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SCHWARZ PROBLEM IN LENS AND HALF LENS

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Key words: Schwarz problem, Cauchy–Pompeiu formula, lens, half lens.

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Abstract. We consider the Schwarz boundary value problem (BVP) for the inhomogeneous Cauchy–Riemann equation in lenses and half lenses. By the technique of parqueting–reflection and the Cauchy–Pompeiu representation formula for lenses and half lenses, the Cauchy–Schwarz representation formula is obtained. Also, the solution of the Schwarz BVP is explicitly obtained.

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1 Introduction

A variety of BVPs for partial differential equations (PDEs) has been considered on special domains. Those special domains include the unit disc [11], half plane [5, 12], quarter plane [1, 2, 6], ring [17, 18], half disc and half ring [8], quarter ring [14, 15], lens and lune [9], half lens and half lune [16] and some convex polygons, e.g. equilateral triangle [10] and half hexagon [15].

The Schwarz BVP is considered for an analytic function with given boundary values of its real part. Also, the Cauchy–Schwarz representation formula is obtained by the technique of parqueting–reflection and the Cauchy–Pompeiu representation formula, see e.g. [3, 4, 7, 19].

In particular, the solution of the Schwarz BVP in a lens is explicitly obtained in [9]. There, a lens is defined by $D = D \cap D_m(r)$, where $D = \{z : \ |z| < 1\}$ and $D_m(r) = \{z : \ |z - m| < r\}, \ 0 < r < 1 < m, \ r^2 + 1 = m^2$. In the complex plane $\mathbb{C}$, a lens is formed by two arcs C_1 and C_2 of the two circles $|z - ib| = r$ and $|z + ib| = r$. The points a and $-a$ lie on the real axis where the arcs C_1 and C_2 meet with the angle $\pi\alpha$. If a and $\pi\alpha$ are known, $b = a \cot \frac{\pi\alpha}{2}$ and $r = \frac{a}{\sin \frac{\pi\alpha}{2}}$, see [13].

Let D be the lens formed. It is formed by two arcs C_1 and C_2 of the two circles $|z - i| = \sqrt{2}$ and $|z + i| = \sqrt{2}$ where $a = 1$ and $\pi\alpha = 90°$. Also, Ω is the half lens (Fig. 1.).

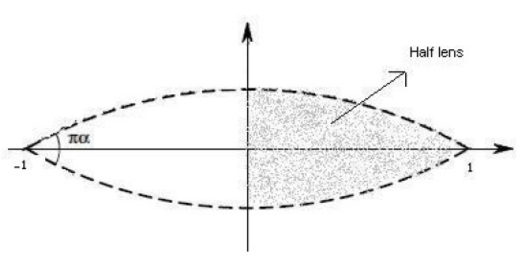

Fig. 1. The lens D and the half lens Ω

This paper is organized as follows. In Section 2, by the technique of parqueting–reflection and the Cauchy–Pompeiu representation formula on the lens D, the Cauchy–Schwarz representation formula is explicitly obtained. Then, the Schwarz BVP in D for the inhomogeneous Cauchy–Riemann equation is studied. The Schwarz BVP for the half lens Ω is considered in Section 3.

2 Schwarz problem for the lens D

The point $z \in D$ is reflected at ∂C_1 onto $\frac{i\bar{z}+1}{\bar{z}+i}$, and both these points are reflected at ∂C_2 onto the points $\frac{-i\bar{z}+1}{\bar{z}-i}$ and $\frac{1}{z}$.

The Cauchy–Schwarz representation formula is derived by combining the Cauchy–Pompeiu representation formula applied to the points described above.

Theorem 2.1. *Any function $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ can be represented as*

$$\begin{aligned} w(z) =& \frac{1}{2\pi i}\int_{\partial D\cap\partial C_1} \mathrm{Re}\, w(\zeta)\left[\frac{2(\zeta-i)}{\zeta-z} - 1 + \frac{2z(\zeta-i)}{\zeta z-1} - 1\right]\frac{d\zeta}{\zeta-i} \\ &+ \frac{1}{2\pi i}\int_{\partial D\cap\partial C_2} \mathrm{Re}\, w(\zeta)\left[\frac{2(\zeta+i)}{\zeta-z} - 1 + \frac{2z(\zeta+i)}{\zeta z-1} - 1\right]\frac{d\zeta}{\zeta+i} \\ &+ \frac{1}{\pi}\int_{\partial D\cap\partial C_1} \mathrm{Im}\, w(\zeta)\frac{d\zeta}{\zeta-i} + \frac{1}{\pi}\int_{\partial D\cap\partial C_2} \mathrm{Im}\, w(\zeta)\frac{d\zeta}{\zeta+i} \\ &- \frac{1}{\pi}\int_D \Big\{ w_{\bar\zeta}(\zeta)\left[\frac{1}{\zeta-z} + \frac{z}{\zeta z-1}\right] \\ &+ \overline{w_{\bar\zeta}(\zeta)}\left[\frac{z-i}{-\bar\zeta(z-i)-iz+1} - \frac{z+i}{\bar\zeta(z+i)-iz-1}\right]\Big\} d\xi d\eta, \end{aligned} \tag{2.1}$$

where $\zeta = \xi + i\eta$.

Proof. The Cauchy–Pompeiu formula

$$\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta-z} - \frac{1}{\pi}\int_D w_{\bar\zeta}(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta-z} = \begin{cases} w(z), & z\in D,\\ 0, & z\notin \overline{D}, \end{cases} \tag{2.2}$$

applied to $z \in D$ and $\frac{i\bar{z}+1}{\bar{z}+i}$, $\frac{1}{z}$, $\frac{-i\bar{z}+1}{\bar{z}-i} \notin \overline{D}$, respectively, gives the following four equalities

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta-z} - \frac{1}{\pi}\int_D w_{\bar\zeta}(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta-z}, \tag{2.3}$$

$$0 = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\tfrac{(\bar z+i)\mathrm{d}\zeta}{\zeta(\bar z+i)-i\bar z-1} - \frac{1}{\pi}\int_D w_{\bar\zeta}(\zeta)\tfrac{(\bar z+i)\mathrm{d}\xi\mathrm{d}\eta}{\zeta(\bar z+i)-i\bar z-1}, \tag{2.4}$$

$$0 = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\tfrac{z\mathrm{d}\zeta}{\zeta z-1} - \frac{1}{\pi}\int_D w_{\bar\zeta}(\zeta)\tfrac{z\mathrm{d}\xi\mathrm{d}\eta}{\zeta z-1}, \tag{2.5}$$

$$0 = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\tfrac{(\bar z-i)\mathrm{d}\zeta}{\zeta(\bar z-i)+i\bar z-1} - \frac{1}{\pi}\int_D w_{\bar\zeta}(\zeta)\tfrac{(\bar z-i)\mathrm{d}\xi\mathrm{d}\eta}{\zeta(\bar z-i)+i\bar z-1}. \tag{2.6}$$

Taking the complex conjugate of (2.4) and (2.6), where $\bar{z}$ appears, and adding the resulting four relations, lead to the claimed representation formula. □

The Cauchy-Schwarz representation formula (2.1), serves to solve the Schwarz BVP for the inhomogeneous Cauchy-Riemann equation in the lens D.

