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ESTIMATES FOR MAXWELL VISCOELASTIC MEDIUM "IN TENSION-RATES"

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Abstract. The fictitious domain method for the Maxwell viscoelastic medium is considered. An estimate via the small parameter α which characterises the convergence of the solution of an auxiliary problem to the solution of the original problem is obtained.

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1 Introduction

Let us consider the dynamic viscoelastic problem based on the Maxwell model: in the cylinder Q=$\{D \times [0 \le t \le t_1]\}$, where $D \subset R^3$ is a simply-connected domain with a sufficiently smooth boundary γ. Let us denote by $\gamma_t = \gamma \times [0, t_1]$ the column-vectors of deformation and stress : $\overrightarrow{\varepsilon} = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23})^T$, $\overrightarrow{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})^T$, where T means transposition, and the column-vector of velocity $\overrightarrow{\upsilon} = (\upsilon_1, \upsilon_2, \upsilon_3)^T$.

As shown in work [1] the statement of the problem in tension-rates may be represented as follows:

$$\frac{\partial \overrightarrow{\upsilon}}{\partial t} + R^* \overrightarrow{\sigma} = \overrightarrow{f}, \tag{1.1}$$

$$\frac{\partial \overrightarrow{\varepsilon}}{\partial t} - R\overrightarrow{V} = 0, \tag{1.2}$$

$$B\frac{\partial \overrightarrow{\sigma}}{\partial t} + C\overrightarrow{\sigma} = \frac{\partial \overrightarrow{\varepsilon}}{\partial t}. \tag{1.3}$$

Here $\overrightarrow{f}$ is the vector of mass force,

$$B = \begin{pmatrix} a_1 & b_1 & b_1 & 0 & 0 & 0 \\ b_1 & a_1 & b_1 & 0 & 0 & 0 \\ b_1 & b_1 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1 \end{pmatrix}, C = \begin{pmatrix} a_2 & b_2 & b_2 & 0 & 0 & 0 \\ b_2 & a_2 & b_2 & 0 & 0 & 0 \\ b_2 & b_2 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 \end{pmatrix},$$

$$a_1 = \frac{1}{E}, \; b_1 = -\frac{\nu}{E}, \; c_1 = \frac{2(1+\nu)}{E}, \; a_2 = \frac{1}{3\Theta}, \; b_2 = -\frac{1}{6\Theta}, \; c_2 = \frac{1}{\Theta},$$

E - the Young modulus , ν - the Poisson ratio. The matrices B, C are permutable and are of the type described in work [2], R is the following linear matrix differential operator:

$$R = \begin{pmatrix} \nabla_1 & 0 & 0 & \nabla_2 & \nabla_3 & 0 \\ 0 & \nabla_2 & 0 & \nabla_1 & 0 & \nabla_3 \\ 0 & 0 & \nabla_3 & 0 & \nabla_1 & \nabla_2 \end{pmatrix}^T, \; R^* = -R^T, \; \nabla_i = \frac{\partial}{\partial x_i}, \; i = 1, 2, 3.$$

Equation (1.1) expresses the law of conservation of momentum, if the volumetric density $\rho \equiv 1$. Relation (1.2) is a corollary of the tension-displacement relation: $\overrightarrow{\varepsilon} = R\overrightarrow{u}$.

Here $\overrightarrow{u} = (u_1, u_2, u_3)^T$ is the displacement vector. The vectors of displacement $\overrightarrow{u}$ and velocity $\overrightarrow{v}$ are connected by the relation $\overrightarrow{v} = \frac{\partial \overrightarrow{u}}{\partial t}$. Relation (1.3) is a constitutive equation for the Maxwell viscoelastic medium. A solution to system (1.1)-(1.3) is sought in the cylinder Q and is such that

$$\overrightarrow{u}(x, 0) = \overrightarrow{\varphi}(x), \; \frac{\partial \overrightarrow{u}}{\partial t}(x, 0) = \overrightarrow{\psi}(x), \; x \in D,$$

hence

$$\overrightarrow{v}(x, 0) = \overrightarrow{\psi}(x), \; \overrightarrow{\varepsilon}(x, 0) = R\overrightarrow{\varphi}(x), \tag{1.4}$$

