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SOME CLASSES OF STATE IDEALS IN STATE MV-ALGEBRAS

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Key words: obstinate, primary, Boolean state ideals; state locally finite, state local, state simple algebras.

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Abstract. In this paper, we introduce some types of state ideals in state MV-algebras such as: obstinate state ideals, primary state ideals and Boolean state ideals in state MV-algebras. We present some characterizations of them and investigate some relations between them. We consider the quotient algebras induced by these state ideals and prove some related theorems.

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1 Introduction

States on MV-algebras were introduced by Mundici [22] with the intent of measuring the average truth-value of propositions in the Łukasiewicz logic, which are generalizations of probability measures on Boolean algebras, states on MV-algebras have been deeply investigated by Flaminio and Montagna [16] and consequently, state pseudo MV-algebras (or state GMV-algebras) were introduced by Rachunek and Šalounova [24]. The concept of a state BL-algebra was introduced by Ciungu et al. [6] as an extension of the concept of a state MV-algebra. Subsequently, this concept was extended by Dvurecenskij et al. [10] to RL-monoids (not necessarily commutative). In Dvurecenskij [11] there are descriptions of completely subdirectly irreducible state-morphism BL-algebras and this generalizes an analogous result for state-morphism MV-algebras presented in Di Nola and Dvurecenskij [7] and Di Nola et al. [9]. Afterwards, Botur and Dvurecenskij [2] presented a complete description of subdirectly irreducible state BL-algebras and general theory of state-morphism algebras, that is, algebras of general type with state-morphism which is a fixed idempotent endomorphism.

In this paper, we introduce some concepts of state ideals in a state MV-algebra, such as: Boolean state ideals, primary state ideals and obstinate state ideals in a state MV-algebra (A, σ) and give some of their properties. We present characterizations of obstinate state ideals and primary state ideals. We study relations between obstinate state ideals and the other state ideals in state MV-algebra (A, σ). We introduce state local, state locally finite, state chain and state simple MV-algebras. Also, we show that M is a maximal state ideal of (A, σ) if and only if $(A/M, \overline{\sigma})$ is state locally finite, if I is an obstinate state ideal of (A, σ), then $(A/I, \overline{\sigma})$ is a state locally finite MV-algebra, I is a primary state ideal of (A, σ) if and only if $(A/I, \overline{\sigma})$ is state local and if P is a prime state ideal of (A, σ), then $(A/P, \overline{\sigma})$ is a state chain. Finally, we give a characterization of state simple MV-algebras.

Definition 1. [3] An MV-algebra is a structure $(A, \oplus, {}^*, 0)$ where $\oplus$ is a binary operation, * is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in A$:

$(MV1)$ $(A, \oplus, 0)$ is an abelian monoid,
$(MV2)$ $(a^*)^* = a$,
$(MV3)$ $0^* \oplus a = 0^*$,
$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that $1 = 0^*$ and the auxiliary operation $\odot$ is as follow:

$$x \odot y = (x^* \oplus y^*)^*.$$

We say that the element $x \in A$ has order n and we write $ord(x) = n$, if n is the smallest natural number such that $nx = 1$, where $nx := x \oplus \cdots \oplus x$ (n-times). In this case we say that the element x has a finite order, and write $ord(x) < \infty$. An MV-algebra A is locally finite if every non-zero element of A has a finite order. We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Lemma 1.1. [4] *In each MV-algebra, the following relations hold for all $x, y, z \in A$:*
(1) $x \leq y$ if and only if $y^* \leq x^*$,
(2) if $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$,
(3) $x \leq y$ if and only if $x^* \oplus y = 1$ if and if $x \odot y^* = 0$,
(4) $x, y \leq x \oplus y$ and $x \odot y \leq x, y$, $x \leq nx = x \oplus x \oplus \cdots \oplus x$ and $x^n = x \odot x \odot \cdots \odot x \leq x$,
(5) $x \oplus x^* = 1$ and $x \odot x^* = 0$,
(6) if $x \in B(A)$, then $x \wedge y = x \odot y$ and $x \vee y = x \oplus y$, for any $y \in A$,
(7) $nx \wedge my \leq nm(x \wedge y)$,
(8) if $x \leq y$ and $z \leq t$, then $x \oplus z \leq y \oplus t$,
(9) $x \wedge (x_1 \oplus x_2 \oplus \cdots \oplus x_n) \leq (x \wedge x_1) \oplus \cdots \oplus (x \wedge x_n)$,
(10) $(x \odot y^*) \wedge (y \odot x^*) = 0$,
where $B(A)$ is the set of all complemented elements of $L(A)$ such that $L(A)$ is distributive lattice with 0 and 1 on A.

Definition 2. [3] An ideal of an MV-algebra A is a nonempty subset I of A satisfying the following conditions:
$(I1)$ if $x \in I$, $y \in A$ and $y \leq x$ then $y \in I$,
$(I2)$ if $x, y \in I$, then $x \oplus y \in I$.
We denote by $Id(A)$ the set of all ideals of an MV-algebra A.

Definition 3. [4] Let I be an ideal of an MV-algebra A. Then I is a proper ideal if $I \neq A$. A proper ideal I is a prime ideal if and only if for all $x, y \in A$, $x \wedge y \in I$ implies $x \in I$ or $y \in I$.
• [1] An ideal I is called a Boolean ideal if $x \wedge x^* \in I$, for all $x \in A$.

• [1] A proper ideal I is called a primary ideal if for every $a, b \in A$ such that $a \odot b \in I$, there exists an integer $n > 0$ such that $a^n \in I$ or $b^n \in I$.

• [13] A proper ideal I of A is called an obstinate ideal of A if $x, y \notin I$ imply $x \odot y^* \in I$ and $y \odot x^* \in I$, for all $x, y \in A$.