Theorem 2.2. *The Schwarz problem*

$$\begin{aligned}&w_{\bar z}=f,\quad in\ \ D,\quad \mathrm{Re}\,w=\gamma\quad on\quad \partial D,\\ &\frac{1}{\pi i}\int_{\partial D\cap\partial C_1}\mathrm{Im}\,w(\zeta)\frac{d\zeta}{\zeta-i}+\frac{1}{\pi i}\int_{\partial D\cap\partial C_2}\mathrm{Im}\,w(\zeta)\frac{d\zeta}{\zeta+i}=c,\end{aligned}$$

with given $f\in L_p(D;\mathbb{C}),\ 2<p,\ \gamma\in C(\partial D;\mathbb{R}),\ c\in\mathbb{R}$ *is uniquely solved by*

$$\begin{aligned}w(z)=&\frac{1}{2\pi i}\int_{\partial D\cap\partial C_1}\gamma(\zeta)\left[\frac{2(\zeta-i)}{\zeta-z}-1+\frac{2z(\zeta-i)}{\zeta z-1}-1\right]\frac{d\zeta}{\zeta-i}\\ &+\frac{1}{2\pi i}\int_{\partial D\cap\partial C_2}\gamma(\zeta)\left[\frac{2(\zeta+i)}{\zeta-z}-1+\frac{2z(\zeta+i)}{\zeta z-1}-1\right]\frac{d\zeta}{\zeta+i}+ic\\ &-\frac{1}{\pi}\int_D\Big\{f(\zeta)\left[\frac{1}{\zeta-z}+\frac{z}{\zeta z-1}\right]\\ &+\overline{f(\zeta)}\left[\frac{z-i}{-\bar\zeta(z-i)-iz+1}-\frac{z+i}{\bar\zeta(z+i)-iz-1}\right]\Big\}d\xi d\eta. \qquad (2.7)\end{aligned}$$

Proof. The right-hand side of (2.7) up to the term

$$Tf(z)=-\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta-z},$$

is an analytic function and $Tf(z)$ is a weak solution to the Cauchy–Riemann equation $w_{\bar z}=f$, so $w(z)$ is a weak solution of the inhomogeneous Cauchy–Riemann equation (see [19]).

Now, we consider the boundary behavior. Let

$$\begin{aligned}w_0(z)=&-\frac{1}{\pi}\int_D\Big\{f(\zeta)\left[\frac{1}{\zeta-z}+\frac{z}{\zeta z-1}\right]\\ &+\overline{f(\zeta)}\left[\frac{z-i}{-\bar\zeta(z-i)-iz+1}-\frac{z+i}{\bar\zeta(z+i)-iz-1}\right]\Big\}d\xi d\eta. \qquad (2.8)\end{aligned}$$

For $z\in\partial D\cap\partial C_1$, i.e. $(z-i)(\bar z+i)=2$,

$$w_0(z)=-\frac{1}{\pi}\int_D\left\{f(\zeta)\left[\frac{1}{\zeta-z}+\frac{z}{\zeta z-1}\right]-\overline{f(\zeta)}\left[\frac{1}{\bar\zeta-\bar z}+\frac{\bar z}{\bar\zeta\bar z-1}\right]\right\}d\xi d\eta.$$

Hence, $\mathrm{Re}\,w_0(z)=0$. Similarly, for $z\in\partial D\cap\partial C_2$, i.e. $(z+i)(\bar z-i)=2$,

$$w_0(z)=-\frac{1}{\pi}\int_D\left\{f(\zeta)\left[\frac{1}{\zeta-z}+\frac{z}{\zeta z-1}\right]-\overline{f(\zeta)}\left[\frac{\bar z}{\bar\zeta\bar z-1}+\frac{1}{\bar\zeta-\bar z}\right]\right\}d\xi d\eta.$$

Hence, $\mathrm{Re}\,w_0(z)=0$. In fact

$$\begin{aligned}\mathrm{Re}\,w(z)=&\frac{1}{2\pi i}\int_{\partial D\cap\partial C_1}\gamma(\zeta)\left[\frac{\zeta-i}{\zeta-z}+\frac{\bar\zeta+i}{\bar\zeta-\bar z}-1+\frac{z(\zeta-i)}{\zeta z-1}+\frac{\bar z(\bar\zeta+i)}{\bar\zeta\bar z-1}-1\right]\frac{d\zeta}{\zeta-i}\\ &+\frac{1}{2\pi i}\int_{\partial D\cap\partial C_2}\gamma(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar\zeta-i}{\bar\zeta-\bar z}-1+\frac{z(\zeta+i)}{\zeta z-1}+\frac{\bar z(\bar\zeta-i)}{\bar\zeta\bar z-1}-1\right]\frac{d\zeta}{\zeta+i}\\ &+\mathrm{Re}\,w_0(z),\end{aligned}$$

where $w_0(z)$ is defined by (2.8). Therefore, on ∂C_1

$$\begin{aligned}\mathrm{Re}w(z) =&\frac{1}{2\pi i}\int_{\partial D\cap\partial C_1}\gamma(\zeta)\left[\frac{\zeta-i}{\zeta-z}+\frac{\bar\zeta+i}{\bar\zeta-\bar z}-1+\frac{z(\zeta-i)}{\zeta z-1}+\frac{iz-1}{\zeta z-1}-1\right]\frac{d\zeta}{\zeta-i}\\ &+\frac{1}{2\pi i}\int_{\partial D\cap\partial C_2}\gamma(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{-iz-1}{\zeta z-1}-1+\frac{z(\zeta+i)}{\zeta z-1}+\frac{-z-i}{\zeta-z}-1\right]\frac{d\zeta}{\zeta+i}\\ &+\mathrm{Re}w_0(z).\end{aligned}$$

Also, $\mathrm{Re}w(z)$ on ∂C_1 could be written as

$$\begin{aligned}\mathrm{Re}w(z) =&\frac{1}{2\pi i}\int_{\partial D\cap\partial C_1}\gamma(\zeta)\left[\frac{\zeta-i}{\zeta-z}+\frac{\bar\zeta+i}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta-i}\\ =&\frac{1}{2\pi i}\int_{\partial C_1}\Gamma_1(\zeta)\left[\frac{\zeta-i}{\zeta-z}+\frac{\bar\zeta+i}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta-i},\end{aligned}$$

where

$$\Gamma_1(\zeta)=\begin{cases}\gamma(\zeta), & \zeta\in\partial D\cap\partial C_1,\\ 0, & \zeta\in\partial C_1\setminus(\partial D\cap\partial C_1).\end{cases}$$

From the properties of the Poisson kernel for C_1, the equality

$$\lim_{z\to\zeta}\mathrm{Re}w(z)=\gamma(\zeta)$$

follows for $\zeta\in\partial D\cap\partial C_1$ up to the tips ±1 of the lens D, because Γ_1 fails to be continuous there if γ does not vanish at these points. Also on ∂C_2, we have

$$\begin{aligned}\mathrm{Re}w(z) =&\frac{1}{2\pi i}\int_{\partial D\cap\partial C_1}\gamma(\zeta)\left[\frac{\zeta-i}{\zeta-z}+\frac{iz-1}{\zeta z-1}-1+\frac{z(\zeta-i)}{\zeta z-1}+\frac{-z+i}{\zeta-z}-1\right]\frac{d\zeta}{\zeta-i}\\ &+\frac{1}{2\pi i}\int_{\partial D\cap\partial C_2}\gamma(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar\zeta-i}{\bar\zeta-\bar z}-1+\frac{z(\zeta+i)}{\zeta z-1}+\frac{-iz-1}{\zeta z-1}-1\right]\frac{d\zeta}{\zeta+i}\\ &+\mathrm{Re}w_0(z).\end{aligned}$$

Also, $\mathrm{Re}w(z)$ on ∂C_2 could be written as

$$\begin{aligned}\mathrm{Re}w(z) =&\frac{1}{2\pi i}\int_{\partial D\cap\partial C_2}\gamma(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar\zeta-i}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta+i}\\ =&\frac{1}{2\pi i}\int_{\partial C_2}\Gamma_2(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar\zeta-i}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta+i},\end{aligned}$$

where

$$\Gamma_2(\zeta)=\begin{cases}\gamma(\zeta), & \zeta\in\partial D\cap\partial C_2,\\ 0, & \zeta\in\partial C_2\setminus(\partial D\cap\partial C_2).\end{cases}$$

From the properties of the Poisson kernel for C_2 as above for $\zeta\in\partial D\cap\partial C_2$ the equality

$$\lim_{z\to\zeta}\mathrm{Re}w(z)=\gamma(\zeta)$$

holds with the possible exception of the tips ±1. In fact, we show that

$$\lim_{z\to\pm1}\mathrm{Re}w(z)=\gamma(\pm1).$$

We have

$$\begin{aligned}1 =&\frac{1}{2\pi i}\int_{\partial D\cap\partial C_1}\left[\frac{\zeta-i}{\zeta-z}+\frac{\bar\zeta+i}{\bar\zeta-\bar z}-1+\frac{z(\zeta-i)}{\zeta z-1}+\frac{\bar z(\bar\zeta+i)}{\bar\zeta\bar z-1}-1\right]\frac{d\zeta}{\zeta-i}\\&+\frac{1}{2\pi i}\int_{\partial D\cap\partial C_2}\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar\zeta-i}{\bar\zeta-\bar z}-1+\frac{z(\zeta+i)}{\zeta z-1}+\frac{\bar z(\bar\zeta-i)}{\bar\zeta\bar z-1}-1\right]\frac{d\zeta}{\zeta+i}.\end{aligned}$$