The displacements $\overrightarrow{u}(x, t)$ are determined from the relation

$$\overrightarrow{u}(x, t) = \overrightarrow{\varphi}(x) + t\overrightarrow{\psi}(x) + \int_0^t (t - s)\overrightarrow{r}(x, s)ds,$$

$$\overrightarrow{r} = \overrightarrow{f} - R^*\overrightarrow{\sigma}.$$

On the lateral surface of the cylinder Q the desired solution satisfies the homogeneous boundary condition

$$\sum_{k=1}^{3} \sigma_{ik}(x, t)n_k = 0, \quad (x, t) \in \gamma_t. \tag{1.5}$$

Here $n = (n_1, n_2, n_3)^T$ is the normal vector to γ.

Determination of the initial data for $\overrightarrow{\sigma}(x, t)$ is described in work [2]. Following [1] we reformulate problem (1.1)-(1.5) in terms of tension; let us set $\overrightarrow{F} = R\overrightarrow{f}$, then

$$B\frac{\partial^2 \overrightarrow{\sigma}}{\partial t^2} + C\frac{\partial \overrightarrow{\sigma}}{\partial t} = -RR^*\overrightarrow{\sigma} + \overrightarrow{F}, \tag{1.6}$$

$$\overrightarrow{\sigma}(x, 0) = \overrightarrow{g}(x), \quad \frac{\partial \overrightarrow{\sigma}}{\partial t}(x, 0) = \overrightarrow{p}(x). \tag{1.7}$$

Let us call problem (1.5) - (1.7) Problem I. Work [2] demonstrates stabilization of the solution to Problem I to the solution of the static elasticity problem:

$$R^*\overrightarrow{\sigma}^y(x) + \overrightarrow{F}(x) = 0, \quad \overrightarrow{\sigma}^y(x) = B\overrightarrow{\varepsilon}^y(x), \quad \sum_{k=1}^{3} \sigma^y_{ik}(x)n_k = 0, \quad x \in \gamma. \tag{1.8}$$

Problem I behaves like a parabolic equation and it is possible to use estimates for solutions of parabolic equations. In [2] there is the estimate

$$\|\overrightarrow{\sigma} - \overrightarrow{\sigma}^y\| \le e^{-\beta t} \|\overrightarrow{\sigma}(x, 0) - \overrightarrow{\sigma}^y(x)\|,$$

where $\beta > 0$ is a constant. The equality

$$\|\overrightarrow{\sigma}\|^2 = (B\frac{\partial \overrightarrow{\sigma}}{\partial t}, \frac{\partial \overrightarrow{\sigma}}{\partial t}) + \beta\frac{d}{dt}(B\overrightarrow{\sigma}, \overrightarrow{\sigma}) + ((A - 2\alpha^2 B - \alpha C)\overrightarrow{\sigma}, \overrightarrow{\sigma}),$$

demonstrates that $A-2\alpha^2B-\alpha C>0, A=-RR^*$.

The existence and uniqueness theorem for the solution to Problem I is proved in [3].

In accordance with the fictitious domain method [5], [6], [3], [7] let us supplement the initial domain D with a certain domain D_1, consider the domain $D_0=D\cup D_1$ with the boundary Γ, $\Gamma_t=\Gamma\times[0,t_1]$, $Q_1=D_1\times[0,t_1]$ and the following auxiliary problem

$$L_\alpha\vec{\sigma}^\alpha=\vec{F},\quad (x,t)\in Q,\qquad L_\alpha\vec{\sigma}^\alpha=0,\quad (x,t)\in Q_1,$$

$$\sum_{k=1}^{3}\vec{\sigma}^\alpha_{jk}n_k=0,\quad (x,t)\in\gamma_t\qquad \vec{\sigma}^\alpha(x,0)=0,\quad x\in D_1, \tag{1.9}$$

$$\vec{\sigma}^\alpha(x,0)=\vec{g}(x),\ x\in D,\qquad \frac{\partial\vec{\sigma}^\alpha}{\partial t}(x,0)=0,\quad x\in D_1,$$

$$\frac{\partial\vec{\sigma}^\alpha}{\partial t}(x,0)=\vec{p}(x),\quad x\in D,\quad \sum_{k=1}^{3}\sigma^\alpha_{ik}(x,t)n_k=0,\quad (x,t)\in\Gamma_t.$$

