3 z_2\left(x\right) + \int\limits_{-\infty}^{+\infty} G\left(x, \, t\right) dt, \tag{3.12}$$

where $G\left(x, t\right)$ is the Green function of the Sturm-Liouville operator, and z_1, z_2 are two linearly independent solutions of the following equation $z'' - p(x)z = 0$. It is known that, by condition (3.2) (see, for example [4]), $z_1 > 0$, $z_2 > 0$, $\lim\limits_{x \to +\infty} z_1\left(x\right) = +\infty$, $\lim\limits_{x \to -\infty} z_1\left(x\right) = 0$, $\lim\limits_{x \to -\infty} z_2\left(x\right) = +\infty$, $\lim\limits_{x \to +\infty} z_2\left(x\right) = 0$ and

$$G\left(x, t\right) > 0, \ \int\limits_R G\left(x, t\right) dt < \infty, \quad x \in R.$$

Therefore, by (3.12), we have $C_2 = 0$ and $C_3 = 0$, and (3.11) implies that $z''(x) \geq 1$ for any $x \in R$. Then there exists $\xi \in R$ such that $z\left(\xi\right) = k > 0$ and $z'\left(\xi\right) = m > 0$. We have

$$z'\left(x\right) - m = \left(x - \xi\right) + \int\limits_{\xi}^{x} pz\left(t\right) dt.$$

Taking into account condition (3.2), we obtain $z - k \geq 1/2(x - \xi)^2 \ \ (x \geq \xi)$. Therefore, $\lim\limits_{x \to +\infty} z(x) = +\infty$, so, $z \notin L_2\left(R\right)$.

2. If $C_1 = 0$, then by (3.10),

$$z''(x) - p(x)z(x) = 0, x \in R. \tag{3.13}$$

The solution z of (3.13) is represented as $z = C_4 z_1 + C_5 z_2$. As mentioned above, $z_1\left(x\right) \to +\infty$, $z_2\left(x\right) \to 0 \ (x \to +\infty)$, and $z_2\left(x\right) \to +\infty$, $z_1\left(x\right) \to 0 \ (x \to -\infty)$. Consequently, $C_4 = 0$ and $C_5 = 0$. So, $z = 0$.

Contradictions obtained by us show that $R(l) = L_2(R)$. □

Remark 2 The condition $p \geq 1$ can be replaced by $p \geq \delta > 0$. To show this, it is enough to put $x = \delta^{-1/2} t$ in equation (3.1).

Lemma 3.2. *Assume that $p \geq 1$ is continuously differentiable and satisfies conditions (a), (b) and (c) of Theorem 1.1. Then there exists $C > 0$ such that for the solution y of equation* (3.1) *the following estimate holds:*

$$\left\|y'''\right\|_2 + \left\|py'\right\|_2 \leq C \left\|f\right\|_2. \tag{3.14}$$

Proof. By (3.4), we have $y, \ y' \in L_2\left(R\right)$. Assume, that $y' = z$, then the expression $ly = -y''' + py'$ will take the form $Vz = -z'' + pz \ (z \in L_2\left(R\right))$. By [12, Chapter 7, Theorem 3], there exists $C_4 > 0$ such that for any $z \in D\left(V\right)$ the following inequality holds:

$$\left\|z''\right\|_2 + \left\|pz\right\|_2 \leq C_4 \left\|Vz\right\|_2.$$

Then, by Lemma 2.5 and Definition 2, we obtain the proof of lemma. □

4 Proof of Theorem 1.1

Proof. We put $x = at$ in equation (1.1), where $a > 0$. If we introduce the notations

$$\tilde{y}(t) = y(at), \ \tilde{p}_0(t) = p(at), \ \tilde{q}_0(t) = q(at), \ \tilde{f}(t) = a^3 f(at),$$

then (1.1) can be written in the following form:

$$\tilde{L}\tilde{y} = -\tilde{y}''' + a^2\tilde{p}_0\tilde{y}' + a^3\tilde{q}_0\tilde{y} = \tilde{f}(t). \tag{4.1}$$

We denote by l_a the closure in $L_2(R)$ of the differential operator $l_{0a}\tilde{y} = -\tilde{y}''' + a^2\tilde{p}_0(t)\tilde{y}'$ defined on $C_0^{(3)}(R)$. Since $a^2|\tilde{p}_0(t)| \geq a^2$, by Lemma 3.1 and Remark 2, the operator l_a is continuously invertible and holds the following estimate:

$$\|\tilde{y}'''\|_2 + \left\|a^2\tilde{p}_0\tilde{y}'\right\|_2 \leq C_{l_a}\|l_a\tilde{y}\|_2 \quad \forall \tilde{y} \in D(l_a), \tag{4.2}$$

where C_{l_a} does not depend on $\tilde{y}$. Taking into account condition (d), by (4.2) and [1, Theorem 6.3], we have

$$\left\|a^3\tilde{q}_0\tilde{y}\right\|_2 \leq a^3\rho_{q,p}C_{l_a}\|l_a\tilde{y}\|_2. \tag{4.3}$$