Multiplying this relation by $\gamma(\pm1)$ and subtracting the resulting quantity from $\mathrm{Re}\,w(z)$, for $z\in\partial D\cap\partial C_1$, we get

$$\mathrm{Re}\,w(z)-\gamma(\pm1)=\frac{1}{2\pi i}\int_{\partial D\cap\partial C_1}\tilde\gamma(\zeta)\left[\frac{\zeta-i}{\zeta-z}+\frac{\bar\zeta+i}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta-i}\ ,$$

where $\tilde\gamma(\zeta)=\gamma(\zeta)-\gamma(\pm1)$ and $\tilde\gamma(\pm1)=0$. So $\lim_{z\to\pm1}\mathrm{Re}\,w(z)=\gamma(\pm1)$.

Similarly, for $z\in\partial D\cap\partial C_2$,

$$\mathrm{Re}\,w(z)-\gamma(\pm1)=\frac{1}{2\pi i}\int_{\partial D\cap\partial C_2}\hat\gamma(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar\zeta-i}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta+i},$$

where $\hat\gamma(\zeta)=\gamma(\zeta)-\gamma(\pm1)$ and $\hat\gamma(\pm1)=0$. So $\lim_{z\to\pm1}\mathrm{Re}\,w(z)=\gamma(\pm1)$. □

3 Schwarz problem for the half lens Ω

The point $z\in\Omega$ is reflected at ∂C_1 onto $\frac{i\bar z+1}{\bar z+i}$, and both these points are reflected at ∂C_2 onto the points $\frac{-i\bar z+1}{\bar z-i}$ and $\frac{1}{z}$. Reflection of these four points at the imaginary axis are $-\bar z$, $\frac{iz-1}{z-i}$, $-\frac{1}{\bar z}$, $-\frac{iz+1}{z+i}$.

The Cauchy–Schwarz representation formula is derived by combining the Cauchy–Pompeiu representation formula applied to the points described above.

Theorem 3.1. *Any function $w\in C^1(\Omega;\mathbb{C})\cap C(\overline{\Omega};\mathbb{C})$ can be represented as*

$$\begin{aligned}w(z) =&\frac{1}{2\pi i}\int_{-(\sqrt2-1)i}^{(\sqrt2-1)i}\mathrm{Re}\,w(t)\Big[\frac{2}{t-z}+\frac{2z}{tz-1}+\frac{2(z-i)}{t(z-i)-iz+1}+\frac{2(z+i)}{t(z+i)+iz+1}\Big]dt\\&+\frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1}\mathrm{Re}\,w(\zeta)\Big[\frac{2(\zeta-i)}{\zeta-z}-1+\frac{2z(\zeta-i)}{\zeta z-1}-1+\frac{2(z-i)(\zeta-i)}{\zeta(z-i)-iz+1}-1\\&+\frac{2(z+i)(\zeta-i)}{\zeta(z+i)+iz+1}-1\Big]\frac{d\zeta}{\zeta-i}\\&+\frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2}\mathrm{Re}\,w(\zeta)\Big[\frac{2(\zeta+i)}{\zeta-z}-1+\frac{2z(\zeta+i)}{\zeta z-1}-1+\frac{2(z-i)(\zeta+i)}{\zeta(z-i)-iz+1}-1\\&+\frac{2(z+i)(\zeta+i)}{\zeta(z+i)+iz+1}-1\Big]\frac{d\zeta}{\zeta+i}\\&+\frac{2}{\pi}\int_{\partial\Omega\cap\partial C_1}\mathrm{Im}\,w(\zeta)\frac{d\zeta}{\zeta-i}+\frac{2}{\pi}\int_{\partial\Omega\cap\partial C_2}\mathrm{Im}\,w(\zeta)\frac{d\zeta}{\zeta+i}\\&-\frac{1}{\pi}\int_{\Omega}\Big\{w_{\bar\zeta}(\zeta)\Big[\frac{1}{\zeta-z}+\frac{z}{\zeta z-1}+\frac{z-i}{\zeta(z-i)-iz+1}+\frac{z+i}{\zeta(z+i)+iz+1}\Big]\\&+\overline{w_{\bar\zeta}(\zeta)}\Big[\frac{z-i}{-\bar\zeta(z-i)-iz+1}-\frac{z+i}{\bar\zeta(z+i)-iz-1}-\frac{1}{\bar\zeta+z}-\frac{z}{\bar\zeta z+1}\Big]\Big\}d\xi d\eta,\end{aligned}\qquad(3.1)$$

where $\zeta=\xi+i\eta$.

Proof. The Cauchy–Pompeiu formula (2.2) is applied to $z \in \Omega$, $-\bar{z}$, $\frac{iz-1}{z-i}$, $\frac{i\bar{z}+1}{\bar{z}+i}$, $\frac{1}{z}$, $-\frac{1}{\bar{z}}$, $-\frac{iz+1}{z+i}$, $\frac{-i\bar{z}+1}{\bar{z}-i} \notin \overline{\Omega}$. Then we have

$$w(z) = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{1}{\pi}\int_{\Omega} w_{\bar{\zeta}}(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta - z}, \tag{3.2}$$

$$0 = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta + \bar{z}} - \frac{1}{\pi}\int_{\Omega} w_{\bar{\zeta}}(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta + \bar{z}}, \tag{3.3}$$

$$0 = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{(z-i)\mathrm{d}\zeta}{\zeta(z-i) - iz + 1} - \frac{1}{\pi}\int_{\Omega} w_{\bar{\zeta}}(\zeta)\frac{(z-i)\mathrm{d}\xi\mathrm{d}\eta}{\zeta(z-i) - iz + 1}, \tag{3.4}$$

$$0 = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{(\bar{z}+i)\mathrm{d}\zeta}{\zeta(\bar{z}+i) - i\bar{z} - 1} - \frac{1}{\pi}\int_{\Omega} w_{\bar{\zeta}}(\zeta)\frac{(\bar{z}+i)\mathrm{d}\xi\mathrm{d}\eta}{\zeta(\bar{z}+i) - i\bar{z} - 1}, \tag{3.5}$$

$$0 = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{z\mathrm{d}\zeta}{\zeta z - 1} - \frac{1}{\pi}\int_{\Omega} w_{\bar{\zeta}}(\zeta)\frac{z\mathrm{d}\xi\mathrm{d}\eta}{\zeta z - 1}, \tag{3.6}$$

$$0 = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{\bar{z}\mathrm{d}\zeta}{\zeta\bar{z} + 1} - \frac{1}{\pi}\int_{\Omega} w_{\bar{\zeta}}(\zeta)\frac{\bar{z}\mathrm{d}\xi\mathrm{d}\eta}{\zeta\bar{z} + 1}, \tag{3.7}$$

$$0 = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{(z+i)\mathrm{d}\zeta}{\zeta(z+i) + iz + 1} - \frac{1}{\pi}\int_{\Omega} w_{\bar{\zeta}}(\zeta)\frac{(z+i)\mathrm{d}\xi\mathrm{d}\eta}{\zeta(z+i) + iz + 1}, \tag{3.8}$$

$$0 = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{(\bar{z}-i)\mathrm{d}\zeta}{\zeta(\bar{z}-i) + i\bar{z} - 1} - \frac{1}{\pi}\int_{\Omega} w_{\bar{\zeta}}(\zeta)\frac{(\bar{z}-i)d\mathrm{d}\xi\mathrm{d}\eta}{\zeta(\bar{z}-i) + i\bar{z} - 1}. \tag{3.9}$$

Taking the complex conjugate of (3.3), (3.5), (3.7) and (3.9), where $\bar{z}$ appears, and adding the resulting eight relations, lead to the claimed representation formula. □

The Cauchy-Schwarz representation formula (3.1) serves to solve the Schwarz BVP for the inhomogeneous Cauchy-Riemann equation in the half lens Ω.