Here

$$L\vec{\sigma}\equiv B\frac{\partial^2\vec{\sigma}}{\partial t^2}+C\frac{\partial\vec{\sigma}}{\partial t}=A\vec{\sigma},A\vec{\sigma}=-RR^*\vec{\sigma},\ \ L_\alpha\vec{\sigma}^\alpha\equiv B\frac{\partial^2\vec{\sigma}^\alpha}{\partial t^2}+C\frac{\partial\vec{\sigma}^\alpha}{\partial t}=a^\alpha A\vec{\sigma}^\alpha,$$

$$a^\alpha=\begin{cases}1,\ x\in D\\ \alpha^{-2},\ x\in D_1,\end{cases}\qquad \alpha>0 \text{ - is the small parameter.}$$

On the curve γ_t of discontinuity of the coefficient let us lay the matching condition

$$\vec{\sigma}^\alpha|^+_{\gamma_t}=\vec{\sigma}^\alpha|^-_{\gamma_t},\qquad \frac{\partial\vec{\sigma}^\alpha}{\partial N}|^+_{\gamma_t}=\frac{M}{\alpha}\frac{\partial\vec{\sigma}^\alpha}{\partial n}|^-_{\gamma_t}, \tag{1.10}$$

where $\frac{\partial}{\partial N}$ is the conormal derivative, $\frac{\partial}{\partial n}$ is the normal derivative.

Signs "+" or "-" mean the limit values of the functions from inside or outside the boundary γ_t. The parameter M takes values 1 or -1. Let us introduce the following series into consideration:

$$S_1=\sum_{k=0}^{\infty}\alpha^k\vec{V}_k\quad\text{in } Q,\qquad S_2=\sum_{k=1}^{\infty}\alpha^k\vec{W}_k\qquad\text{in}\quad Q_1, \tag{1.11}$$

If we substitute (1.11) in (1.9), then we get the following relations for determination of $\vec{V}_k$ and $\vec{W}_k$:

$$L\vec{V}_0=\vec{F},\ (x,t)\in Q,\qquad L_\alpha\vec{W}_1=0,\ (x,t)\in Q_1,$$

$$\vec{V}_0(x,0)=\vec{g}(x),\ \frac{\partial\vec{V_0}}{\partial t}(x,0)=\vec{p}(x),\ \ x\in D,\qquad \vec{W}_1(x,0)=0,\ \frac{\partial\vec{W}_1}{\partial t}(x,0)=0,\ \ x\in D_1,$$

$$\sum_{k=1}^{3}(V_0)_{ik}n_k=0,\ (x,t)\in\gamma_t,\qquad \frac{\partial\vec{W}_1}{\partial n}=\frac{M}{\alpha}\frac{\partial\vec{V}_0}{\partial N},\ (x,t)\in\gamma_t,$$

$$\sum_{k=1}^{3}(\vec{W}_1)_{ik}n_k=0,\ (x,t)\in\Gamma_t,$$

and for $k \geq 1$

$$L\overrightarrow{V}_k = 0,\ (x,t) \in Q, \qquad L_\alpha \overrightarrow{W}_{k+1} = 0,\ (x,t) \in Q_1, \tag{1.12}$$

$$\overrightarrow{V}_k(x,0) = 0,\ \frac{\partial \overrightarrow{V}_k}{\partial t}(x,0) = 0,\ x \in D, \qquad \overrightarrow{W}_{k+1}(x,0) = 0,\ \frac{\partial \overrightarrow{W}_{k+1}}{\partial t}(x,0) = 0,\ x \in D_1,$$

$$\overrightarrow{V}_k = \overrightarrow{W}_k,\ (x,t) \in \gamma_t, \qquad \sum_{i=1}^{3} (\overrightarrow{W}_{k+1})_{mi} n_i = 0,\ (x,t) \in \Gamma_t.$$

