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CRITERIA FOR EMBEDDING OF GENERALIZED BESSEL AND RIESZ POTENTIAL SPACES IN REARRANGEMENT INVARIANT SPACES

N.A. Bokayev, M.L. Goldman, G.Zh. Karshygina

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Key words: functional norm, cones of functions with monotonicity conditions, embedding of potential spaces in rearrangement-invariant spaces, generalized Bessel and Riesz potentials.

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Abstract. We consider the spaces of generalized Bessel and Riesz potentials and establish criteria for the embedding of these spaces in rearrangements invariant spaces. To do this we obtain constructive equivalent descriptions for the cones of decreasing rearrangement of potentials. Covering and equivalence of cones are studied with respect to order relations which allows to weaken substantially the assumptions on the kernels of potentials.

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1 Introduction

We study the potential space $H_E^G \equiv H_E^G(\mathbb{R}^n)$ on the n-dimensional Euclidean space $\mathbb{R}^n$

$$H_E^G(\mathbb{R}^n) = \{u = G * f : f \in E(\mathbb{R}^n)\}, \tag{1.1}$$

where $E(\mathbb{R}^n)$ is a rearrangement-invariant space (RIS) in the axiomatics of the book C. Bennet, R. Sharpley [1]. The kernel G is called admissible if

$$G \in L_1(\mathbb{R}^n) + E'(\mathbb{R}^n), \tag{1.2}$$

where $E'(\mathbb{R}^n)$ is the associated space for an RIS $E(\mathbb{R}^n)$, that is, $E'(\mathbb{R}^n)$ consists of all measurable functions g for which

$$\|g\|_{E'(\mathbb{R}^n)} = sup \left\{ \left| \int\limits_{R^n} g\, f d\mu \right| : \quad f \in E(\mathbb{R}^n),\ \|f\|_{E(\mathbb{R}^n)} \le 1 \right\} < \infty. \tag{1.3}$$

For Banach spaces of measurable functions on $\mathbb{R}^n$ the sum of spaces $A_0 + A_1$ (as in (1.2)) is determined in the following way:

$$A_0 + A_1 = \{f = f_0 + f_1 : \quad f_0 \in A_0, \quad f_1 \in A_1\} \tag{1.4}$$

with

$$\|f\|_{A_0+A_1} = \inf_{f_0+f_1=f} \left\{ \|f_0\|_{A_0} + \|f_1\|_{A_1} \right\}, \tag{1.5}$$

where the infimum is taken over all representations $f = f_0 + f_1,\ f_i \in A_i; \quad i = 0, 1$ (see [1]). Two types of conditions on admissible kernels will be considered that lead to the spaces of the generalized Bessel and Riesz potentials. These constructions generalize both classical Bessel and Riesz potentials, studied in the books of S.M. Nikolskii [9], E.M. Stein [11], and V.G. Maz'ya [8], as well as their generalizations, considered in [5, 6] by M.L. Goldman and in [7] by A. Gogatishvili, H. Neves and B. Opitz.

In this paper, we develop the results obtained in our work [2]. In the general case, representation (1.1) for the potential $u \in H_E^G(\mathbb{R}^n)$ may not be unique, so the norm of u is defined by the formula

$$\|u\|_{H_E^G} = inf\{\|f\|_E : \quad f \in E(\mathbb{R}^n), \;\; G * f = u\}, \tag{1.6}$$

where the infimum is taken over all representations of form (1.1) for a given potential u (factor norm). Here the convolution $G * f$ is defined in a standard way as the integral

$$(G * f)(x) = \int\limits_{\mathbb{R}^n} G(x-y) f(y) dy, \tag{1.7}$$

Theorem 1.1. ([5; Theorem 2.1]). *Let G be an admissible kernel. Then, integral (1.7) converges for almost all $x \in \mathbb{R}^n$. Furthermore, $H_E^G(\mathbb{R}^n)$ is a Banach space,*

$$H_E^G(\mathbb{R}^n) \subset E(\mathbb{R}^n) + L_\infty(\mathbb{R}^n), \tag{1.8}$$

$$\|u\|_{E+L_\infty} \le \|G\|_{L_1+E'} \|u\|_{H_E^G}, \quad , \quad u \in H_E^G. \tag{1.9}$$

For the case of admissible kernels, we consider the decreasing rearrangements u^* of potentials u with respect to the Lebesque measure in $\mathbb{R}^n$, and

$$u^{**}(t) = t^{-1} \int\limits_0^t u^*(\tau) d\tau, \quad t \in \mathbb{R}_+ = (0, \infty).$$

We introduce the following cones of decreasing rearrangements on $(0, T)$ with $T \in (0, \infty]$ equipped with positively homogeneous functionals:

$$M(T) = M_E^G(T) = \{h(t) = u^*(t), \quad t \in (0,T) : \quad u \in H_E^G(\mathbb{R}^n)\}, \tag{1.10}$$

$$\rho_{M(T)}(h) = \inf\{\|u\|_{H_E^G} : \; u \in H_E^G(\mathbb{R}^n); \quad u^*(t) = h(t), \quad t \in (0,T)\} \tag{1.11}$$

and

$$\tilde{M}(T) = \tilde{M}_E^G(T) = \{h(t) = u^{**}(t), \quad t \in (0,T) : \quad u \in H_E^G(\mathbb{R}^n)\}, \tag{1.12}$$

$$\rho_{\tilde{M}(T)}(h) = \inf\{\|u\|_{H_E^G} : \; u \in H_E^G(\mathbb{R}^n); \quad u^{**}(t) = h(t), \quad t \in (0,T)\}. \tag{1.13}$$

The cones $M_E^G(T)$ and $\tilde{M}_E^G(T)$ define local (for $T \in \mathbb{R}_+$) or global (for $T = \infty$) integral properties of potentials $u \in H_E^G$ and of their maximal functions Mu by using the relation $(Mu)^* \cong u^{**}$, see [1]. Thus, for RIS $X(\mathbb{R}^n)$ with $T = \infty$

$$H_E^G(\mathbb{R}^n) \subset X(\mathbb{R}^n) \Leftrightarrow M_E^G(\infty) \mapsto \tilde{X}(\mathbb{R}_+). \tag{1.14}$$