Theorem 3.2. *The Schwarz problem*

$$w_{\bar{z}} = f \text{ in } \Omega, \ \mathrm{Re}\, w = \gamma \text{ on } \partial\Omega, \ \gamma\Big(\pm(\sqrt{2}-1)i\Big) = 0,$$

$$\frac{2}{\pi i}\int_{\partial\Omega\cap\partial C_1} \mathrm{Im}\, w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - i} + \frac{2}{\pi i}\int_{\partial\Omega\cap\partial C_2} \mathrm{Im}\, w(\zeta)\frac{\mathrm{d}\zeta}{\zeta + i} = c,$$

with given $f \in L_p(\Omega;\mathbb{C})$, $2 < p$, $\gamma \in C(\partial\Omega;\mathbb{R})$, $c \in \mathbb{R}$ *is uniquely solved by*

$$\begin{aligned} w(z) =& \frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}-1)i} \gamma(t)\Big[\frac{2}{t-z} + \frac{2z}{tz-1} + \frac{2(z-i)}{t(z-i) - iz + 1} + \frac{2(z+i)}{t(z+i) + iz + 1}\Big]\mathrm{d}t \\ &+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\Big[\frac{2(\zeta - i)}{\zeta - z} - 1 + \frac{2z(\zeta - i)}{\zeta z - 1} - 1 + \frac{2(z-i)(\zeta - i)}{\zeta(z-i) - iz + 1} - 1 \\ &+ \frac{2(z+i)(\zeta - i)}{\zeta(z+i) + iz + 1} - 1\Big]\frac{\mathrm{d}\zeta}{\zeta - i} \\ &+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\Big[\frac{2(\zeta + i)}{\zeta - z} - 1 + \frac{2z(\zeta + i)}{\zeta z - 1} - 1 + \frac{2(z-i)(\zeta + i)}{\zeta(z-i) - iz + 1} - 1 \\ &+ \frac{2(z+i)(\zeta + i)}{\zeta(z+i) + iz + 1} - 1\Big]\frac{\mathrm{d}\zeta}{\zeta + i} \\ &+ ic \\ &- \frac{1}{\pi}\int_{\Omega}\Big\{f(\zeta)\Big[\frac{1}{\zeta - z} + \frac{z}{\zeta z - 1} + \frac{z-i}{\zeta(z-i) - iz + 1} + \frac{z+i}{\zeta(z+i) + iz + 1}\Big] \\ &+ \overline{f(\zeta)}\Big[\frac{z-i}{-\bar{\zeta}(z-i) - iz + 1} - \frac{z+i}{\bar{\zeta}(z+i) - iz - 1} - \frac{1}{\bar{\zeta} + z} - \frac{z}{\bar{\zeta}z + 1}\Big]\Big\}\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{3.10}$$

Proof. Let

$$w_0(z) = -\frac{1}{\pi}\int_\Omega \Big\{f(\zeta)\big[\frac{1}{\zeta - z} + \frac{z}{\zeta z - 1} + \frac{z-i}{\zeta(z-i) - iz + 1} + \frac{z+i}{\zeta(z+i)+iz+1}\big] + \overline{f(\zeta)}\big[\frac{z-i}{-\bar\zeta(z-i) - iz + 1} - \frac{z+i}{\bar\zeta(z+i) - iz - 1} - \frac{1}{\bar\zeta + z} - \frac{z}{\bar\zeta z + 1}\big]\Big\}\mathrm{d}\xi\mathrm{d}\eta. \tag{3.11}$$

For $z \in \big(-(\sqrt{2}-1)i, (\sqrt{2}-1)i\big)$, i.e. $z = -\bar z$,

$$w_0(z) = -\frac{1}{\pi}\int_\Omega \Big\{f(\zeta)\big[\frac{1}{\zeta - z} + \frac{z}{\zeta z - 1} + \frac{z-i}{\zeta(z-i) - iz + 1} + \frac{z+i}{\zeta(z+i)+iz+1}\big] - \overline{f(\zeta)}\big[\frac{1}{\bar\zeta - \bar z} + \frac{\bar z}{\bar\zeta\bar z - 1} + \frac{\bar z + i}{\bar\zeta(\bar z + i) + i\bar z + 1} + \frac{\bar z - i}{\bar\zeta(\bar z - i) - i\bar z + 1}\big]\Big\}\mathrm{d}\xi\mathrm{d}\eta.$$

So, $\mathrm{Re}\, w_0(z) = 0$. For $z \in \partial\Omega \cap \partial C_1$ i.e. $(z-i)(\bar z + i) = 2$,

$$w_0(z) = -\frac{1}{\pi}\int_\Omega \Big\{f(\zeta)\big[\frac{1}{\zeta - z} + \frac{z}{\zeta z - 1} + \frac{z-i}{\zeta(z-i) - iz + 1} + \frac{z+i}{\zeta(z+i)+iz+1}\big] - \overline{f(\zeta)}\big[\frac{1}{\bar\zeta - \bar z} + \frac{\bar z}{\bar\zeta\bar z - 1} + \frac{\bar z + i}{\bar\zeta(\bar z + i) + i\bar z + 1} + \frac{\bar z - i}{\bar\zeta(\bar z - i) - i\bar z + 1}\big]\Big\}\mathrm{d}\xi\mathrm{d}\eta.$$

So, $\mathrm{Re}\, w_0(z) = 0$. Similarly, for $z \in \partial\Omega \cap \partial C_2$, i.e. $(z+i)(\bar z - i) = 2$,

$$w_0(z) = -\frac{1}{\pi}\int_\Omega \Big\{f(\zeta)\big[\frac{1}{\zeta - z} + \frac{z}{\zeta z - 1} + \frac{z-i}{\zeta(z-i) - iz + 1} + \frac{z+i}{\zeta(z+i)+iz+1}\big] - \overline{f(\zeta)}\big[\frac{\bar z}{\bar\zeta\bar z - 1} + \frac{1}{\bar\zeta - \bar z} + \frac{\bar z - i}{\bar\zeta(\bar z - i) - i\bar z + 1} + \frac{\bar z + i}{\bar\zeta(\bar z + i) + i\bar z + 1}\big]\Big\}\mathrm{d}\xi\mathrm{d}\eta.$$

So, $\mathrm{Re}\, w_0(z) = 0$. In fact,

$$\begin{aligned}\mathrm{Re}\, w(z) =& \frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}-1)i} \gamma(t)\Big[\frac{1}{t-z} + \frac{1}{\bar t - \bar z} + \frac{z}{tz-1} + \frac{\bar z}{\bar t\bar z - 1} + \frac{z-i}{t(z-i) - iz + 1} \\ &+ \frac{\bar z + i}{\bar t(\bar z + i) + i\bar z + 1} + \frac{z+i}{t(z+i) + iz + 1} + \frac{\bar z - i}{\bar t(\bar z - i) - i\bar z + 1}\Big]\mathrm{d}t \\ &+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\Big[\frac{\zeta - i}{\zeta - z} + \frac{\bar\zeta + i}{\bar\zeta - \bar z} - 1 + \frac{z(\zeta - i)}{\zeta z - 1} + \frac{\bar z(\bar\zeta + i)}{\bar\zeta\bar z - 1} - 1 \\ &+ \frac{(z-i)(\zeta - i)}{\zeta(z-i) - iz + 1} + \frac{(\bar z + i)(\bar\zeta + i)}{\bar\zeta(\bar z + i) + i\bar z + 1} - 1 + \frac{(z+i)(\zeta - i)}{\zeta(z+i) + iz + 1} \\ &+ \frac{(\bar z - i)(\bar\zeta + i)}{\bar\zeta(\bar z - i) - i\bar z + 1} - 1\Big]\frac{\mathrm{d}\zeta}{\zeta - i} \\ &+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\Big[\frac{\zeta + i}{\zeta - z} + \frac{\bar\zeta - i}{\bar\zeta - \bar z} - 1 + \frac{z(\zeta + i)}{\zeta z - 1} + \frac{\bar z(\bar\zeta - i)}{\bar\zeta\bar z - 1} - 1 \\ &+ \frac{(z-i)(\zeta + i)}{\zeta(z-i) - iz + 1} + \frac{(\bar z + i)(\bar\zeta - i)}{\bar\zeta(\bar z + i) + i\bar z + 1} - 1 + \frac{(z+i)(\zeta + i)}{\zeta(z+i) + iz + 1} \\ &+ \frac{(\bar z - i)(\bar\zeta - i)}{\bar\zeta(\bar z - i) - i\bar z + 1} - 1\Big]\frac{\mathrm{d}\zeta}{\zeta + i} \\ &+ \mathrm{Re}\, w_0(z),\end{aligned}$$