We note that $\overrightarrow{V}_k \in W_2^{2,1}(Q)$, $\quad k = 0, 1...$,$\overrightarrow{W}_k \quad W_2^{2,1}(Q_1)$, $\quad k = 1, 2, ...$

Theorem 1.1. *There exists α_0 such that for all $0 < \alpha < \alpha_0$, the series S_1 and S_2 are absolutely convergent in the spaces $W_2^{2,1}(Q)$, $W_2^{2,1}(Q_1)$ respectively, and the following equalities take place:*

$$\overrightarrow{\sigma}^\alpha = S_1,\ (x,t) \in Q, \qquad \overrightarrow{\sigma}^\alpha = S_2,\ (x,t) \in Q_1, \tag{1.13}$$

where $\overrightarrow{\sigma}^\alpha$ is the solution to problem (1.9).

Proof. We have the evident a priori estimates

$$\left\|\overrightarrow{W}_k\right\|_{W_2^{2,1}(Q_1)} \leq C_2 \left\|\frac{\partial \overrightarrow{W}_k}{\partial n}\right\|_{W_2^{1/2,1}(\gamma_t)} \leq C_2 \left\|\frac{\partial \overrightarrow{V}_{K-1}}{\partial N}\right\|_{W_2^{1/2,1}(\gamma_t)} \leq C_2 C_3 \left\|\overrightarrow{V}_{k-1}\right\|_{W_2^{2,1}(Q)}, \tag{1.14}$$

where C_2, C_3 are constants which depend on the domains D, D_1 and do not depend on α. Let us prove the convergence of the series S_1 in $W_2^{2,1}(Q)$ and S_2 in $W_2^{2,1}(Q_1)$. We note that

$$\left\|\overrightarrow{V}_k\right\|_{W_2^{2,1}(Q)} \leq C_4 \left\|\overrightarrow{V}\right\|_{W_2^{3/2,1}(\gamma_t)} = C_4 \left\|\overrightarrow{W}_k\right\|_{W_2^{3/2,1}(\gamma_t)} \leq C_4 C_5 \left\|\overrightarrow{W}_k\right\|_{W_2^{2,1}(Q_1)},$$

and using (1.8), (1.14) we have

$$\left\|\overrightarrow{V}_k\right\|_{W_2^{2,1}(Q)} \leq C_6 \left\|\overrightarrow{V}_{k-1}\right\|_{W_2^{2,1}(Q)},\ k \geq 1,$$

$$\left\|\overrightarrow{V}_0\right\|_{W_2^{2,1}(Q)} \leq C_1 (\left\|\overrightarrow{F}\right\|_{L_2(Q)} + \|p\|_{L_2(D)} + \int_0^{t_1} \|g\|_{W_2^1(D)}\, dt), \tag{1.15}$$

where $C_6 = C_2 C_3 C_4 C_5$.

Assuming that $\alpha < \alpha_0 = C_6^{-1}$, we obtain that the series S_1 converges absolutely in $W_2^{2,1}(Q)$ and, respectivaly, the series S_2 absolutely converges in $W_2^{2,1}(Q_1)$. By multiplying the first equality in (1.12) by α^k and the second one by α^{k+1} and $\overrightarrow{W}_k$ by α_k and by summing over k, we have

$$LS_1 = \overrightarrow{F},\ (x,t) \in Q, \qquad L_\alpha S_2 = 0,\ (x,t) \in Q_1.$$