In an MV-algebra M, the distance function is defined as

$$d: \quad M \times M \longrightarrow M, \qquad d(x, y) := (x \odot y^*) \oplus (y \odot x^*).$$

Suppose that I is an ideal of an MV-algebra A. Define $x \sim_I y$ if and only if $d(x, y) \in I$ if and only if $x \odot y^* \in I$ and $y \odot x^* \in I$. Then $\sim_I$ is a congruence relation on A. The set of all congruence classes is denoted by A/I, so $A/I = \{[x] : x \in A\}$, where $[x] = \{y \in A : x \sim_I y\}$. We can easily see that $x \in I$ if and only if $x/I = 0/I$. The MV-algebra operations on A/I, given by $x/I \oplus y/I = (x \oplus y)/I$ and $(x/I)^* = x^*/I$, are well defined. Hence $(A/I, \oplus, *, [0])$ becomes an MV-algebra [4, 23].

Definition 4. [15] A state MV-algebra is a pair (A, σ) such that A is an MV-algebra and σ is a unary operation on A satisfying the following conditions:
(1) $\sigma(1) = 1$,
(2) $\sigma(x^*) = \sigma(x)^*$,
(3) $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$,
(4) $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$.

Lemma 1.2. [15] *In a state MV-algebra (A, σ) the following conditions hold:*
(a) $\sigma(0) = 0$,
(b) if $x \leq y$, *then* $\sigma(x) \leq \sigma(y)$.
(c) $\sigma(x \oplus y) \leq \sigma(x) \oplus \sigma(y)$, *and if* $x \odot y = 0$, *then* $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$,
(d) $\sigma(x \ominus y) \geq \sigma(x) \ominus \sigma(y)$ *and if* $y \leq x$, *then* $\sigma(x \ominus y) = \sigma(x) \ominus \sigma(y)$,
(e) letting $d(x, y) = (x \ominus y) \oplus (y \ominus x)$, *we have that* $d(\sigma(x), \sigma(y)) \leq \sigma(d(x, y))$,
(f) $\sigma(x) \odot \sigma(y) \leq \sigma(x \odot y)$. *Thus if* $x \odot y = 0$, *then* $\sigma(x) \odot \sigma(y) = 0$,
(g) $\sigma(\sigma(x)) = \sigma(x)$,
(h) $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$,
(i) the image $\sigma(A)$ *of* A *under* σ *is the domain of an* MV*-subalgebra of* A.

Definition 5. [15] A σ-ideal (or state ideal) of a state MV-algebra (A, σ) is an MV-ideal closed under σ. We will denote the set of all σ-ideals of (A, σ) by $I_\sigma(A)$.

A proper state ideal of (A, σ) is called a maximal state ideal if it not strictly contained in any proper state ideal of (A, σ).

Let $Ml_\sigma(A)$ be the set of all maximal state ideals of A. We set

$$Rad_\sigma(A) = \bigcap\{I \in Ml_\sigma(A)\}.$$

Lemma 1.3. [7] *A state ideal* I *is a maximal state ideal if and only if for any* $a \notin I$ *there exists an integer* $n > 0$ *such that* $(n\sigma(a))^* = \sigma(a^*)^n \in I$.

Lemma 1.4. [7] *Let* (A, σ) *be a state* MV*-algebra,* I *be a state ideal of* A *and* $a \notin I$. *The state ideal generated by* I *and* a *is the set*

$$(I, a]_\sigma = \{x \in A : x \leq i \oplus n(a \oplus \sigma(a)), i \in I, n \geq 1\}.$$

In particular, $(a]_\sigma = \{x \in A : x \leq n(a \oplus \sigma(a)), n \geq 1\}$.

Definition 6. [14] A proper σ-ideal P of (A, σ) is called a prime σ-ideal (prime state ideal), if for any $a, b \in A$ such that $(a \oplus \sigma(a)) \wedge (b \oplus \sigma(b)) \in P$, $a \in P$ or $b \in P$.

2 Obstinate state ideals of a state MV-algebra

We firstly present some definitions related to state ideals of a state MV-algebra.

Definition 7. A proper ideal I of A is called an obstinate state ideal of A if $x, y \notin I$ imply $\sigma(x) \odot \sigma(y)^* \in I$ and $\sigma(y) \odot \sigma(x)^* \in I$, for all $x, y \in A$.

The following example shows that there exist obstinate state ideals and a state ideal may not be an obstinate state ideal of (A, σ).

Example 1. Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [23].
(i) Define

$$\sigma(x) = \begin{cases} 1 & \text{if } x = 1 \\ b & \text{if } x = a \\ a & \text{if } x = b \\ 0 & \text{if } x = 0 \end{cases}$$

We can easily prove that (A, σ) is a state MV-algebra, $I = \{0\}$ and all A are state ideals of (A, σ). Since $b = b \odot b = \sigma(a) \odot \sigma(b)^* \notin I$ and $a = a \odot a = \sigma(b) \odot \sigma(a)^* \notin I$, hence I is not obstinate state ideal of (A, σ).
(ii) Define

$$\sigma(x) = \begin{cases} 1 & \text{if } x = 1, \quad \text{or } a \\ 0 & \text{if } x = 0, \quad \text{or } b \end{cases}$$

We can easily show that (A, σ) is a state MV-algebra. Also, $I_1 = \{0\}$, $I_2 = \{0, b\}$ and all A are state ideals of (A, σ). We can check that $I_2 = \{0, b\}$ is an obstinate state ideal of (A, σ).

In the following proposition, we give necessary and sufficient conditions on a proper state ideal to be an obstinate state ideal.