where $w_0(z)$ is defined by (3.11). Therefore, on ∂C_1

$$\begin{aligned}
\mathrm{Re}w(z) =& \frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}-1)i} \gamma(t)\Big[\frac{1}{t-z} + \frac{i-z}{t(z-i)-iz+1} + \frac{z}{tz-1} + \frac{-(z+i)}{t(z+i)+iz+1} \\
&+ \frac{z-i}{t(z-i)-iz+1} + \frac{-1}{t-z} + \frac{z+i}{t(z+i)+iz+1} + \frac{-z}{tz-1}\Big]\mathrm{d}t \\
&+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\Big[\frac{\zeta-i}{\zeta-z} + \frac{\bar{\zeta}+i}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta-i)}{\zeta z-1} + \frac{iz-1}{\zeta z-1} - 1 \\
&+ \frac{(z-i)(\zeta-i)}{\zeta(z-i)-iz+1} + \frac{2}{\zeta(z-i)-iz+1} - 1 + \frac{(z+i)(\zeta-i)}{\zeta(z+i)+iz+1} \\
&+ \frac{2iz}{\zeta(z+i)+iz+1} - 1\Big]\frac{\mathrm{d}\zeta}{\zeta-i} \\
&+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\Big[\frac{\zeta+i}{\zeta-z} + \frac{-iz-1}{\zeta z-1} - 1 + \frac{z(\zeta+i)}{\zeta z-1} + \frac{-z-i}{\zeta-z} - 1 \\
&+ \frac{(z-i)(\zeta+i)}{\zeta(z-i)-iz+1} + \frac{2}{\zeta(z+i)+iz+1} - 1 + \frac{(z+i)(\zeta+i)}{\zeta(z+i)+iz+1} \\
&+ \frac{-2iz}{\zeta(z-i)-iz+1} - 1\Big]\frac{\mathrm{d}\zeta}{\zeta+i} \\
&+ \mathrm{Re}w_0(z).
\end{aligned}$$

Also, $\mathrm{Re}w(z)$ on ∂C_1 could be written as

$$\begin{aligned}
\mathrm{Re}w(z) =& \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\left[\frac{\zeta-i}{\zeta-z} + \frac{\bar{\zeta}+i}{\bar{\zeta}-\bar{z}} - 1\right]\frac{\mathrm{d}\zeta}{\zeta-i} \\
=& \frac{1}{2\pi i}\int_{\partial C_1} \Gamma_1(\zeta)\left[\frac{\zeta-i}{\zeta-z} + \frac{\bar{\zeta}+i}{\bar{\zeta}-\bar{z}} - 1\right]\frac{\mathrm{d}\zeta}{\zeta-i},
\end{aligned}$$

where

$$\Gamma_1(\zeta) = \begin{cases} \gamma(\zeta), & \zeta\in\partial\Omega\cap\partial C_1, \\ 0, & \zeta\in\partial C_1\setminus(\partial\Omega\cap\partial C_1). \end{cases}$$

From the properties of the Poisson kernel for C_1, the equality

$$\lim_{z\to\zeta}\mathrm{Re}w(z) = \gamma(\zeta)$$

follows for $\zeta\in\partial\Omega\cap\partial C_1$ up to the corner points $-(\sqrt{2}-1)i$ and 1 of the half lens Ω, because

Γ_1 fails to be continuous there if γ does not vanish at these points. Also, on ∂C_2 we have

$$\begin{aligned}
\mathrm{Re}w(z) =& \frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}-1)i} \gamma(t)\Big[\frac{1}{t-z} + \frac{-(z+i)}{t(z+i)+iz+1} + \frac{z}{tz-1} + \frac{-z+i}{t(z-i)-iz+1} \\
&+ \frac{z-i}{t(z-i)-iz+1} + \frac{-z}{tz-1} + \frac{z+i}{t(z+i)+iz+1} + \frac{-1}{t-z}\Big]\mathrm{d}t \\
&+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\Big[\frac{\zeta-i}{\zeta-z} + \frac{iz-1}{\zeta z-1} - 1 + \frac{z(\zeta-i)}{\zeta z-1} + \frac{-z+i}{\zeta-z} - 1 \\
&+ \frac{(z-i)(\zeta-i)}{\zeta(z-i)-iz+1} + \frac{2iz}{\zeta(z+i)+iz+1} - 1 + \frac{(z+i)(\zeta-i)}{\zeta(z+i)+iz+1} \\
&+ \frac{2}{\zeta(z-i)-iz+1} - 1\Big]\frac{\mathrm{d}\zeta}{\zeta-i} \\
&+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\Big[\frac{\zeta+i}{\zeta-z} + \frac{\bar\zeta-i}{\bar\zeta-\bar z} - 1 + \frac{z(\zeta+i)}{\zeta z-1} + \frac{-iz-1}{\zeta z-1} - 1 \\
&+ \frac{(z-i)(\zeta+i)}{\zeta(z-i)-iz+1} + \frac{-2iz}{\zeta(z-i)-iz+1} - 1 + \frac{(z+i)(\zeta+i)}{\zeta(z+i)+iz+1} \\
&+ \frac{2}{\zeta(z+i)+iz+1} - 1\Big]\frac{\mathrm{d}\zeta}{\zeta+i} \\
&+ \mathrm{Re}w_0(z).
\end{aligned}$$

Also, $\mathrm{Re}w(z)$ on ∂C_2 could be written as

$$\begin{aligned}
\mathrm{Re}w(z) =& \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\left[\frac{\zeta+i}{\zeta-z} + \frac{\bar\zeta-i}{\bar\zeta-\bar z} - 1\right]\frac{\mathrm{d}\zeta}{\zeta+i} \\
=& \frac{1}{2\pi i}\int_{\partial C_2} \Gamma_2(\zeta)\left[\frac{\zeta+i}{\zeta-z} + \frac{\bar\zeta-i}{\bar\zeta-\bar z} - 1\right]\frac{\mathrm{d}\zeta}{\zeta+i},
\end{aligned}$$

where

$$\Gamma_2(\zeta) = \begin{cases} \gamma(\zeta), & \zeta\in\partial\Omega\cap\partial C_2, \\ 0, & \zeta\in\partial C_2\setminus(\partial\Omega\cap\partial C_2). \end{cases}$$

From the properties of the Poisson kernel for C_2, the equality

$$\lim_{z\to\zeta}\mathrm{Re}w(z) = \gamma(\zeta)$$

follows for $\zeta\in\partial\Omega\cap\partial C_2$ up to the corner points 1 and $(\sqrt{2}-1)i$ of the half lens Ω, because Γ_2 fails to be continuous there if γ does not vanish at these points. For $t\in\big(-(\sqrt{2}-1)i,(\sqrt{2}-1)i\big)$, we have

$$\begin{aligned}\mathrm{Re}w(z) =&\frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}-1)i}\gamma(t)\Big[\frac{1}{t-z}+\frac{1}{\bar{t}-\bar{z}}+\frac{z}{tz-1}+\frac{-z}{tz-1}+\frac{z-i}{t(z-i)-iz+1}\\&+\frac{-z+i}{t(z-i)-iz+1}+\frac{z+i}{t(z+i)+iz+1}+\frac{-(z+i)}{t(z+i)+iz+1}\Big]\mathrm{d}t\\&+\frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1}\gamma(\zeta)\Big[\frac{\zeta-i}{\zeta-z}+\frac{2}{\zeta(z-i)-iz+1}-1+\frac{z(\zeta-i)}{\zeta z-1}\\&+\frac{2iz}{\zeta(z+i)+iz+1}-1+\frac{(z-i)(\zeta-i)}{\zeta(z-i)-iz+1}+\frac{-z+i}{\zeta-z}-1\\&+\frac{(z+i)(\zeta-i)}{\zeta(z+i)+iz+1}+\frac{iz-1}{\zeta z-1}-1\Big]\frac{\mathrm{d}\zeta}{\zeta-i}\\&+\frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2}\gamma(\zeta)\Big[\frac{\zeta+i}{\zeta-z}+\frac{2}{\zeta(z+i)+iz+1}-1+\frac{z(\zeta+i)}{\zeta z-1}\\&+\frac{-2iz}{\zeta(z-i)-iz+1}-1+\frac{(z-i)(\zeta+i)}{\zeta(z-i)-iz+1}+\frac{-iz-1}{\zeta z-1}-1\\&+\frac{(z+i)(\zeta+i)}{\zeta(z+i)+iz+1}+\frac{-z-i}{\zeta-z}-1\Big]\frac{\mathrm{d}\zeta}{\zeta+i}\\&+\mathrm{Re}w_0(z).\end{aligned}$$