Similarly we get that

$$S_1(x,0) = \overrightarrow{g}(x),\ \frac{\partial S_1}{\partial t}(x,0) = \overrightarrow{p}(x),\ x \in D, \qquad S_2(x,0) = 0,\ \frac{\partial S_2}{\partial t}(x,0) = 0,\ x \in D,$$

$$S_1 = S_2,\ (x,t) \in \gamma_t, \qquad \frac{\partial S_2}{\partial n} = \frac{M}{\alpha}\frac{\partial S_1}{\partial N},\ (x,t) \in \gamma_t, \tag{1.16}$$

$$S_2(x,t) = 0, \quad (x,t) \in \Gamma_t.$$

Hence we obtain that $\overrightarrow{\sigma}^\alpha = S_1$ in Q, $\overrightarrow{\sigma}^\alpha = S_2$ in Q_1, if $0 < \alpha < \alpha_0$. □

From the proof of this theorem it follows that the following estimates are true

$$\|\overrightarrow{\sigma} - \overrightarrow{\sigma}_+^\alpha\|_{W_2^{2,1}(Q)} \le C_7\alpha,$$

$$\|\overrightarrow{\sigma} - \overrightarrow{\sigma}_-^\alpha\|_{W_2^{2,1}(Q)} \le C_8\alpha. \tag{1.17}$$

Here $\overrightarrow{\sigma}_+^\alpha = \overrightarrow{\sigma}^\alpha$, if $M = 1$, $\overrightarrow{\sigma}_-^\alpha = \overrightarrow{\sigma}^\alpha$ if $M = -1$, $\overrightarrow{\sigma}$ is the solution to Problem I, and $C_7, C_8 > 0$ depend on D but do not depend on α.

Theorem 1.2. *For all $0 < \alpha < \alpha_0$ the following estimates take place, where $\bar{\sigma}$ is the solution to Problem I, $\bar{\sigma}_+^\alpha$, $\bar{\sigma}_-^\alpha$ are the solutions to auxiliary problem (1.9) for $M = 1$, and $M = -1$ respectively:*

$$\left\|\overrightarrow{\sigma} - \frac{1}{2}(\overrightarrow{\sigma}_+^\alpha + \overrightarrow{\sigma}_-^\alpha)\right\|_{W_2^{2,1}(Q)} \le C_9\alpha^2, \tag{1.18}$$

$$\overrightarrow{\sigma}^\alpha = S_1, \quad (x,t) \in Q, \qquad \overrightarrow{\sigma}^\alpha = S_2, \quad (x,t) \in Q_1. \tag{1.19}$$

where $\overrightarrow{\sigma}^\alpha$ is the solution to Problem I, and $C_9 > 0$ depends on D but does not depend on α.

Proof. By using Theorem 1.1 we have

$$\overrightarrow{\sigma}_+^\alpha = \sum_{k=0}^{\infty} \alpha^k \overrightarrow{V}_k^+,\ (x,t) \in Q, \qquad \overrightarrow{\sigma}_-^\alpha = \sum_{k=0}^{\infty} \alpha^k \overrightarrow{W}_k^+, \quad (x,t) \in Q_1. \tag{1.20}$$

Here $\overrightarrow{V}_k^+$, $\overrightarrow{W}_k^+$ are solutions to (1.12) for M=1.

In accordance with Theorem 1.2. let us represent $\overrightarrow{\sigma}_-^\alpha$ in the following form

$$\overrightarrow{\sigma}_-^\alpha = \sum_{k=0}^{\infty} \alpha^k \overrightarrow{V}_k^-, \quad (x,t) \in Q, \qquad \overrightarrow{\sigma}_+^\alpha = \sum_{k=0}^{\infty} \alpha^k \overrightarrow{W}_k^-, \quad (x,t) \in Q_1, \tag{1.21}$$

where $\overrightarrow{V}_k^-, \overrightarrow{W}_k^-$ are solutions to (1.11) for M= -1.

We obtain that $\overrightarrow{V}_0^+ \equiv \overrightarrow{V}_0^- \equiv \overrightarrow{\sigma}$ is a solution to Problem I.