Proposition 2.1. *A proper state ideal I of A is an obstinate state ideal if and only if for any $x \in A$ if $x \notin I$ there exists $n \geq 1$ such that $n\sigma(x)^* \in I$.*

Proof. Suppose that I is a proper obstinate state ideal and $x \notin I$. Since $1 \notin I$, then $0 = \sigma(x) \odot \sigma(1)^* \in I$ and $\sigma(x)^* = \sigma(1) \odot \sigma(x)^* \in I$. So $n\sigma(x)^* \in I$, for $n = 1$.
Conversely, let $x, y \notin I$. We show that $\sigma(x) \odot \sigma(y)^* \in I$ and $\sigma(y) \odot \sigma(x)^* \in I$.
By hypothesis $n\sigma(x)^* \in I$ and $m\sigma(y)^* \in I$, for some $n, m \geq 1$. We know that $\sigma(x)^* \leq n\sigma(x)^*$ and $\sigma(y)^* \leq n\sigma(y)^*$. By ideal property $\sigma(x)^* \in I$ and $\sigma(y)^* \in I$. Since $\sigma(y) \odot \sigma(x)^* \leq \sigma(x)^*$ and $\sigma(x) \odot \sigma(y)^* \leq \sigma(y)^*$, then $\sigma(x) \odot \sigma(y)^* \in I$ and $\sigma(y) \odot \sigma(x)^* \in I$. □

Theorem 2.1. *Let I be an obstinate state ideal of (A, σ). Then I is a maximal state ideal of (A, σ).*

Proof. Let I be an obstinate state ideal which is not a maximal. So there exists a proper state ideal J such that $I \subset J$. Suppose that $a \in J \setminus I$. Then by Proposition 2.1, $n\sigma(a)^* \in I$ for some $n \geq 1$. We know that $\sigma(a)^* \leq n\sigma(a)^*$. By the ideal property $\sigma(a)^* \in I$ and also $\sigma(a)^* \in J$. Since $a \in J$, so $\sigma(a) \in J$, hence $\sigma(a) \oplus \sigma(a)^* = 1 \in J$, which is a contradiction. □

In the following example we show that the converse of the above theorem may not hold.

Example 2. In Example 1 (i), we can check that $I = \{0\}$ is a maximal state ideal. But I is not an obstinate state ideal of (A, σ).

Lemma 2.1. *Let I be a proper state ideal of (A, σ). Then I is an obstinate state ideal if and only if either $x \in I$ or $\sigma(x)^* \in I$ for all $x \in A$.*

Proof. Assume that I is an obstinate state ideal and $x \notin I$. By Proposition 2.1, we get that $n\sigma(x)^* \in I$, for some $n \geq 1$. We know that $\sigma(x)^* \leq n\sigma(x)^*$, then $\sigma(x)^* \in I$ and obtain the result.
Conversely, let $x \notin I$. We need to show that $n\sigma(x)^* \in I$, for some $n \geq 1$. By hypothesis, we get that $1\sigma(x)^* \in I$. Hence I is an obstinate state ideal of (A, σ). □

Definition 8. A state ideal I of (A, σ) is called Boolean state ideal of (A, σ) if $(x \oplus \sigma(x)) \wedge (x^* \oplus \sigma(x^*)) \in I$.

Definition 9. A state MV-algebra (A, σ) is called a state Boolean algebra if for $x, y \in A$, $\sigma(x) \wedge \sigma(x^*) = 0$ or $\sigma(x) \vee \sigma(x^*) = 1$.

The following example shows that there exist a Boolean state ideals and a state ideal may not be a Boolean state of (A, σ).

Example 3. Let $A = \{0, a, b, c, d, 1\}$, where $0 < a, b < c < 1$ and $0 < b < d < 1$. Define $\oplus$, $\odot$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, *, 0, 1)$ is an MV-algebra [18]. Define

$$\sigma(x) = \begin{cases} 0 & \text{if } x = 0, b, \text{ or } d \\ 1 & \text{if } x = 1, c, \text{ or } a \end{cases}$$

We can easily check that (A, σ) is a state MV-algebra and $I_1 = \{0\}$, $I_2 = \{0, b, d\}$ and A are state ideals of (A, σ). Routine calculation shows that I_2 is a Boolean state ideal of (A, σ) but $\{0\}$ is not Boolean state ideal of (A, σ), because $(c \oplus \sigma(c)) \wedge (b \oplus \sigma(b)) = b \notin \{0\}$.

Proposition 2.2. *If (A, σ) is state MV-algebra, then $(A/I, \overline{\sigma})$ is a state MV-algebra, where $\overline{\sigma}(x/I) = \sigma(x)/I$.*

Proof. First, we show that $\overline{\sigma}$ is well defined. Suppose that $x/I = y/I$. Then $d(x, y) \in I$. Since I is a state ideal of (A, σ), $\sigma(d(x, y)) \in I$. It follows from Lemma 1.2 (e) that $d(\sigma(x), \sigma(y)) \leq \sigma(d(x, y)) \in I$, hence $d(\sigma(x), \sigma(y)) \in I$. Thus $\sigma(x)/I = \sigma(y)/I$. Therefore $\overline{\sigma}(x/I) = \overline{\sigma}(y/I)$.
Now, we prove that $(A/I, \overline{\sigma})$ is a state MV-algebra. Since (A, σ) is a state MV-algebra, we have
(1) $\overline{\sigma}(1/I) = \sigma(1)/I = 1/I$.
(2) $\overline{\sigma}((x/I)^*) = \overline{\sigma}(x^*/I) = \sigma(x^*)/I = (\sigma(x))^*/I = (\sigma(x)/I)^* = (\overline{\sigma}(x/I))^*$,
(3) $\overline{\sigma}((x \oplus y)/I) = \sigma(x \oplus y)/I = (\sigma(x) \oplus \sigma(y \ominus (x \odot y)))/I = \sigma(x)/I \oplus \sigma(y \ominus (x \odot y))/I = \overline{\sigma}(x/I) \oplus \overline{\sigma}((y \ominus (x \odot y))/I)$,
(4) $\overline{\sigma}(\overline{\sigma}(x/I) \oplus \overline{\sigma}(y/I)) = \overline{\sigma}(\sigma(x)/I \oplus \sigma(y)/I) = \overline{\sigma}((\sigma(x) \oplus \sigma(y))/I) = (\sigma(\sigma(x) \oplus \sigma(y)))/I = (\sigma(x) \oplus \sigma(y))/I = \sigma(x)/I \oplus \sigma(y)/I = \overline{\sigma}(x/I) \oplus \overline{\sigma}(y/I)$. □