Here $\tilde{X}(\mathbb{R}_+)$ is the Luxemburg representation for the RIS $X(\mathbb{R}^n)$ (see [1]). The embedding of the cones in a RIS:

$$M_E^G(T) \mapsto \tilde{X}(0,T) \tag{1.15}$$

means that $h \in M_E^G(T) \Rightarrow h \in \tilde{X}(0,T)$ and there is a constant $d = d(E, G, T, \tilde{X}) \in \mathbb{R}_+$, such that

$$\|h\|_{\tilde{X}(0,T)} \le d \rho_{M(T)}(h), \qquad h \in M_E^G(T). \tag{1.16}$$

It should be noted that the cones of decreasing rearrangements for potentials are very complicated. Therefore, the problem of their equivalent descriptions in more transparent terms, and the equivalent constructive conditions for embeddings of type (1.15) is of particular interest. This work is devoted to this problem.

2 On the covering of cones of monotonic functions on the positive semi-axis

Let $T \in (0, \infty]$. We denote by $\Omega(T)$ the set of functions φ on $\mathbb{R}_+ = (0, \infty)$ with the properties

$$\begin{cases} 1) \quad 0 < \varphi(t) \downarrow, \quad \varphi(t+0) = \varphi(t), \ \int\limits_0^t \varphi d\xi < \infty, \quad t \in (0, T), \\ 2) \quad \text{if } \ T < \infty, \text{then} \quad \varphi(t) = 0, \quad t \in [T, \infty). \end{cases} \tag{2.1}$$

For $n \in \mathbb{N}$, $\varphi \in \Omega(T)$ we introduce the functions

$$f_\varphi(t,\tau) = \varphi(\max\{t,\tau\}) = \begin{cases} \varphi(t), & 0 < \tau \le t, \\ \varphi(\tau), & t < \tau < \infty; \end{cases} \tag{2.2}$$

$$\check{f}_\varphi(t,\tau) = \varphi(\max\{2^n t, 2^n \tau\}) = \begin{cases} \varphi(2^n t), & 0 < \tau \le t, \\ \varphi(2^n \tau), & t < \tau < \infty; \end{cases} \tag{2.3}$$

$$\tilde{f}_\varphi(t,\tau) = \begin{cases} \frac{1}{t} \int\limits_0^t \varphi(\xi) d\xi, & 0 < \tau \le t, \\ \varphi(\tau), & t < \tau < \infty. \end{cases} \tag{2.4}$$

We note that if $T < \infty$ then

$$f_\varphi(t,\tau) = 0, \quad t \ge T, \ \tau \in \mathbb{R}_+; \quad \check{f}_\varphi(t,\tau) = 0, \quad t \ge 2^{-n}T, \ \tau \in \mathbb{R}_+. \tag{2.5}$$

Let $\tilde{E}(\mathbb{R}_+)$ be RIS with the norm $\|\cdot\|_{\tilde{E}(\mathbb{R}_+)}$, $\tilde{E}'(\mathbb{R}_+)$ be the associated RIS, and

$$\tilde{E}^{\downarrow}(\mathbb{R}_+) = \left\{ g \in \tilde{E}(\mathbb{R}_+) : \quad 0 \le g \downarrow; \quad g(t+0) = g(t), \quad t \in \mathbb{R}_+ \right\}, \tag{2.6}$$

and if $T < \infty$

$$\tilde{E}^{\downarrow}(0,T) = \left\{ g \in \tilde{E}^{\downarrow}(\mathbb{R}_+) : \quad g(t) = 0, \ t \in [T, \infty) \right\}. \tag{2.7}$$

We introduce the following cones of non-negative functions on $\mathbb{R}_+$:

$$K(T) = K_{\varphi,\tilde{E}}(T) = \left\{ h(t) \equiv h(g;t) := \int\limits_0^\infty f_\varphi(t,\tau) g(\tau) d\tau : \ g \in \tilde{E}^{\downarrow}(0,T) \right\}, \tag{2.8}$$

$$\check{K}(T) = \check{K}_{\varphi,\tilde{E}}(T) = \left\{ \check{h}(t) \equiv \check{h}(\check{g};t) := \int\limits_0^\infty \check{f}_\varphi(t,\tau) \check{g}(\tau) d\tau : \ \check{g} \in \tilde{E}^{\downarrow}(0,T) \right\}, \tag{2.9}$$

$$\tilde{K}(T) = \tilde{K}_{\varphi,\tilde{E}}(T) = \left\{ \tilde{h}(t) \equiv \tilde{h}(\tilde{g};t) := \int\limits_0^\infty \tilde{f}_\varphi(t,\tau) \tilde{g}(\tau) d\tau : \ \tilde{g} \in \tilde{E}^{\downarrow}(0,T) \right\}, \tag{2.10}$$

equipped respectively with the positive homogeneous functionals

$$\rho_{K(T)}(h) = \inf \left\{ \|g\|_{\tilde{E}(\mathbb{R}_+)} : \ g \in \tilde{E}^{\downarrow}(0,T); \quad h(g;t) = h(t), \quad t \in \mathbb{R}_+ \right\} \tag{2.11}$$

$$\rho_{\check{K}(T)}(\check{h}) = \inf \left\{ \|\check{g}\|_{\tilde{E}(\mathbb{R}_+)} : \ \check{g} \in \tilde{E}^{\downarrow}(0,T); \ \check{h}(\check{g};t) = \check{h}(t), \quad t \in \mathbb{R}_+ \right\} \tag{2.12}$$

$$\rho_{\tilde{K}}(T)(\tilde{h}) = \inf\left\{\|\tilde{g}\|_{\tilde{E}(\mathbb{R}_+)}: \quad \tilde{g} \in \tilde{E}^{\downarrow}(0,T); \quad \tilde{h}(\tilde{g};t) = \tilde{h}(t), \qquad t \in \mathbb{R}_+\right\}. \tag{2.13}$$

In functional (2.11), the lower bound is taken over all functions $g \in \tilde{E}^{\downarrow}(0,T)$, such that the function $h(g;t)$ in the form of integral (2.8) coincides with a given function $h \in K(T)$. The same refers to functionals (2.12), (2.13).