Let us introduce the following notation $\overrightarrow{W}_1 = \overrightarrow{W}_1^+ + \overrightarrow{W}_1^-$, where the function $\overrightarrow{W}_1$ satisfies the following conditions

$$L_\alpha \overrightarrow{W}_1 = 0, \quad (x,t) \in Q_1, \qquad \frac{\partial \overrightarrow{W}_1}{\partial n} = 0,\ (x,t) \in \gamma_t,$$

$$\overrightarrow{W}_1(x,0) = 0, \quad \frac{\partial \overrightarrow{W}_1}{\partial t}(x,0) = 0, x \in D_1, \qquad \overrightarrow{W}_1(x,t) = 0, \quad (x,t) \in \Gamma_t.$$

Hence we obtain $\overrightarrow{W}_1 = 0$ or $\overrightarrow{W}_1^+ = -\overrightarrow{W}_1^-$

Further we introduce $\overrightarrow{V} = \overrightarrow{V}_1^+ + \overrightarrow{V}_1^-$, where the function $\overrightarrow{V}_1$ satisfies the problem

$$L\overrightarrow{V}_1 = 0,\ (x,t) \in Q, \qquad \overrightarrow{V}_1(x,0) = 0,\ \frac{\partial \overrightarrow{V}_1}{\partial t}(x,0) = 0,\ x \in D,$$

$$\overrightarrow{V}_1(x,t) = 0,\ (x,t) \in \Gamma_t.$$

From here we have $\overrightarrow{V}_1 = 0$, or $\overrightarrow{V}_1^+ = -\overrightarrow{V}_1^-$.

By introducing next $\overrightarrow{W}_2 = \overrightarrow{W}_2^+ - \overrightarrow{W}_2^-$, $\overrightarrow{V}_2 = \overrightarrow{V}_2^+ - \overrightarrow{V}_2^-$, we obtain that

$$\overrightarrow{W}_2^+ = \overrightarrow{W}_2^-, \quad \overrightarrow{V}_2^+ = \overrightarrow{V}_2^-. \tag{1.22}$$

By continuing the process we come to the equalities $\overrightarrow{V}_k^+ = \overrightarrow{V}_k^-$ if k is even, $\overrightarrow{V}_k^+ = -\overrightarrow{V}_k^-$ if k is odd (1.21).

By using (1.21) and substituting (1.19),(1.20), we have

$$\overrightarrow{\sigma}_+^\alpha = \overrightarrow{\sigma} + \alpha \overrightarrow{V}_1^+ + \alpha^2 \overrightarrow{V}_2^+ + ... \tag{1.23}$$

$$\overrightarrow{\sigma}_-^\alpha = \overrightarrow{\sigma} - \alpha \overrightarrow{V}_1^+ + \alpha^2 \overrightarrow{V}_2^+ + ... \tag{1.24}$$

By adding equalities (1.22),(1.23) and by using estimates (1.17) for $0 < \alpha < \alpha_1$, we obtain that

$$\left\| \overrightarrow{\sigma} - \frac{1}{2}(\overrightarrow{\sigma}_+^\alpha + \overrightarrow{\sigma}_-^\alpha) \right\|_{W_2^{2,1}(Q)} \leq \alpha^2 \left\| \overrightarrow{V}_2^+ + \alpha^2 \overrightarrow{V}_4^+ + ... \right\|_{W_2^{2,1}(Q)} \leq C_8 \alpha^2 \left\| \overrightarrow{V}_0^+ \right\|_{W_2^{2,1}(Q)}$$

$$\leq C_9 \alpha^2,$$

where $C_8 = C_6^2$.

Then for $x \in D$ and $0 < \alpha < \alpha_0$ there is a pointwise two-sided inequality:

$$O(\alpha^2) + \min \left\langle \bar{\sigma}_+^\alpha, \bar{\sigma}_-^\alpha \right\rangle \leqslant \bar{\sigma} \leqslant \max \left\langle \bar{\sigma}_+^\alpha, \bar{\sigma}_-^\alpha \right\rangle + O(\alpha^2)$$

□

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