Also, $\mathrm{Re}w(z)$ for $t\in\big(-(\sqrt{2}-1)i,(\sqrt{2}-1)i\big)$ could be written as

$$\begin{aligned}\mathrm{Re}w(z) =&\frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}-1)i}\gamma(t)\left[\frac{1}{t-z}+\frac{1}{\bar{t}-\bar{z}}-1\right]\mathrm{d}t\\=&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma_3(t)\left[\frac{1}{t-z}+\frac{1}{\bar{t}-\bar{z}}-1\right]\mathrm{d}t,\end{aligned}$$

where

$$\Gamma_3(t)=\begin{cases}\gamma(t), & t\in\big(-(\sqrt{2}-1)i,(\sqrt{2}-1)i\big),\\0, & t\in i\mathbb{R}\setminus\big(-(\sqrt{2}-1)i,(\sqrt{2}-1)i\big).\end{cases}$$

From the properties of the Poisson kernel, we have

$$\lim_{z\to\zeta}\mathrm{Re}w(z)=\gamma(\zeta).$$

Now, we consider the boundary behavior at the corner points $-(\sqrt{2}-1)i$ and $(\sqrt{2}-1)i$. Let

$$w_1(z)=\frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}-1)i}\gamma(t)\Big[\frac{2}{t-z}+\frac{2z}{tz-1}+\frac{2(z-i)}{t(z-i)-iz+1}+\frac{2(z+i)}{t(z+i)+iz+1}\Big]\mathrm{d}t,$$

$$\begin{aligned}w_2(z) =&\frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1}\gamma(\zeta)\Big[\frac{2(\zeta-i)}{\zeta-z}-1+\frac{2z(\zeta-i)}{\zeta z-1}-1+\frac{2(z-i)(\zeta-i)}{\zeta(z-i)-iz+1}-1\\&+\frac{2(z+i)(\zeta-i)}{\zeta(z+i)+iz+1}-1\Big]\frac{\mathrm{d}\zeta}{\zeta-i},\\w_3(z) =&\frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2}\gamma(\zeta)\Big[\frac{2(\zeta+i)}{\zeta-z}-1+\frac{2z(\zeta+i)}{\zeta z-1}-1+\frac{2(z-i)(\zeta+i)}{\zeta(z-i)-iz+1}-1\\&+\frac{2(z+i)(\zeta+i)}{\zeta(z+i)+iz+1}-1\Big]\frac{\mathrm{d}\zeta}{\zeta+i}.\end{aligned}$$

We will prove that $\lim_{z\to\pm(\sqrt{2}-1)i}\mathrm{Re}\,w(z)=\gamma\left(\pm(\sqrt{2}-1)i\right)=0$. We have

$$\begin{aligned}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}-1)i}\gamma(u)\frac{2(z-i)\mathrm{d}u}{u(z-i)-iz+1}&=\int_{-(\sqrt{2}-1)i}^{-(\sqrt{2}+1)i}\gamma\left(\frac{it-1}{t-i}\right)\frac{2(z-i)\mathrm{d}t}{(t-z)(t-i)}\\&=\int_{-(\sqrt{2}+1)i}^{-(\sqrt{2}-1)i}-\gamma\left(\frac{it-1}{t-i}\right)\frac{2\mathrm{d}t}{t-z}\\&\quad+\int_{-(\sqrt{2}+1)i}^{-(\sqrt{2}-1)i}\gamma\left(\frac{it-1}{t-i}\right)\frac{2\mathrm{d}t}{t-i},\end{aligned}$$

and

$$\begin{aligned}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}-1)i}\gamma(u)\frac{2(z+i)\mathrm{d}u}{u(z+i)+iz+1}&=\int_{-(\sqrt{2}-1)i}^{-(\sqrt{2}+1)i}\gamma\left(\frac{it-1}{t-i}\right)\frac{2(-iz+1)\mathrm{d}t}{(tz-1)(t-i)}\\&=\int_{-(\sqrt{2}+1)i}^{-(\sqrt{2}-1)i}-\gamma\left(\frac{it-1}{t-i}\right)\frac{2z\mathrm{d}t}{tz-1}\\&\quad+\int_{-(\sqrt{2}+1)i}^{-(\sqrt{2}-1)i}\gamma\left(\frac{it-1}{t-i}\right)\frac{2\mathrm{d}t}{t-i}.\end{aligned}$$

Thus,

$$\begin{aligned}w_1(z)=&\frac{1}{2\pi i}\int_{-(\sqrt{2}+1)i}^{(\sqrt{2}-1)i}\Gamma_4(t)\frac{2\mathrm{d}t}{t-z}+\frac{1}{2\pi i}\int_{-(\sqrt{2}+1)i}^{(\sqrt{2}-1)i}\Gamma_4(t)\frac{2z\mathrm{d}t}{tz-1}\\&+\frac{1}{\pi i}\int_{-(\sqrt{2}+1)i}^{-(\sqrt{2}-1)i}\gamma\left(\frac{it-1}{t-i}\right)\frac{2\mathrm{d}t}{t-i},\end{aligned}$$

where

$$\Gamma_4(it)=\begin{cases}-\gamma\left(\frac{-t-1}{i(t-1)}\right), & -(\sqrt{2}+1)\le t\le-(\sqrt{2}-1),\\ \gamma(it), & -(\sqrt{2}-1)\le t\le(\sqrt{2}-1).\end{cases}$$

Also,

$$\mathrm{Re}\,w_1(z)=\frac{1}{2\pi i}\int_{-(\sqrt{2}+1)i}^{(\sqrt{2}-1)i}\Gamma_4(t)\left[\frac{1}{t-z}+\frac{1}{\bar{t}-\bar{z}}\right]\mathrm{d}t+\frac{1}{2\pi i}\int_{-(\sqrt{2}+1)i}^{(\sqrt{2}-1)i}\Gamma_4(t)\left[\frac{z}{tz-1}+\frac{\bar{z}}{\bar{t}\bar{z}-1}\right]\mathrm{d}t.$$

So, for $t\in\left(-(\sqrt{2}+1)i,(\sqrt{2}-1)i\right)$,

$$\lim_{z\to t}\mathrm{Re}\,w_1(z)=\Gamma_4(t).$$

In particular,

$$\lim_{z\to-(\sqrt{2}-1)i}\mathrm{Re}\,w_1(z)=\gamma\left(-(\sqrt{2}-1)i\right)=0,$$

because of the continuity of Γ_4 at $-(\sqrt{2}-1)i$.