The positive homogeneity of the functionals means that $h \in K(T)$, $\alpha \geq 0 \Rightarrow \quad \rho_{K(T)}(\alpha h) = \alpha\rho_{K(T)}(h)$; similarly for $\rho_{\check{K}(T)}$ and $\rho_{\tilde{K}(T)}$.

Remark 1. For $\varphi \in \Omega(T)$ the following inequalities hold:

$$\varphi(2^n t) \leq \varphi(t); \quad \varphi(t) \leq \frac{1}{t}\int_0^t \varphi d\xi, \quad t \in \mathbb{R}_+. \tag{2.14}$$

Therefore,

$$\check{f}_\varphi(t,\tau) \leq f_\varphi(t,\tau) \leq \tilde{f}_\varphi(t,\tau), \quad t,\tau \in \mathbb{R}_+. \tag{2.15}$$

Remark 2. Everywhere in this paper we require that for any $t \in \mathbb{R}_+$

$$f_\varphi(t,\cdot) \in \tilde{E}'(\mathbb{R}_+), \quad \check{f}_\varphi(t,\cdot) \in \tilde{E}'(\mathbb{R}_+), \quad \tilde{f}_\varphi(t,\cdot) \in \tilde{E}'(\mathbb{R}_+). \tag{2.16}$$

In the case $T < \infty$ conditions (2.16) hold for any $\varphi \in \Omega(T)$ and for any RIS $\tilde{E}(\mathbb{R}_+)$, since in this case $0 \leq f_\varphi(t,\tau)$, $\check{f}_\varphi(t,\tau)$, $\tilde{f}_\varphi(t,\tau)$ are bounded decreasing functions of the variable τ with a compact support. Such functions belong to $L_1(\mathbb{R}_+) \cap L_\infty(\mathbb{R}_+)$, and, therefore, to any RIS $\tilde{E}'(\mathbb{R}_+)$.

In the case $T = \infty$ each conditions in (2.16) is equivalent to the fact that for $t \in \mathbb{R}_+$

$$\varphi(\cdot)\chi_{(t,\infty)}(\cdot) \in \tilde{E}'(\mathbb{R}_+). \tag{2.17}$$

Conditions (2.16) imply the following estimates for $t \in \mathbb{R}_+$:

$$0 \leq h(t) \leq \| f_\varphi(t,\cdot) \|_{\tilde{E}'(\mathbb{R}_+)} \rho_{K(T)}(h), \;\; h \in K(T); \tag{2.18}$$

$$0 \leq \check{h}(t) \leq \| \check{f}_\varphi(t,\cdot) \|_{\tilde{E}'(\mathbb{R}_+)} \rho_{\check{K}(T)}(\check{h}), \;\; \check{h} \in \check{K}(T); \tag{2.19}$$

$$0 \leq \tilde{h}(t) \leq \| \tilde{f}_\varphi(t,\cdot) \|_{\tilde{E}'(\mathbb{R}_+)} \rho_{\tilde{K}(T)}(\tilde{h}), \;\; \tilde{h} \in \tilde{K}(T). \tag{2.20}$$

Indeed, for $h \in K(T)$ by the *Hölder* inequality from representation (2.8) it follows that

$$0 \leq h(t) \leq \| f_\varphi(t,\cdot) \|_{\tilde{E}'(\mathbb{R}_+)} \| g \|_{\tilde{E}(\mathbb{R}_+)}, \quad \forall g \in \tilde{E}^{\downarrow}(0,T).$$

Passing to the lower bound over all $g \in \tilde{E}^{\downarrow}(0,T)$ for which $h(g;t) = h(t)$, we obtain, by (2.11), inequality (2.18). Similarly, inequalities (2.19) and (2.20) are derived.

Remark 3. From estimates (2.18), (2.19), (2.20) it follows the non-degeneracy of functionals (2.11), (2.12), (2.13). Indeed, if for $h \in K(T)$ we have $\rho_{K(T)}(h) = 0$, then from (2.18) it follows that $h(t) = 0$, $t \in \mathbb{R}_+$. Similarly, for $\check{h} \in \check{K}(T)$ and $\tilde{h} \in \tilde{K}(T)$.

Suppose that on a subset $\mathfrak{L} \subset L_0^+(\mathbb{R}_+)$ we introduce a partial order relation $\prec$, subordinate to a pointwise estimate almost everywhere: h_1, $h_2 \in \mathfrak{L}$, $h_1 \leq h_2$ a.e. on $\mathbb{R}_+$ implies that $h_1 \prec h_2$. Let $K, M \subset \mathfrak{L}$ be some cones, equipped with non-degenerate positively homogeneous functionals ρ_K and ρ_M.

Definition 1. A cone M covers a cone K with respect to the order $\prec$ with covering constants $c_0 \in (0,\infty),\ c_1 \in [0,\infty)$, if for any $h_1 \in K$ there is $h_2 \in M$, such that

$$\rho_M(h_2) \le c_0\rho_K(h_1); \quad h_1 \prec h_2 + c_1\rho_K(h_1). \tag{2.21}$$

In the case when the order relation is determined by a pointwise estimate, we consider a pointwise covering of cones with covering constants c_0, c_1.

Notation: $K \prec M$ means that a cone M covers a cone K; $K \approx M \Leftrightarrow K \prec M \prec K$ means that equivalence of cones.

Pointwise covering: $K \le M$, pointwise equivalence: $K \cong M$.

In the case of a pointwise covering, relations (2.21) take the form

$$\rho_M(h_2) \le c_0\rho_K(h_1), \quad h_1(t) \le h_2(t) + c_1\rho_K(h_1)(\text{a.e.}). \tag{2.21$'$}$$

For an order relation subordinated to a pointwise estimate, we have

$$K \le M \Rightarrow K \prec M; \quad K \cong M \Rightarrow K \approx M. \tag{2.22}$$

We will be interested, first of all, in the order relation equivalent to the pointwise estimate: for $f_1,\ f_2 \in L_0^+(\mathbb{R}_+)$ we have $f_1 \prec f_2 \Leftrightarrow f_1 \le f_2$ almost everywhere on $\mathbb{R}_+$.