By (4.1), $\tilde{L} = l_a + a^3\tilde{q}_0E$. If we choose $a = [2\rho_{q,p}C_{l_a}]^{-\frac{1}{3}}$, then (4.3) implies that the following estimate holds

$$\left\|a^3\tilde{q}_0\tilde{y}\right\|_2 \leq \alpha\,\|l_a\tilde{y}\|_2, \tag{4.4}$$

where $\alpha \in \left(0, \ \frac{1}{2}\right]$. By the known perturbation theorem (see, for example, [8, Chapter 4, Theorem 1.16]) there exists the inverse operator $\left(l_a + a^3\tilde{q}_0E\right)^{-1}$ and the equality $R\left(l_a + a^3\tilde{q}_0E\right) = L_2(R)$ is true. So, there exists the solution $\tilde{y}$ of equation (4.1) and it is unique. By inequalities (4.2) and (4.4), we obtain

$$\|\tilde{y}'''\|_2 + \left\|a^2\tilde{p}_0\tilde{y}'\right\|_2 + \left\|a^3\tilde{q}_0\tilde{y}\right\|_2 \leq \left(\frac{1}{2} + C_{l_a}\right)\|l_a\tilde{y}\|_2. \tag{4.5}$$

Taking into account (4.4), we get

$$\|l_a\tilde{y}\|_2 \leq \left\|\left(l_a + a^3\tilde{q}_0E\right)\tilde{y}\right\|_2 + \left\|a^3\tilde{q}_0\tilde{y}\right\|_2 \leq \left\|\left(l_a + a^3\tilde{q}_0E\right)\tilde{y}\right\|_2 + \frac{1}{2}\|l_a\tilde{y}\|_2,$$

that implies

$$\|l_a\tilde{y}\|_2 \leq 2\left\|\left(l_a + a^3\tilde{q}_0E\right)\tilde{y}\right\|_2. \tag{4.6}$$

By (4.5) and (4.6), we obtain that

$$\|\tilde{y}'''\|_2 + \left\|a^2\tilde{p}_0\tilde{y}'\right\|_2 + \left\|a^3\tilde{q}_0\tilde{y}\right\|_2 \leq C\left\|\tilde{f}\right\|_2, \ C = 1 + 2C_{l_a}.$$

Putting $t = a^{-1}x$, we get the estimate (1.2). □

5 On the well-posedness and maximal regularity of solution of the fourth-order differential equation

In this section we announce one result on the correct solvability and maximal regularity of the following equation:

$$-y^{(4)} + p(x)y'' + s(x)y' + \theta(x)y = F, \tag{5.1}$$

where $x \in R = (-\infty, +\infty)$, and $F \in L_2(R)$.

We denote by M the closure of the operator $M_0 y = -y^{(4)} + p\,(x)\,y'' + s\,(x)\,y' + \theta\,(x)\,y$ defined on the set $C_0^{(4)}(R)$ of four times continuously differentiable functions with compact support. We say that y is a solution of (5.1), if $y \in D(M)$ and $My = F$. We define

$$\nu_{g,\,h,\,\delta_+}(t) = \left(\int\limits_0^t g^2(\xi)d\xi\right)^{1/2} \left(\int\limits_{t-\delta_+}^{+\infty} \xi h^{-2}(\xi)d\xi\right)^{1/2} \quad (t > 0),$$

$$\mu_{g,\,h,\,\delta_-}(\tau) = \left(\int\limits_\tau^0 g^2(\zeta)d\zeta\right)^{1/2} \left(\int\limits_{-\infty}^{\tau+\delta_-} \zeta h^{-2}(\zeta)d\zeta\right)^{1/2} \quad (\tau < 0),$$

and

$$\omega_{g,\,h,\,\delta_+,\,\delta_-} = \max\left(\sup_{t>0} \nu_{g,\,h,\,\delta_+}(t),\ \sup_{\tau<0} \mu_{g,\,h,\,\delta_-}(\tau)\right).$$

Teorem 5.1. *Let $p \geq 1$ be a twice continuously differentiable function that satisfies conditions (b) and (c) of Theorem 1.1 and $\omega_{1,\,\left(\sqrt{p}\right)_1^*,\,0,\,0} < +\infty$. Assume that s is a continuously differentiable function, and θ is a continuous function such that* $\max\left[\omega_{\theta,\,p_2^*,\,\delta_+,\,\delta_-},\, \gamma_{s,\,p_2^*,\,\delta_+,\,\delta_-}\right] < +\infty$ *for some $\delta_+ > 0$ and $\delta_- > 0$. Then for any $f \in L_2(R)$ there exists the unique solution y of equation* (5.1). *Moreover, the following estimate holds:*

$$\|y^{(4)}\|_2 + \|py''\|_2 + \|sy'\|_2 + \|\theta y\|_2 \leq C\|f\|_2,$$

where $C > 0$ does not depend on y.

This theorem is proved similarly to the proof of Theorem 1.1 using Lemma 2.5, [1, Theorem 6.1] (for cases $n = 2$ and $k = 1,\,2$) and [12, Theorem 3].

Acknowledgments

The authors were supported by grant no. AP05131649 of the Ministry of Education and Science of the Republic of Kazakhstan.

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Received: 30.04.2018