Lemma 2.2. *If I is a Boolean state ideal of (A, σ), then $(A/I, \overline{\sigma})$ is a state Boolean algebra.*

Proof. Let I be a state Boolean ideal. Then $(x \oplus \sigma(x)) \wedge (x^* \oplus \sigma(x^*)) \in I$. It follows that $\sigma(x)/I \wedge \sigma(x^*)/I \leq (x \oplus \sigma(x))/I \wedge (x^* \oplus \sigma(x^*))/I = 0/I$. Hence $\sigma(x)/I \wedge \sigma(x^*)/I = 0/I$. Thus $(A/I, \overline{\sigma})$ is a state Boolean algebra. □

Theorem 2.2. *Let I be a prime and Boolean state ideal of (A, σ). Then I is an obstinate state ideal of (A, σ).*

Proof. Let I be a prime and Boolean state ideal of (A, σ). Then we have $(x \oplus \sigma(x) \wedge (x^* \oplus \sigma(x)^*) \in I$, for any $x \in A$. Since I is prime state, it follows that $x \in I$ or $x^* \in I$. So $x \in I$ or $\sigma(x)^* \in I$. By Lemma 2.1, I is an obstinate state ideal of (A, σ). □

Corollary 2.1. (*Extension property for obstinate state ideals*) *Suppose that I and J are two proper state ideals such that $I \subseteq J$. If I is an obstinate state ideal, then J is also an obstinate state ideal of (A, σ).*

Proof. Let I be an obstinate state ideal and $I \subseteq J$. Then by Theorem 2.1, I is a maximal state ideal. Since J is a proper ideal, we get that $I = J$. Hence J is an obstinate state ideal of (A, σ). □

Remark 1. Let I and J be state ideals of A. We have

$$I \vee J = (I \cup J] = \{a \in A : a \leq b \oplus c, \quad \text{for some } b \in I \text{ and } c \in J\}.$$

It is a state ideal of (A, σ). If I or J is an obstinate state ideal, then by Corollary 2.1, we get that $I \vee J$ is an obstinate state ideal (A, σ).

Lemma 2.3. $\{0\}$ *is an obstinate state ideal of (A, σ) if and only if every state ideal I of (A, σ) is an obstinate state ideal.*

Proof. Suppose that I is a arbitrary state ideal of (A, σ). Since $\{0\} \subseteq I$ and $\{0\}$ is an obstinate state, then by Corollary 2.1, I is an obstinate state ideal of (A, σ). The converse is clear. □

Definition 10. A state MV-algebra (A, σ) is called state locally finite if for every non-zero element $a \in A$, $\sigma(a)$ has a finite order.

Example 4. *State MV-algebra (A, σ) of Example 1 (i) is state locally finite.*

Theorem 2.3. *M is a maximal state ideal of (A, σ) if and only if $(A/M, \overline{\sigma})$ is state locally finite.*

Proof. Let M is a maximal state ideal of (A, σ). We prove that for every nonzero element a/M, $\overline{\sigma}(a/M)$ has a finite order. Let $a/M \neq 0/M$. Then $a \notin M$. So there is $n \in \mathbb{N}$ such that $(n\sigma(a))^* = \sigma(a^*)^n \in M$. We deduce that $\sigma(a^*)^n/M = 0/M$, so $(\sigma(a^*)^n/M)^* = 1/M$. We obtain $n\sigma(a)/M = 1/M$, hence $n\overline{\sigma}(a/M) = 1/M$. Hence $\overline{\sigma}(a/M)$ has a finite order. Thus $(A/M, \sigma)$ is state locally finite.
Conversely, let $I \neq M$ be a state ideal of (A, σ) such that $M \subset I$ and consider $a \in I \setminus M$. Since $a/M \neq 0/M$, hence we have $n\overline{\sigma}(a/M) = 1/M$, for some $n \in \mathbb{N}$, so $n\sigma(a)/M = 1/M$, hence $(n\sigma(a))^*/M = 0/M$, so $(n\sigma(a))^* \in M \subseteq I$. We have $n\sigma(a), (n\sigma(a))^* \in I$. Thus $I = A$ and M is a maximal state ideal of (A, σ). □

Corollary 2.2. *Let I be an obstinate state ideal of (A, σ). Then $(A/I, \overline{\sigma})$ is state locally finite.*

Proof. Using Theorem 2.1, we obtain I is a maximal state ideal of (A, σ). Also, by Theorem 2.3, we deduce that A/I is state locally finite. □

Lemma 2.4. *If $\{0\}$ is an obstinate state ideal of (A, σ), then (A, σ) is a state locally finite MV-algebra.*

Proof. Suppose that $\{0\}$ is an obstinate state ideal of (A, σ). It follows that from Theorem 2.1, $\{0\}$ is a maximal state ideal of A. Hence $A/\{0\} \simeq A$ is state locally finite. $\square$

In the following example, we show that the converse of the above lemma is not true.