We also consider the set $\mathfrak{L}$ of all functions $f \in L_0^+(\mathbb{R}_+)$, for which their Lebesgue distribution functions

$$\lambda_f(y) = \mu\{x \in \mathbb{R}_+ : (x) > y\},\ y \in [0,\infty)$$

are not identical to infinity, i.e. $\exists y_0 \in [0,\infty):\ \lambda_f(y_0) < \infty$. For $f \in \mathfrak{L}$ we introduce a decreasing rearrangement f^* as a right inverse function of a decreasing function λ_f, i.e.

$$f^*(t) = inf\{y \in [0,\infty):\ \lambda_f(y) \le t\},\ t \in \mathbb{R}_+.$$

We define the order relation for $f_1,\ f_2 \in \mathfrak{L}(\mathbb{R}_+)$: we say that $f_1 \prec f_2$, if

$$\int_0^t f_1^*(\tau)d\tau \le \int_0^t f_2^*(\tau)d\tau, \quad t \in \mathbb{R}_+. \tag{2.23}$$

It is subordinated to the order with respect to the pointwise estimate:

$$0 \le f_1 \le f_2 \quad \text{a.e. on} \quad \mathbb{R}_+ \Rightarrow f_1^* \le f_2^* \quad \text{a.e. on} \quad \mathbb{R}_+ \Rightarrow (2.23).$$

We give the result on the mutual pointwise covering of cones $K(T),\ \check{K}(T),\ \tilde{K}(T)$.

Theorem 2.1. ***1.*** *In the notation and under assumptions (2.1) - (2.16) the pointwise coverings*

$$(A): \check{K}(T) \le K(T); \quad K(T) \le \tilde{K}(T) \tag{2.24}$$

hold with the covering constants: $c_0(A) = 1+\varepsilon,\ \forall \varepsilon > 0;\ c_1(A) = 0.$

2. *Suppose that the assumptions of Part 1 are satisfied and there is also a constant* $c \in [1,\infty)$*, such that*

$$\varphi(t) \le c\varphi(2^n t), \quad t \in (0, 2^{-n}T). \tag{2.25}$$

If $T = \infty$*, then (2.25) holds for any* $t \in \mathbb{R}_+$*. Then the covering*

$$(B): K(T) \le \check{K}(T) \tag{2.26}$$

takes place with the constants of covering $c_0(B) = c2^n \parallel \sigma_{2^n} \parallel$,

$$c_1(B) = 0, \quad \textit{if} \quad T = \infty; \quad c_1(B) = \parallel f_\varphi(2^{-n}T, \cdot) \parallel_{\tilde{E}'(\mathbb{R}_+)} \textit{ if } \quad T < \infty. \tag{2.27}$$

Here for $\alpha > 0$, $(\sigma_\alpha g)(\tau) = g(\alpha\tau)$, $\tau \in \mathbb{R}_+$ *is an stretching operator,* $\parallel \sigma_{2^n} \parallel$ *is the norm of this operator* $\sigma_{2^n} : \tilde{E}(\mathbb{R}_+) \to \tilde{E}(\mathbb{R}_+)$.

3. *Let the assumptions of Part 1 be satisfied and moreover*

$$A_\varphi \equiv A_\varphi(T) := \sup_{t\in(0,T)} \frac{\int_0^t \varphi d\tau}{t\varphi(t)} < \infty. \tag{2.28}$$

Then, we have the covering

$$(\mathcal{D}) : \tilde{K}(T) \leq K(T) \tag{2.29}$$

with the covering constants

$$c_0(\mathcal{D}) = (1+\varepsilon)A_\varphi, \quad \forall\varepsilon > 0; \tag{2.30}$$

$$c_1(\mathcal{D}) = 0, \quad \textit{if} \quad T = \infty; \quad c_1(\mathcal{D}) = \parallel \tilde{f}_\varphi(T, \cdot) \parallel_{\tilde{E}'(\mathbb{R}_+)}, \quad \textit{if} \quad T < \infty. \tag{2.31}$$

Remark 4. Under the assumptions of Part 3 of Theorem 2.1, estimate (2.25) holds with $c = 2^n A_\varphi$ and, correspondingly, there is covering (2.26) with the covering constants $c_0(B) = A_\varphi 2^{2n} \parallel \sigma_{2^n} \parallel$ and $c_1(B)$ of form (2.27).

Next, we give the result on the order covering of the cones $K(T), \check{K}(T), \tilde{K}(T)$.

Theorem 2.2. ***1.*** *In the notation and under assumptions, (2.1) - (2.16), the order covering*

$$(E) : K(T) \prec \check{K}(T) \tag{2.32}$$

is valid with respect to order relation (2.23) with the constants of covering

$$c_0(E) = 2^{2n+1} \parallel \sigma_{2^n} \parallel; \quad c_1(E) = 0. \tag{2.33}$$

2. *Suppose that the assumptions of Part 1 are satisfied and*

$$B_\varphi \equiv B_\varphi(T) := \sup_{t\in(0,T)} \left[\frac{\int_0^t \varphi d\tau}{\frac{1}{t}\int_0^t \varphi(\tau)\tau d\tau} \right] < \infty. \tag{2.34}$$

Then, the order covering

$$(F) : \tilde{K}(T) \prec K(T) \tag{2.35}$$

is valid with respect to order relation (2.23) with the constants of covering

$$c_0(F) = 2(B_\varphi + 1)(2B_\varphi^2 + 1); \quad c_1(F) = 0. \tag{2.36}$$

Corollary 2.1. *Under the assumptions of Part 3 of Theorem 2.1, there is a pointwise equivalence of the cones*

$$\check{K}(T) \cong \tilde{K}(T) \cong K(T). \tag{2.37}$$

Indeed, taking into account, Remark 4, we have the coverings

$$\tilde{K}(T) \leq K(T) \leq \check{K}(T),$$

which together with covering (2.24) give equivalences (2.37).