Similarly,

$$\begin{aligned}w_1(z)=&\frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}+1)i}\Gamma_5(t)\frac{2z\mathrm{d}t}{tz-1}+\frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}+1)i}\Gamma_5(t)\frac{2\mathrm{d}t}{t-z}\\&+\frac{1}{\pi i}\int_{(\sqrt{2}-1)i}^{(\sqrt{2}+1)i}\gamma\left(\frac{-it-1}{t+i}\right)\frac{2\mathrm{d}t}{t+i},\end{aligned}$$

where

$$\Gamma_5(it) = \begin{cases} \gamma(it), & -(\sqrt{2}-1) \le t \le (\sqrt{2}-1), \\ -\gamma\left(\frac{t-1}{it+i}\right), & (\sqrt{2}-1) \le t \le (\sqrt{2}+1). \end{cases}$$

Also,

$$\mathrm{Re}w_1(z) = \frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}+1)i} \Gamma_5(t)\left[\frac{z}{tz-1} + \frac{\bar{z}}{\bar{t}\bar{z}-1}\right]\mathrm{d}t + \frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}+1)i} \Gamma_5(t)\left[\frac{1}{t-z} + \frac{1}{\bar{t}-\bar{z}}\right]\mathrm{d}t.$$

So, for $t \in \big(-(\sqrt{2}-1)i, (\sqrt{2}+1)i\big)$,

$$\lim_{z\to t} \mathrm{Re}w_1(z) = \Gamma_5(t).$$

In particular,

$$\lim_{z\to(\sqrt{2}-1)i} \mathrm{Re}w_1(z) = \gamma\left((\sqrt{2}-1)i\right) = 0,$$

because of the continuity of Γ_5 at $(\sqrt{2}-1)i$.

We have

$$\begin{aligned} w_2(z) =& \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\frac{\zeta+z-2i}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta-i} + \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\frac{-\bar{\zeta}+z-2i}{-\bar{\zeta}-z}\frac{\mathrm{d}(-\bar{\zeta})}{-\bar{\zeta}-i} \\ &+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\frac{\zeta z-2zi+1}{\zeta z-1}\frac{\mathrm{d}\zeta}{\zeta-i} + \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\frac{-\bar{\zeta}z-2zi+1}{-\bar{\zeta}z-1}\frac{\mathrm{d}(-\bar{\zeta})}{-\bar{\zeta}-i} \\ =& \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\frac{\zeta+z-2i}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta-i} - \frac{1}{2\pi i}\int_{\overline{\partial\Omega\cap\partial C_1}} \gamma(-\bar{\zeta})\frac{\zeta+z-2i}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta-i} \\ &+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1} \gamma(\zeta)\frac{\zeta z-2zi+1}{\zeta z-1}\frac{\mathrm{d}\zeta}{\zeta-i} - \frac{1}{2\pi i}\int_{\overline{\partial\Omega\cap\partial C_1}} \gamma(-\bar{\zeta})\frac{\zeta z-2zi+1}{\zeta z-1}\frac{\mathrm{d}\zeta}{\zeta-i} \\ =& \frac{1}{2\pi i}\int_{\partial C_1} \Gamma_6(\zeta)\left[\frac{2(\zeta-i)}{\zeta-z}-1\right]\frac{\mathrm{d}\zeta}{\zeta-i} + \frac{1}{2\pi i}\int_{\partial C_1} \Gamma_6(\zeta)\left[\frac{2z(\zeta-i)}{\zeta z-1}-1\right]\frac{\mathrm{d}\zeta}{\zeta-i}, \end{aligned}$$

where $\overline{\partial\Omega\cap\partial C_1} = \{-\bar{\zeta} :\ \zeta\in\partial\Omega\cap\partial C_1\}$ and

$$\Gamma_6(\zeta) = \begin{cases} \gamma(\zeta), & \zeta\in\partial\Omega\cap\partial C_1, \\ -\gamma(-\bar{\zeta}), & \zeta\in\overline{\partial\Omega\cap\partial C_1}, \\ 0, & \zeta\in\partial C_1\setminus\partial D. \end{cases}$$

Also,

$$\begin{aligned} \mathrm{Re}w_2(z) =& \frac{1}{2\pi i}\int_{\partial C_1} \Gamma_6(\zeta)\left[\frac{\zeta-i}{\zeta-z} + \frac{\bar{\zeta}+i}{\bar{\zeta}-\bar{z}} - 1\right]\frac{\mathrm{d}\zeta}{\zeta-i} \\ &+ \frac{1}{2\pi i}\int_{\partial C_1} \Gamma_6(\zeta)\left[\frac{z(\zeta-i)}{\zeta z-1} + \frac{\bar{z}(\bar{\zeta}+i)}{\bar{\zeta}\bar{z}-1} - 1\right]\frac{\mathrm{d}\zeta}{\zeta-i}. \end{aligned}$$

So, for $\zeta\in\partial D\cap\partial C_1$,

$$\lim_{z\to\zeta} \mathrm{Re}w_2(z) = \Gamma_6(\zeta).$$

In particular,

$$\lim_{z\to-(\sqrt{2}-1)i} \mathrm{Re}w_2(z) = \gamma\left(-(\sqrt{2}-1)i\right) = 0,$$

because of the continuity of Γ_6 at $-(\sqrt{2}-1)i$. Also, we have

$$\lim_{z\to(\sqrt{2}-1)i} \mathrm{Re}\,w_2(z) = 0.$$

Similarly,

$$\begin{aligned}
w_3(z) =& \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\frac{\zeta+z+2i}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta+i} + \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\frac{-\bar\zeta+z+2i}{-\bar\zeta-z}\frac{\mathrm{d}(-\bar\zeta)}{-\bar\zeta+i}\\
&+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\frac{\zeta z+2zi+1}{\zeta z-1}\frac{\mathrm{d}\zeta}{\zeta+i} + \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\frac{-\bar\zeta z+2zi+1}{-\bar\zeta z-1}\frac{\mathrm{d}(-\bar\zeta)}{-\bar\zeta+i}\\
=& \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\frac{\zeta+z+2i}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta+i} - \frac{1}{2\pi i}\int_{\overline{\partial\Omega\cap\partial C_2}} \gamma(-\bar\zeta)\frac{\zeta+z+2i}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta+i}\\
&+ \frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2} \gamma(\zeta)\frac{\zeta z+2zi+1}{\zeta z-1}\frac{\mathrm{d}\zeta}{\zeta+i} - \frac{1}{2\pi i}\int_{\overline{\partial\Omega\cap\partial C_2}} \gamma(-\bar\zeta)\frac{\zeta z+2zi+1}{\zeta z-1}\frac{\mathrm{d}\zeta}{\zeta+i}\\
=& \frac{1}{2\pi i}\int_{\partial C_2} \Gamma_7(\zeta)\left[\frac{2(\zeta+i)}{\zeta-z}-1\right]\frac{\mathrm{d}\zeta}{\zeta+i} + \frac{1}{2\pi i}\int_{\partial C_2} \Gamma_7(\zeta)\left[\frac{2z(\zeta+i)}{\zeta z-1}-1\right]\frac{\mathrm{d}\zeta}{\zeta+i},
\end{aligned}$$

where $\overline{\partial\Omega\cap\partial C_2} = \{-\bar\zeta :\ \zeta\in\partial\Omega\cap\partial C_2\}$ and

$$\Gamma_7(\zeta) = \begin{cases} \gamma(\zeta), & \zeta\in\partial\Omega\cap\partial C_2,\\ -\gamma(-\bar\zeta), & \zeta\in\overline{\partial\Omega\cap\partial C_2},\\ 0, & \zeta\in\partial C_2\setminus\partial D.\end{cases}$$

Also,

$$\begin{aligned}
\mathrm{Re}\,w_3(z) =& \frac{1}{2\pi i}\int_{\partial C_2} \Gamma_7(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar\zeta-i}{\bar\zeta-\bar z}-1\right]\frac{\mathrm{d}\zeta}{\zeta+i}\\
&+ \frac{1}{2\pi i}\int_{\partial C_2} \Gamma_7(\zeta)\left[\frac{z(\zeta+i)}{\zeta z-1}+\frac{\bar z(\bar\zeta-i)}{\bar\zeta\bar z-1}-1\right]\frac{\mathrm{d}\zeta}{\zeta+i}.
\end{aligned}$$