Example 5. Let $A = \{0, 1, 2\}$, where $0 < 1 < 2$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$\oplus$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

$*$	0	1	2
	2	1	0

Then $(A, \oplus, *, 0, 2)$ is a locally finite MV-algebra. Let σ be the identity on A. Then (A, σ) is a state MV-algebra. But $I = \{0\}$ is not an obstinate state ideal of (A, σ), since $2 \odot 1^* = 2 \odot 1 = 1 \notin I$.

We recall that An MV-algebra A is said to be state semisimple if and only if it is nontrivial and $Rad_\sigma(A) = \{0\}$ [14].

Corollary 2.3. *If $\{0\}$ is an obstinate state ideal of (A, σ), then (A, σ) is a state semisimple .*

Proof. Since $\{0\}$ is an obstinate state ideal of (A, σ), it follows that from Theorem 2.1, $\{0\}$ is a maximal state ideal of (A, σ). Hence $Rad_\sigma(A) = \{0\}$. Thus (A, σ) is a state semisimple MV-algebra. $\square$

In the following example, we show that the converse of the above corollary may not hold.

Example 6. In Example 1 (i), we have $I = \{0\}$ is a state ideal of (A, σ) such that $Rad_\sigma(A) = \{0\}$. Hence (A, σ) is a state semisimple MV-algebra but $I = \{0\}$ is not an obstinate state ideal of (A, σ).

Theorem 2.4. *Let I be a state ideal of (A, σ). Then I is an obstinate state ideal if and only if every ideal of A/I is an obstinate state ideal.*

Proof. Assume that I is an obstinate state ideal of (A, σ). Let $x/I \notin \{[0]\}$, from Lemma 2.1, it is suffices to show $(\sigma(x)/I)^* \in \{[0]\}$. Since $x/I \notin \{[0]\}$, $x/I \neq 0/I$, hence $x \notin I$. We apply the hypothesis and obtain $\sigma(x^*) \in I$, then $\sigma(x^*) = d(\sigma(x^*), 0) \in I$. On the other hand $\sigma(x^*)/I = 0/I$ or $\overline{\sigma}((x/I)^*) = \sigma(x^*)/I \in \{[0]\}$. Hence $\{[0]\}$ is an obstinate state ideal of A/I. Hence by Lemma 2.3, we coclude that every ideal of A/I is an obstinate state ideal.
Conversely, assume that every state ideal of the quotient algebra $(A/I, \overline{\sigma})$ is an obstinate state ideal and $x \in A$ such that $x \notin I$.
We must show that $\sigma(x^*) \in I$. By hypothesis, we get that $x/I \neq 0/I$, hence $x/I \notin \{[0]\}$. Since $\{[0]\}$ is a state ideal of $(A/I, \overline{\sigma})$. By hypothesis, $\{[0]\}$ is an obstinate state ideal. Therefore $\overline{\sigma}(x/I)^* = \sigma(x^*)/I \in \{[0]\}$, then $\sigma(x^*)/I = 0/I$. So $\sigma(x^*) \in I$. Hence I is an obstinate state ideal of (A, σ). $\square$

Definition 11. A state MV-algebra (A, σ) is called local if it has only a maximal state ideal.

Definition 12. A proper σ-ideal I of (A, σ) is called a primary state ideal if for every $a, b \in A$ for which $a \odot b \in I$, there is $n \geq 1$ such that $\sigma(a)^n \in I$ or $\sigma(b)^n \in I$.

Example 7. (i) Consider state MV-algebra of Example 3. Since $I_2 = \{0, b, d\}$ is only maximal state ideal of (A, σ), hence (A, σ) is a local state MV-algebra. We can easily show that I_2 is a primary state ideal of (A, σ).
(ii) Consider the state MV-algebra in Example 1 (i). $I = \{0\}$ is not a primary state ideal of (A, σ), because $a \odot b = 0 \in I$ but $b = \sigma(a)^n \notin I$ and $a = \sigma(b)^n \notin I$.

Proposition 2.3. *A state MV-algebra (A, σ) is local if and only if $ord\sigma(x) < \infty$ or $ord(\sigma(x^*)) < \infty$, for every $x \in A$.*

Proof. Let (A, σ) be local. Then it has only a maximal state ideal I. Let $x \in A$ and $ord(\sigma(x)) = ord(\sigma(x^*)) = \infty$. If $(x]_\sigma = A$, then there is $n \geq 1$ such that $1 = n(x \oplus \sigma(x))$, so $1 = \sigma(1) = \sigma(n(x \oplus \sigma(x))$, and since $1 = \sigma(n(x \oplus \sigma(x)) \leq 2n\sigma(x)$, it follows that $2n\sigma(x) = 1$, which is a contradiction. Thus $(x]_\sigma$ is proper. Similarly, $(x^*]_\sigma$ is proper. Then $(x]_\sigma, (x^*]_\sigma \subseteq I$, so $x, x^* \in I$, which is a contradiction. Conversely, suppose that there are $I_1, I_2 \in Ml_\sigma(A)$, $I_1 \neq I_2$ and let $a \in I_1 \setminus I_2$, for example. Then from Lemma 1.3, there is $n \geq 1$ such that $(n\sigma(a))^* \in I_2$, so $\sigma(n\sigma(a))^*) \in I_2$. Let $x = n\sigma(a)$. Since $\sigma(x^*) \in I_2$, we deduce that $ord(\sigma(x^*)) = ord(\sigma(x)^*) = \infty$ and from hypothesis, it follows that $ord(\sigma(x)) < \infty$, so there is $m \geq 1$ such that $m\sigma(x) = 1$, that is, $m\sigma(n\sigma(a)) = 1$. Since $m\sigma(n\sigma(a)) \leq m(n\sigma(\sigma(a))$, we get that $mn\sigma(a) = 1$. But $a \in I_1$, thus $1 = mn\sigma(a) \in I_1$, which is a contradiction. Thus (A, σ) has only a maximal state ideal and so (A, σ) is local. □