Corollary 2.2. *Under the assumptions of Part 2 of Theorem 2.1, there is a pointwise equivalence of the cones*

$$\check{K}(T) \cong K(T). \tag{2.38}$$

This follows from coverings (2.24) and (2.26).

Corollary 2.3. *Under the assumptions of Part 1 of Theorem 2.2. there is an order equivalence of the cones*

$$\check{K}(T) \approx K(T) \tag{2.39}$$

with respect to order relation (2.23).

This follows from the pointwise covering $\check{K}(T) \leq K(T)$ and order covering (2.32).

Corollary 2.4. *Under the assumptions of Theorem 2.2, Part 2 an ordinal equivalence of the cones*

$$\tilde{K}(T) \approx K(T) \tag{2.40}$$

holds with respect to order relation (2.23).

This follows from the pointwise covering $K(T) \leq \tilde{K}(T)$ and order covering (2.35).

3 Proofs of the results of Section 2

3.1 Proof of Theorem 2.1, Part 1

We prove the first covering property in (2.24). For $\check{h} \in \check{K}(T)$ for any constant $c > 1$ we find a function $\check{g} \in \tilde{E}^{\downarrow}(0, T)$, such that

$$\check{h}(t) = \check{h}(\check{g}; t), \ t \in \mathbb{R}_+$$

and

$$\| \check{g} \|_{\tilde{E}(\mathbb{R}_+)} \leq c\rho_{\check{K}(T)}(\check{h}),$$

(see (2.9), (2.12)). We set $h(t) := h(\check{g}; t)$, i.e.

$$h(t) = \int\limits_0^\infty f_\varphi(t, \tau)\check{g}(\tau)d\tau \in K(T); \rho_{K(T)}(h) \leq \| \check{g} \|_{\tilde{E}(\mathbb{R}_+)}$$

(see (2.8), (2.11)). Thus, (2.15) implies the inequality

$$\check{h}(t) = \int\limits_0^\infty \check{f}_\varphi(t, \tau)\check{g}(\tau)d\tau \leq \int_0^\infty f_\varphi(t, \tau)\check{g}(\tau)d\tau = h(t), t \in \mathbb{R}_+,$$

and moreover,

$$\rho_{K(T)}(h) \leq \| \check{g} \|_{\tilde{E}(\mathbb{R}_+)} \leq c\rho_{\check{K}(T)}(\check{h}).$$

These estimates prove the covering $\check{K}(T) \leq K(T)$ with the following covering constants: any $c_0(A) > 1$; $c_1(A) = 0$.

Similarly, we prove the second covering in (2.24): $K(T) \leq \tilde{K}(T)$ with the same covering constants (we use the second inequality in (2.15)).

3.2 Proof of Theorem 2.1, Part 2

Let $h \in K(T)$. Then

$$\exists g \in \tilde{E}^{\downarrow}(0,T) : h(t) = h(g;t),\ \| g \|_{\tilde{E}(\mathbb{R}_+)} \leq 2\rho_{K(T)}(h).$$

We introduce

$$\breve{g}(\tau) = c2^n g(2^n\tau) \in \tilde{E}^{\downarrow}(0, 2^{-n}T). \tag{3.1}$$

Here $c \in [1,\infty)$ is the constant from condition (2.25).
We have, by (2.8)

$$h \in K(T) \Rightarrow h(t) = \int_0^\infty f_\varphi(t,\tau)g(\tau)d\tau = \int_0^\infty \check{f}_\varphi(2^{-n}t, 2^{-n}\tau)g(\tau)d\tau =$$

$$2^n \int_0^\infty \check{f}_\varphi(2^{-n}t, s)g(2^n s)ds = \frac{1}{c}\int_0^\infty \check{f}_\varphi(2^{-n}t, s)\breve{g}(s)ds. \tag{3.2}$$

We define

$$\check{h}(t) = \int_0^\infty \check{f}_\varphi(t,s)\breve{g}(s)ds \in \check{K}(T), \tag{3.3}$$

then, by (3.1)

$$\begin{aligned}\rho_{\check{K}(T)}(\check{h}) &\leq \| \breve{g} \|_{\tilde{E}(\mathbb{R}_+)} = c2^n \| \sigma_{2^n}(g) \|_{\tilde{E}(\mathbb{R}_+)} \\ &\leq c2^n \| \sigma_{2^n} \| \| g \|_{\tilde{E}(\mathbb{R}_+)} \leq c2^{n+1} \| \sigma_{2^n} \| \rho_{K(T)}(h).\end{aligned} \tag{3.4}$$

Here $\sigma_{2^n} : \tilde{E}(\mathbb{R}_+) \to \tilde{E}(\mathbb{R}_+)$ is a bounded operator: $\| \sigma_{2^n} \| < \infty$. Thus, for any $h \in K(T)$ we found $\check{h} \in \check{K}(T)$, such that $\rho_{\check{K}(T)}(\check{h}) \leq c_0\rho_{K(T)}(h)$, where

$$c_0 = c2^{n+1} \| \sigma_{2^n} \| \in \mathbb{R}_+.$$

Further, it follows from (3.2) and (3.3) that

$$h(t) = \frac{1}{c}\check{h}(2^{-n}t) = \frac{1}{c}\int_0^\infty \check{f}_\varphi(2^{-n}t, s)\breve{g}(s)ds, \quad \forall t \in \mathbb{R}_+,$$

i.e.

$$\frac{1}{c}\check{h}(t) = h(2^n t) = \int_0^\infty f\varphi(2^n t, \tau)g(\tau)d\tau,\ t \in \mathbb{R}_+. \tag{3.5}$$