So, for $\zeta\in\partial D\cap\partial C_2$,

$$\lim_{z\to\zeta} \mathrm{Re}\,w_3(z) = \Gamma_7(\zeta).$$

In particular,

$$\lim_{z\to(\sqrt{2}-1)i} \mathrm{Re}\,w_3(z) = \gamma\left((\sqrt{2}-1)i\right) = 0,$$

because of the continuity of Γ_7 at $(\sqrt{2}-1)i$. Also, we have

$$\lim_{z\to-(\sqrt{2}-1)i} \mathrm{Re}\,w_3(z) = 0.$$

Now, we show that $\lim_{z\to 1}\mathrm{Re}w(z)=\gamma(1)$. We have

$$\begin{aligned}1=&\frac{1}{2\pi i}\int_{-(\sqrt{2}-1)i}^{(\sqrt{2}-1)i}\Big[\frac{1}{t-z}+\frac{1}{\bar{t}-\bar{z}}+\frac{z}{tz-1}+\frac{\bar{z}}{\bar{t}\bar{z}-1}+\frac{z-i}{t(z-i)-iz+1}\\&+\frac{\bar{z}+i}{\bar{t}(\bar{z}+i)+i\bar{z}+1}+\frac{z+i}{t(z+i)+iz+1}+\frac{\bar{z}-i}{\bar{t}(\bar{z}-i)-i\bar{z}+1}\Big]\mathrm{d}t\\&+\frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1}\Big[\frac{\zeta-i}{\zeta-z}+\frac{\bar{\zeta}+i}{\bar{\zeta}-\bar{z}}-1+\frac{z(\zeta-i)}{\zeta z-1}+\frac{\bar{z}(\bar{\zeta}+i)}{\bar{\zeta}\bar{z}-1}-1+\frac{(z-i)(\zeta-i)}{\zeta(z-i)-iz+1}\\&+\frac{(\bar{z}+i)(\bar{\zeta}+i)}{\bar{\zeta}(\bar{z}+i)+i\bar{z}+1}-1+\frac{(z+i)(\zeta-i)}{\zeta(z+i)+iz+1}+\frac{(\bar{z}-i)(\bar{\zeta}+i)}{\bar{\zeta}(\bar{z}-i)-i\bar{z}+1}-1\Big]\frac{\mathrm{d}\zeta}{\zeta-i}\\&+\frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2}\Big[\frac{\zeta+i}{\zeta-z}+\frac{\bar{\zeta}-i}{\bar{\zeta}-\bar{z}}-1+\frac{z(\zeta+i)}{\zeta z-1}+\frac{\bar{z}(\bar{\zeta}-i)}{\bar{\zeta}\bar{z}-1}-1+\frac{(z-i)(\zeta+i)}{\zeta(z-i)-iz+1}\\&+\frac{(\bar{z}+i)(\bar{\zeta}-i)}{\bar{\zeta}(\bar{z}+i)+i\bar{z}+1}-1+\frac{(z+i)(\zeta+i)}{\zeta(z+i)+iz+1}+\frac{(\bar{z}-i)(\bar{\zeta}-i)}{\bar{\zeta}(\bar{z}-i)-i\bar{z}+1}-1\Big]\frac{\mathrm{d}\zeta}{\zeta+i}.\end{aligned}$$

Multiplying this relation by $\gamma(1)$ and subtracting the resulting quantity from $\mathrm{Re}w(z)$, for $z\in\partial\Omega\cap\partial C_1$, we get

$$\mathrm{Re}w(z)-\gamma(1)=\frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_1}\tilde{\gamma}(\zeta)\left[\frac{\zeta-i}{\zeta-z}+\frac{\bar{\zeta}+i}{\bar{\zeta}-\bar{z}}-1\right]\frac{\mathrm{d}\zeta}{\zeta-i},$$

where $\tilde{\gamma}(\zeta)=\gamma(\zeta)-\gamma(1)$ and $\tilde{\gamma}(1)=0$. So $\lim_{z\to 1}\mathrm{Re}w(z)=\gamma(1)$.

Similarly, for $z\in\partial\Omega\cap\partial C_2$,

$$\mathrm{Re}w(z)-\gamma(1)=\frac{1}{2\pi i}\int_{\partial\Omega\cap\partial C_2}\hat{\gamma}(\zeta)\left[\frac{\zeta+i}{\zeta-z}+\frac{\bar{\zeta}-i}{\bar{\zeta}-\bar{z}}-1\right]\frac{\mathrm{d}\zeta}{\zeta+i},$$

where $\hat{\gamma}(\zeta)=\gamma(\zeta)-\gamma(1)$ and $\hat{\gamma}(1)=0$. So $\lim_{z\to 1}\mathrm{Re}w(z)=\gamma(1)$. □

References

[1] S.A. Abdymanapov, H. Begehr, G. Harutyunyan, A.B. Tungatarov, *Four boundary value problems for the Cauchy-Riemann equation in a quarter plane*, More Progresses in Anal. In: Begehr H, Nicolosi F, editors. Proc. 5. Intern. ISAAC Congress, Catania, Italy, (2005), Singapore. World Sci. (2009), 1137–1147.

[2] S.A. Abdymanapov, H. Begehr, A.B. Tungatarov, *Some Schwarz problems in a quarter plane*, Eurasian. Math. J. 3 (2005), 22–35.

[3] H. Begehr, *Boundary value problems in complex analysis I*, Bol. Asoc. Math. Venez. 12 (2005), 65–85. (II. Bol. Asoc. Math. Venez. 12 (2005), 217–250.

[4] H. Begehr, *Complex analytic methods for partial differential equations: an introductory text*, World Scientific, Singapore, 1994.

[5] H. Begehr, E.A. Gaertner, *Dirichlet problem for the inhomogeneous polyharmonic equation in the upper half plane*, Georg. Math. J. 14 (2007), no. 1, 33–52.

[6] H. Begehr, G. Harutyunyan, *Complex baundary value problems in a quarter plane*, In: Wang Y. et al., editors. Proc. 13. Intern. Conf. Finite Infinite Complex Anal. Appl. Shantou, China, (2005), NJ. World Sci. (2006), 1–10.

[7] H. Begehr, D. Schmersau, *The Schwarz problem for polyanalytic functions*, Z. Anal. Anwend. 24 (2005), no. 2, 341–351.

[8] H. Begehr, T. Vaitekhovich, *Harmonic boundary value problems in the half disc and half ring*, Funct. Approx. 40 (2009), no. 2, 251–282.

[9] H. Begehr, T. Vaitekhovich, *Schwarz problem in lens and lune*, Complex Var. Elliptic Equ. 59 (2014), no. 1, 76–84.

[10] H. Begehr, T. Vaitekhovich, *Harmonic Dirichlet problem for some equilateral triangle*, Complex Var. Elliptic Equ. 57 (2012), nos. 2-4, 185–196.

[11] S. Burgumbayeva, *Boundary value problems for Tri-harmonic functions in the unit disc*, Ph.D. thesis, FU Berlin, 2009. http://www.diss.fu-berlin.de/diss/receive/FUDISS-thesis-000000012636

[12] E.A. Gaertner, *Basic complex boundary value problems in the upper half plane*, Ph.D. thesis, FU Berlin, 2006. http://www.diss.fu-berlin.de/diss/receive/FUDISS-thesis-000000002129

[13] V. Mityushev, P.M. Adler, *Darcy flow around a two-dimensional permeable lens*, J. Phys. A: Math. Gen. 39 (2006), 3545–3560.

[14] B. Shupeyeva, *Harmonic boundary value problems in a quarter ring domain*, Adv. Pure Appl. Math. 3 (2012), 393–419.

[15] B. Shupeyeva, *Some basic boundary value problems complex partial differential equation in quarter ring and half hexagon*, Ph.D. thesis, FU Berlin, 2013. http://www.diss.fu-berlin.de/diss/receive/FUDISS-thesis-000000094596

[16] N. Taghizadeh, V. Soltani Mohammadi, M. Najand Foumani, *Schwarz boundary value problem in half lens and half lune*, Complex Var. Elliptic Equ. DOI: 10.1080/17476933.2015.1101076 (2015).

[17] T. Vaitekhovich, *Boundary value problems to second order complex partial differential equations in a ring domain*, Sauliai Math. Semin. 2 (2007), 117–146.

[18] T. Vaitekhovich, *Boundary value problems to first order complex partial differential equations in a ring domain*, Integr. Transf. Special Funct. 19 (2008), 211–233.

[19] I.N. Vekua, *Generalized analytic functions*, Oxford, Pergamon Press, 1962.

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