Theorem 2.5. *Let (A, σ) be a state MV-algebra. Then the following statements are equivalent: (i) (A, σ) is local, (ii) every proper σ-ideal of (A, σ) is a primary σ-ideal.*

Proof. $(i) \Rightarrow (ii)$ Suppose that (A, σ) is local and let I_0 be the only maximal σ-ideal, let I be a proper σ-ideal of (A, σ) and $a, b \in A$ such that $a \odot b \in I$. Since $I \subseteq I_0$, it follows that $a \odot b \in I_0$. So $(a \odot b)^* \notin I_0$, thus $a^* \notin I_0$ or $b^* \notin I_0$, (because if $a^* \in I_0$, then $b \leq a^* \oplus (a \odot b) = a^* \vee b \in I_0$, so $b \in I_0$. Hence $b^* \notin I_0$). Suppose that $a^* \notin I_0$. Then $I_0 \subsetneqq (a^*]_\sigma$, so $(a^*]_\sigma = A$, so there is $n \geq 1$ such that $1 = n(a^* \oplus \sigma(a^*))$. Since $1 = \sigma(1) = \sigma(n(a^* \oplus \sigma(a^*)) \leq 2n\sigma(a^*)$, we conclude that $2n\sigma(a^*) = 1$, that is $(2n\sigma(a^*))^* = 0 \in I$, hence $(\sigma(a))^{2n} \in I$. Similarly, if $b^* \notin I_0$, there is $n \geq 1$ such that $(\sigma(b))^{2n} = 0 \in I$. Thus I is a primary σ-ideal.
$(ii) \Rightarrow (i)$ Let $I = \{0\}$ be a proper σ-ideal of (A, σ) and $x \in A$. Then $x \odot x^* = 0 \in I$, so there is $n \geq 1$ such that $\sigma(x)^n \in I$ or $\sigma(x^*)^n \in I$, that is $(\sigma(x)^n)^* = 1$ or$(\sigma(x^*)^n)^* = 1$. These mean that $n\sigma(x^*) = 1$ or $n\sigma(x) = 1$. Thus $ord(\sigma(x)) < \infty$ or $ord(\sigma(x)^*) = ord(\sigma(x^*)) < \infty$. It follows by Proposition 2.3 that (A, σ) is local. □

Theorem 2.6. *I is a primary state ideal of (A, σ) if and only if $(A/I, \overline{\sigma})$ is state local.*

Proof. Let I be a primary state ideal. By Proposition 2.3, it is sufficient to show that $ord(\overline{\sigma}(x/I)) < \infty$ or $ord(\overline{\sigma}(x^*/I)) < \infty$, for every $x/I \in A/I$. Suppose that there exists $a/I \in A/I$, such that $ord(\overline{\sigma}(a/I)) = ord(\overline{\sigma}(a^*/I)) = \infty$. We have $a \odot a^* = 0 \in I$ and since I is a primary state ideal of (A, σ), there is $n \geq 1$ such that $\sigma(a)^n \in I$ or $\sigma(a^*)^n \in I$. Hence $\sigma(a)^n/I = 0/I$ or $\sigma(a^*)^n/I = 0/I$. It follows that $n\overline{\sigma}(a^*/I) = n\sigma(a^*)/I = (\sigma(a)^n)^*/I = 1/I$ or $n\overline{\sigma}(a/I) = n\sigma(a)/I = (\sigma(a^*)^n)^*/I = 1/I$, which is a contradiction.

Conversely, let $(A/I, \overline{\sigma})$ be a locall state MV-algebra and $a \odot b \in I$ such that $\sigma(a)^n \notin I$, for all $n \in \mathbb{N}$. Hence $\sigma(a)^n/I \neq 0/I$, for all $n \in \mathbb{N}$. Thus $(\sigma(a)^n/I)^* \neq 1/I$. It follows that $n\overline{\sigma}(a^*/I) = n\sigma(a^*)/I \neq 1/I$, for all $n \in \mathbb{N}$, and so $ord(\overline{\sigma}(a^*/I)) = \infty$. Since $(A/I, \overline{\sigma})$ is a local state MV-algebra, by Proposition 2.3, we conclude that $ord(\overline{\sigma}(a/I)) < \infty$. This means there exists $m \in \mathbb{N}$ such that $m\overline{\sigma}(a/I) = 1/I$. Hence $m\sigma(a)/I = 1/I$ and so$(m\sigma(a)/I)^* = 0/I$. It follows that $\sigma(a^*)^m/I = 0/I$. We get $\sigma(a^*)^m \in I$, for some $m \in \mathbb{N}$. Thus I is a primary state ideal of (A, σ). □

Theorem 2.7. *If P is an obstinate state ideal of (A, σ), then P is a primary state ideal of (A, σ).*

Proof. Let $a \odot b \in P$ such that for every $n \geq 1$, $\sigma(a)^n \notin P$. Since P is an obstinate state ideal and we have $a \notin P$ and $1 \notin P$, so $\sigma(1) \odot \sigma(a)^* \in P$ and $\sigma(a) \odot \sigma(1)^* \in P$. Hence $\sigma(a)^* \in P$, for every $n \geq 1$. By Lemma 1.2 (f), we have $\sigma(a) \odot \sigma(b) \leq \sigma(a \odot b) \in P$. Hence $\sigma(a) \odot \sigma(b) \in P$ and so, $\sigma(a^*) \oplus (\sigma(a) \odot \sigma(b) \in P$. On the other hand $\sigma(b) \leq \sigma(a)^* \vee \sigma(b) \in P$. Hence $\sigma(b)^1 \in P$, for $n = 1$. Therefore P is a primary state ideal of (A, σ). $\square$

The following example shows that a primary state ideal may not be an obstinate state ideal.