We show that for $t \in (0, 2^{-n}T)$ the following estimate holds for all $\tau \in \mathbb{R}_+$:

$$f_\varphi(2^n t, \tau) \geq \frac{1}{c}f_\varphi(t,\tau). \tag{3.6}$$

Indeed, if $t \in (0, 2^{-n}T)$ we have

$$f_\varphi(2^n t,\tau) = \left\{\begin{matrix}\varphi(2^n t), & 0 < \tau \leq 2^n t \\ \varphi(\tau), & \tau > 2^n t\end{matrix}\right\} \geq \left\{\begin{matrix}\frac{1}{c}\varphi(t), & 0 < \tau \leq 2^n t \\ \varphi(\tau), & \tau > 2^n t\end{matrix}\right\} \geq \frac{1}{c}f_\varphi(t,\tau).$$

It follows from (3.5) and (3.6) that

$$\frac{1}{c}\check{h}(t) \geq \frac{1}{c}\int\limits_0^\infty f_\varphi(t,\tau)g(\tau)d\tau = \frac{1}{c}h(t)$$

i.e.

$$h(t) \leq \check{h}(t), t \in (0, 2^{-n}T). \tag{3.7}$$

If $T = \infty$ this inequality is true for all $t \in \mathbb{R}_+$. Now, let $T < \infty$. If $t \in [2^{-n}T, T)$ we have $\check{h}(t) = 0$ (since $\check{f}_\varphi(t,\tau) = 0, \tau \in \mathbb{R}_+$), and, by virtue of (2.18), for all $t \in \mathbb{R}_+$,

$$h(t) \leq h(2^{-n}T) \leq \| f_\varphi(2^{-n}T, \cdot) \|_{\tilde{E}'(\mathbb{R}_+)} \rho_{K(T)}(h) = \check{h}(t) + c_1\rho_{K(T)}(h). \tag{3.8}$$

Inequalities (3.7) with $T = \infty$ or (3.8) with $T < \infty$ together with (3.4) prove (2.26)-(2.27).

3.3 Proof of Theorem 2.1, Part 3

Let $\tilde{h} \in \tilde{K}(T)$. According to (2.10), (2.13), for any $\varepsilon > 0$ there exists $\tilde{g} \in \tilde{E}^{\downarrow}(0,T)$, such that

$$\tilde{h}(t) = \tilde{h}(\tilde{g}; t) = \int\limits_0^\infty \tilde{f}_\varphi(t,\tau)\tilde{g}(\tau)d\tau; \quad \| \tilde{g} \|_{\tilde{E}(\mathbb{R}_+)} \leq (1+\varepsilon)\rho_{\tilde{K}(T)}(\tilde{h}).$$

We then define $h \in K(T)$ by the formula

$$h(t) = h(\tilde{g}; t) = \int\limits_0^\infty f_\varphi(t,\tau)A_\varphi\tilde{g}(\tau)d\tau.$$

Here $A_\varphi\tilde{g} \in \tilde{E}^{\downarrow}(0,T)$ and

$$\rho_{K(T)}(h) \leq \| A_\varphi\tilde{g} \|_{\tilde{E}(\mathbb{R}_+)} = A_\varphi \| \tilde{g} \|_{\tilde{E}(\mathbb{R}_+)} \leq (1+\varepsilon)A_\varphi\rho_{\tilde{K}(T)}(\tilde{h}). \tag{3.9}$$

Let us note that $A_\varphi \geq 1$, by virtue of inequality (2.14), so that from (2.4) we obtain for $t \in (0,T)$

$$\tilde{f}_\varphi(t,\tau) \leq \begin{Bmatrix} A_\varphi\varphi(t), & 0 < \tau \leq t, \\ \varphi(\tau), & t < \tau < \infty \end{Bmatrix} \leq A_\varphi \begin{Bmatrix} \varphi(t), & 0 < \tau \leq t, \\ \varphi(\tau), & t < \tau < \infty \end{Bmatrix} = A_\varphi f_\varphi(t,\tau).$$

So if $t \in (0,T)$, then

$$\tilde{h}(t) = \int_0^\infty \tilde{f}_\varphi(t,\tau)\tilde{g}(\tau)d\tau \leq A_\varphi\int_0^\infty f_\varphi(t,\tau)\tilde{g}(\tau)d\tau = h(t). \tag{3.10}$$

The obtained estimates for $T = \infty$ prove the covering $\tilde{K}(\infty) \preceq K(\infty)$ with the covering constants $c_0 = (1+\varepsilon)A_\varphi$ (for any $\varepsilon > 0$,) $c_1 = 0$. If $T < \infty$ inequality (3.10) must be supplemented by the corresponding estimate for $t \in [T, \infty)$. For such values of t we have $f_\varphi(t,\tau) = 0, \tau \in \mathbb{R}_+$, so that $h(t) = 0, t \in [T,\infty)$. At the same time, $\tilde{g}(\tau) = 0,\ \tau \in [T,\infty)$, so that $\tilde{h}(t) = \int\limits_0^T \tilde{f}_\varphi(t,\tau)g(\tau)d\tau$. In formula (2.4) with $\tau \in (0,T),\ t \in [T,\infty)$ we take into account that $\varphi(\xi) = 0,\ \ \xi \in [T,t]$, so that

$$\tilde{f}_\varphi(t,\tau) = \frac{1}{t}\int\limits_0^T \varphi(\xi)d\xi \leq \frac{1}{T}\int\limits_0^T \varphi(\xi)d\xi = \tilde{f}_\varphi(T,\tau) \Rightarrow \tilde{h}(t) \leq \tilde{h}(T).$$

From this and (2.20) it follows that for $t \in [T, \infty)$

$$\tilde{h}(t) \leq \| \tilde{f}_\varphi(T,\cdot) \|_{\tilde{E}'(\mathbb{R}_+)} \rho_{\tilde{K}(T)}(\tilde{h}) = h(t) + \| \tilde{f}_\varphi(T,\cdot) \|_{\tilde{E}'(\mathbb{R}_+)} \rho_{\tilde{K}(T)}(\tilde{h}).$$

Together with inequality (3.10), this gives

$$\tilde{h}(t) \leq h(t) + \| \tilde{f}_\varphi(T,\cdot) \|_{\tilde{E}'(\mathbb{R}_+)} \rho_{\tilde{K}(T)}(\tilde{h}), t \in \mathbb{R}_+. \tag{3.11}$$

From (3.9) and (3.11) follows covering (2.29) with covering constants (2.30), (2.31).

3.4 Proof of Theorem 2.2, Part 1

For any $h \in K(T)$ we find $g \in \tilde{E}^{\downarrow}(0,T)$, such that (see (2.8), (2.11))

$$h(g;t) = h(t), t \in \mathbb{R}_+; \| g \|_{\tilde{E}(\mathbb{R}_+)} \leq 2\rho_{K(T)}(h). \tag{3.12}$$

We set

$$\check{g}(\tau) = 2^{2n} g(2^n\tau), \tau \in \mathbb{R}_+; \;\; \check{h}(t) = \check{h}(\check{g};t) = \int\limits_0^\infty \check{f}_\varphi(t,\tau)\check{g}(\tau)d\tau. \tag{3.13}$$

Then, $0 \leq \check{g}(\tau) \downarrow, \check{g}(\tau+0) = \check{g}(\tau), \tau \in \mathbb{R}_+; \check{g}(\tau) = 0, \tau \in [2^{-n}T, \infty)$ (the latter if $T < \infty$), and

$$\| \check{g} \|_{\tilde{E}(\mathbb{R}_+)} = 2^{2n} \| g(2^n\cdot) \|_{\tilde{E}(\mathbb{R}_+)} \leq 2^{2n} \| \sigma_{2^n} \| \| g \|_{\tilde{E}(\mathbb{R}_+)} \leq 2^{2n+1} \| \sigma_{2^n} \| \rho_{K(T)}(h).$$

Thus, $\check{g} \in \tilde{E}^{\downarrow}(0, 2^{-n}T)$ and $\check{h}(t) = \check{h}(g;t) \in \check{K}(T)$, moreover

$$\rho_{\check{K}(T)}(\check{h}) \leq \| \check{g} \|_{\tilde{E}(\mathbb{R}_+)} \leq 2^{2n+1} \| \sigma_{2^n} \| \rho_{K(T)}(h). \tag{3.14}$$

Further,

$$0 \leq h \downarrow; \;\; h(t+0) = h(t), t \in \mathbb{R}_+ \Rightarrow h^* = h;$$

$$0 \leq \check{h} \downarrow; \;\; \check{h}(t+0) = \check{h}(t), \; t \in \mathbb{R}_+ \Rightarrow (\check{h})^* = \check{h}.$$

Therefore, to prove the ordinal covering for relation of order (2.23), it suffices to verify that

$$\int\limits_0^t h d\xi \leq \int_0^t \check{h} d\xi, \; t \in \mathbb{R}_+. \tag{3.15}$$

Together with (3.11) this proves the covering $K(T) \prec \check{K}(T)$ with the covering constants $c_0(E) = 2^{2n+1} \| \sigma_{2^n} \|, \;\; c_1(E) = 0$ (see (2.30)).

According to (2.8), for $h \in K(T)$

$$H(t) := \int\limits_0^t h(\xi)d\xi = \int_0^t \left(\int_0^\infty f_\varphi(\xi,\tau)g(\tau)d\tau \right) d\xi =$$

$$\int_0^t \left(\int_0^\infty \check{f}_\varphi(2^{-n}\xi, 2^{-n}\tau) g(\tau) d\tau \right) d\xi.$$

We make the substitution $\lambda = 2^{-n}\xi; d\xi = 2^n d\lambda$ and $s = 2^{-n}\tau, d\tau = 2^n ds$. Then, taking into account equalities (3.10)

$$H(t) = 2^n \int_0^{2^{-n}t} (\int_0^\infty \check{f}_\varphi(\lambda, 2^{-n}\tau) g(\tau) d\tau) d\lambda = 2^{2n} \int_0^{2^{-n}t} (\int_0^\infty \check{f}_\varphi(\lambda, s) g(2^n s) ds) d\lambda =$$

$$\int_0^{2^{-n}t} (\int_0^\infty \check{f}_\varphi(\lambda, s) \check{g}(s) ds) d\lambda = \int_0^{2^{-n}t} \check{h}(\lambda) d\lambda.$$

So, for $h \in K(T)$ we find $\check{h} \in \check{K}(T)$, such that estimate (3.11) holds and

$$\int_0^t h(\xi) d\xi = \int_0^{2^{-n}t} \check{h}(\lambda) d\lambda \leq \int_0^t \check{h}(\lambda) d\lambda, \quad t \in \mathbb{R}_+.$$

In the last estimate, we took into account that $\check{h}(\lambda) \geq 0, \lambda \in \mathbb{R}_+$. Thus, inequalities (3.11) and (3.12) are valid, and for $h \in K(T)$, $\check{h} \in \check{K}(T)$ (3.12) coincides with the condition that $h \prec \check{h}$ with respect to order relation (3.23). Thus, we have proved order covering (2.30) with the covering constants

$$c_0(E) \leq 2^{2n+1} \parallel \sigma_{2^n} \parallel, \quad c_1(E) = 0.$$

Remark 5. Theorem 2.2, Part 2, coincides with Theorem 2 of [2], where we also give a (non-trivial) proof of this theorem.

4 On the covering of cones of decreasing rearrangements for potentials

In this section we obtain results on equivalent descriptions for cones of decreasing rearrangements of potentials $u \in H_E^G(\mathbb{R}^n)$ in terms of the cones studied in Sections 2 and 3: $K(T)$, $\check{K}(T)$, $\tilde{K}(T)$. A summary of these results is given in [2] (Section 6).

Let $R \in (0, \infty]$. We introduce the class $I_n(R)$ of all functions Φ on $\mathbb{R}_+$ with the properties:

$$1)\ 0 < \Phi(r) \downarrow,\ \Phi(r+0) = \Phi(r);\ \int_0^r \Phi(\rho)\rho^{n-1} d\rho < \infty,\ r \in (0, R); \tag{4.1}$$

$$2)\ \text{if} \quad R < \infty, \quad \text{then} \quad \Phi(r) = 0, \quad r \in [R, \infty). \tag{4.2}$$

We denote

$$T = V_n \mathbb{R}^n,\ \text{if } R < \infty;\ T = \infty,\ \text{if } R = \infty. \tag{4.3}$$

Here V_n is the volume of the unit ball in $\mathbb{R}^n$. We also define

$$\varphi(t) = \Phi\left(\left(\frac{t}{V_n} \right)^{\frac{1}{n}} \right), \quad t \in \mathbb{R}_+. \tag{4.4}$$

Then,

$$\Phi \in I_n(R) \Rightarrow \varphi \in \Omega(T), \tag{4.5}$$

see (2.1).