Example 8. Let $A = \{0, a, b, c, d, 1\}$. where $0 < a, c < d < 1$ and $0 < a < b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra. Suppose that σ is identity, and so (A, σ) is a state MV-algebra. It is clear that $I = \{0, c\}$ is a primary state ideal but I is not an obstinate state ideal, since if $a, b \notin I$, we have $b \odot a^* = b \odot d = a \notin I$.

Definition 13. A state MV-algebra (A, σ) is called a state chain, if for any $x, y \in A$, $\sigma(x) \leq \sigma(y)$ or $\sigma(y) \leq \sigma(x)$.

Theorem 2.8. *If P is a prime state ideal of (A, σ), then $(A/P, \overline{\sigma})$ is a state chain.*

Proof. Let P be a prime state ideal of (A, σ). Let $x/P, y/P \in A/P$. We show that $\overline{\sigma}(x/P) \leq \overline{\sigma}(y/P)$ or $\overline{\sigma}(y/P) \leq \overline{\sigma}(x/P)$. This means $\sigma(x)/P \leq \sigma(y)/P$ or $\sigma(y)/P \leq \sigma(x)/P$, that is $\sigma(x) \odot \sigma(y)^* \in P$ or $\sigma(y) \odot \sigma(x)^* \in P$.

Consider $a = \sigma(x) \odot \sigma(y)^*$ and $b = \sigma(y) \odot \sigma(x)^*$. By Lemma 1.2 (h), we can easily show that $\sigma(a) = a$ and $\sigma(b) = b$, and so $(a \oplus \sigma(a)) \wedge (b \oplus \sigma(b)) = 2a \wedge 2b \leq 4(a \wedge b) = 0 \in P$, by hypothesis, we obtain $a \in P$ or $b \in P$, that is $\sigma(x) \odot \sigma(y)^* \in P$ or $\sigma(y) \odot \sigma(x)^* \in P$. This means $\sigma(x)/P \leq \sigma(y)/P$ or $\sigma(y)/P \leq \sigma(x)/P$. Thus $(A/P, \overline{\sigma})$ is a state chain. $\square$

Proposition 2.4. *Let (A, σ) be a state MV-algebra.*
(i) If I is an obstinate ideal of $\sigma(A)$, then $\sigma^{-1}(I)$ is an obstinate state of (A, σ).
(ii) If I is an obstinate state ideal of (A, σ), then $\sigma(I)$ is an obstinate ideal of $\sigma(A)$.

Proof. (1) If I is an ideal of $\sigma(A)$, then by Proposition 4.6 in [7], it is proved that $\sigma^{-1}(I)$ is a state ideal of A.
Now, suppose that I is an obstinate ideal of $\sigma(A)$. Let $a \notin \sigma^{-1}(I)$, so $\sigma(a) \notin I$. Since I is an obstinate ideal, we have $\sigma(\sigma(a)^*) = \sigma(\sigma(a))^* = \sigma(a)^* \in I$. Hence $\sigma(a)^* \in \sigma^{-1}(I)$. Thus $\sigma^{-1}(I)$ is an obstinate state ideal of (A, σ).
(2) Let I be a state obstinate ideal of (A, σ). By Proposition 4.6, [7], we have $\sigma(I) = I \cap \sigma(A)$. Let $\sigma(a) \notin \sigma(I)$. Since $\sigma(a) \in \sigma(A)$ and $\sigma(I) = I \cap \sigma(A)$, so $\sigma(a) \notin I$, hence $a \notin I$. Also, since I is an obstinate state ideal of (A, σ), so $\sigma(a)^* \in I$ and hence $\sigma(a)^* = \sigma(a^*) = \sigma(\sigma(a^*)) = \sigma(\sigma(a)^*) \in \sigma(I)$. Thus $\sigma(I)$ is an obstinate state ideal of $\sigma(A)$. $\square$

Now, we introduce another kind of state MV-algebras and give a characterization of it.

Definition 14. A state MV-algebra (A, σ) is called state simple if it has exactly two ideals:$\{0\}$ and A.

We recall that for any state operator σ of A, the kernel of σ is the set $Ker(\sigma) = \{x \in A | \sigma(x) = 0\}$. A state operator σ on A is called faithful if $Ker(\sigma) = \{0\}$.

Theorem 2.9. *Let (A, σ) be a state MV-algebra. Then the following are equivalent:*
(1) (A, σ) *is state simple,*
(2) $\sigma(A)$ *is simple and* σ *is faithful.*

Proof. (1) $\Rightarrow$ (2) Let I be an ideal of $\sigma(A)$ and $I \neq \{0\}$. It follows by Proposition 2.4 (1) that $\sigma^{-1}(I)$ is a state ideal of (A, σ). Since (A, σ) is state simple, we have $\sigma^{-1}(I) = \{0\}$ or $\sigma^{-1}(I) = A$. We know that $I \subseteq \sigma^{-1}(I)$ (if $x \in I$, then $\sigma(x) = x$, that is, $x \in \sigma^{-1}(I)$), we obtain $\sigma^{-1}(I) \neq \{0\}$. Thus $\sigma^{-1}(I) = A$. Then $1 \in \sigma^{-1}(I)$, that is $1 = \sigma(1) \in I$. So we get $I = \sigma(A)$. Thus $\sigma(A)$ is simple.
Now, we prove that σ is faithful. We know that $Ker(\sigma)$ is a state ideal of (A, σ) and $Ker(\sigma) \neq A$. It follows that $Ker(\sigma) = \{0\}$. Thus σ is faithful.
(2) $\Rightarrow$ (1) Let I be a state ideal of (A, σ) and $I \neq \{0\}$. By Proposition 2.4 (2), we have $\sigma(I)$ is an ideal of $\sigma(A)$. Since $\sigma(A)$ is simple, we obtain that $\sigma(I) = \{0\}$ or $\sigma(I) = \sigma(A)$. Since σ is faithful and $I \neq \{0\}$, we obtain $\sigma(I) \neq \{0\}$. Thus $\sigma(I) = \sigma(A)$. Then $1 \in \sigma(I)$, that is, $1 \in I$. It follows that $I = A$. Thus (A, σ) is state simple. □

The following example shows that if $\sigma(A)$ is simple it does not necessarily mean that σ is faithful and (A, σ) is state simple.

Example 9. *Let $\Omega = \{1, 2\}$ and $\mathcal{A} = P(\Omega) = \{\{1\}, \{2\}, \{1, 2\}, \emptyset\}$, with operations $\oplus = \cup$, $\odot = \cap$ and $A^* = \Omega - A$, for any $A \in \mathcal{A}$ be an MV-algebra. Define $\sigma(A) = \emptyset$, if $A = \emptyset$, $\{1\}$ and $\sigma(A) = \{1, 2\}$, if $A = \{2\}, \{1, 2\}$, for all $A \in \mathcal{A}$. We can easily check that (A, σ) is a state MV-algebra. Clearly $\sigma(A) = \{\emptyset, \Omega\}$ is simple. But (A, σ) is not state simple, because $\{\emptyset, \{1\}\}$ and $\{\emptyset\}$ are state ideals of (A, σ). Also, σ is not faithful.*

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References

[1] L.P. Belluce, A. Di Nola, A. Lettieri, *Local MV-algebras.* Rend. Circ. Mat. Palermo. (2) 42 (1993), 347-361.

[2] M. Botur, A. Dvurecenskij, *State-morphism algebras general approach.* Fuzzy Sets Sys. 218 (2013), 90-102.

[3] C.C. Chang, *Algebraic analysis of many valued logic,* Trans. Amer. Math. Soc., 88 (1958), 467-490.

[4] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, *Algebraic foundations of many-Valued reasoning.* Kluwer Academic, Dordrecht, (2000).

[5] LC. Ciungu, *Bosbach and RieBŸcan states on residuated lattices.* J Appl Funct Anal 2 (2008), 175–188.

[6] LC. Ciungu, A. Dvurecenskij, M. Hycko, *State BL-algebras.* Soft Comput., 15 (2011), 619-634.

[7] A. Di Nola, A. Dvurečenskij, *State-morphism MV-algebras.* Ann Pure Appl Logic 161 (2009), 161-173.

[8] A. Di Nola, A. Dvurečenskij, *On some classes of state-morphism MV-algebras.* Math Slovaca 59 (2009), 517-534.

[9] A. Di Nola, A. Dvurečenskij, A. Lettieri, *On varieties of MV-algebras with internal states.* Inter. J. Approx. Reasoning, (2010), 5 (2010), 680-694.

[10] A. Dvurečenskij, J. Rachunek, D. Salounova, *State operators on generalizations of fuzzy structures.* Fuzzy Sets Syst. 187 (2012), 58-76.

[11] A. Dvurečenskij, *Subdirectly irreducible state-morphism BL-algebras.* Arch. Math. Log. 50 (2010), 145-160.

[12] A. Dvurečenskij, *States on pseudo MV-algebras.* Stud Logica 68 (2001), 301–327.

[13] F. Forouzesh, E. Eslami, A. Borumand Saeid, *On obstinate ideals in MV-algebras.* Politehn. Univ. Bucharest. Sci Bull Series A, Appli Math. Phys. 76, (2014), 53-62.

[14] F. Forouzesh, *State radical of state ideals in state MV-algebras.* submitted.

[15] T. Flaminio, F. Montagna , *An algebraic approach to states on MV-algebras.* In: Novák V (ed) Fuzzy Logic 2, proceedings of the 5th EUSFLAT conference, September 11–14, Ostrava, vol II (2007), 201–206.

[16] T. Flaminio, F. Montagna, *MV-algebras with internal states and probabilistic fuzzy logic.* Int. J. Approx. Reason. 50 (2009), 138-152.

[17] G. Georgescu, *Bosbach states on fuzzy structures.* Soft Comput. 8 (2004), 217-230.

[18] A. Iorgulescu, *Algebras of logic as BCK algebras.* Academy of economic studies Bucharest, Romania, (2008).

[19] LZ. Liu, XY. Zhang, *States on finite linearly ordered I MT Lalgebras.* Soft Comput. 15 (2011), 2021–2028.

[20] LZ. Liu, XY. Zhang, *States on R0-algebras.* Soft Comput. 12 (2008), 1099–1104.

[21] LZ. Liu, *On the existence of states on MT L-algebras.* Inf. Sci. (2013) 220, 559–567.

[22] D. Mundici, *Averaging the truth value in Lukasiewicz sentential logic.* Studia Logica 55 (1995), 113-127.

[23] D. Piciu, *Algebras of fuzzy logic.* Ed. Universitaria Craiova (2007).

[24] J. Rachunek, D. Salounova, *State operators on GMV-algebras.* Soft Comput. 15 (2011), 327-334.

[25] E. Turunen , J. Mertanen, *States on semi-divisible residuated lattices.* Soft Comput. 12 (2008), 353–357.

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