For a basic RIS $E(\mathbb{R}^n)$ denote its Luxemburg representation by $\tilde{E}(\mathbb{R}_+)$, (see [1]), i.e. $\tilde{E}(\mathbb{R}_+)$ is an RIS on $\mathbb{R}_+$, such that

$$\| f \|_{E(\mathbb{R}^n)} = \| f^* \|_{\tilde{E}(\mathbb{R}_+)},\ f \in E(\mathbb{R}^n), \tag{4.6}$$

where f^* is a decreasing rearrangement of the function f with respect to the n -dimensional Lebesgue measure. Note that the associated RIS $\tilde{E}'(\mathbb{R}_+)$ coincides with the Luxemburg representation for the RIS $E'(\mathbb{R}^n)$, associated with the RIS $E(\mathbb{R}^n)$.

Consider the space H_E^G of potentials (1.1) - (1.3) with admissible kernels G.

Our considerations cover the generalization of the classical Riesz and Bessel potentials introduced in [5, 6]. For some $0 < d_0 \le d_1 < \infty$ let

$$d_0\Phi(r) \le G(x) \le d_1\Phi(r),\ r = |x| \in (0, R);\ \Phi \in I_n(R). \tag{4.7}$$

If here $R = \infty$, and therefore $T = \infty$, then, in addition to (4.5), condition (2.17) is assumed to be satisfied. In this case we call the potentials $u \in H_E^G(\mathbb{R}^n)$ generalized Riesz potentials. It follows from (4.1) and (2.17) that the kernels of generalized Riesz potentials are admissible, that is, they satisfy the condition (1.2). The classical Riesz potentials correspond to the case

$$G(x) = r^{\alpha-n},\quad r = |x| \in \mathbb{R}_+;\ 0 < \alpha < n. \tag{4.8}$$

For them conditions (4.1), (4.7) are satisfied with $\Phi(r) = r^{\alpha-n}$, and condition (2.17) has the form

$$\tau^{\frac{\alpha}{n}-1}\chi_{(t,\infty)}(\tau) \in \tilde{E}'(\mathbb{R}_+),\quad t \in \mathbb{R}_+. \tag{4.9}$$

If in (4.7) $R < \infty$, $B_R = \{x \in \mathbb{R}^n :\ |x| < R\}$, $G_R^0 \equiv G\chi_{B_R}$,

$$G_R^1 \equiv G\chi_{\mathbb{R}^n\setminus B_R} \in E'(\mathbb{R}^n) \cap L_1(\mathbb{R}^n),\ \int_{\mathbb{R}^n} Gdx \ne 0, \tag{4.10}$$

then the potentials with such kernels G are called generalized Bessel potentials. It follows from (4.1), (4.7) and (4.10) that the kernels of generalized Bessel potentials are admissible. The classical Bessel potentials correspond to the case

$G(x) = c(\alpha, n)r^{-\nu}K_\nu(r)$, $\quad r = |x| \in \mathbb{R}_+$, $\quad \alpha \in (0, n]$, $\quad \nu = (n-\alpha)/2$, $c(\alpha, n) \in \mathbb{R}_+$, where K_ν are Bessel-Macdonald functions, see [8, 9]. From their asymptotics in the neighborhood of the origin and at infinity, it follows the fulfillment of properties (4.8) with $\Phi(r) = r^{\alpha-n}$, $r = |x| \le R$ and (4.10).

Theorem 4.1. *In the notations and conditions (4.1)-(4.7), let the estimate*

$$G(x) \ge d_0\Phi(r),\quad r = |x| \in \mathbb{R}_+ \tag{4.11}$$

be valid for some $d_0 \in \mathbb{R}_+$. Then, the pointwise covering takes place for cones (1.6), (2.9):

$$(A):\qquad \check{K}_{\varphi,\tilde{E}}(T) \le M_E^G(T), \tag{4.12}$$

with the covering constants (see (2.21))

$$c_0(A) = (1+\varepsilon)d_0^{-1},\quad \forall\varepsilon > 0;\quad c_1(A) = 0. \tag{4.13}$$

Proof. We follow the scheme of the second step of the proof of Theorem 2.16 in [5] (see there rephrase formulae (3.12)-(3.26)), but we substantially weaken the conditions imposed there.

Let $\check{h} \in \check{K}(T) \equiv \check{K}_{\varphi,\tilde{E}}(T)$. There exists a function $g \in \tilde{E}^{\downarrow}(0,T)$, such that

$$\check{h}(t) = \int_0^\infty \check{f}_\varphi(t,\tau)g(\tau)d\tau, \qquad \|g\|_{\tilde{E}(\mathbb{R}_+)} \le (1+\varepsilon)\rho_{\check{K}(T)}(\check{h}). \tag{4.14}$$

We set

$$f(x) = d_0^{-1}g(V_n r^n), \qquad r = |x| \in \mathbb{R}_+. \tag{4.15}$$

The function $f \ge 0$, is radially symmetric, decreasing and right continuous as a function of r. Therefore, for its symmetric rearrangement $f^\sharp(r)$ and decreasing rearrangement $f^*(t)$ we have

$$f^\sharp(r) = d_0^{-1}g(V_n r^n), \qquad r \in \mathbb{R}_+; \quad f^*(t) = d_0^{-1}g(t), \ \ t \in \mathbb{R}_+. \tag{4.16}$$

Therefore,

$$\|f\|_{E(\mathbb{R}^n)} = \|f^*\|_{\tilde{E}(\mathbb{R}_+)} = d_0^{-1}\|g\|_{\tilde{E}(\mathbb{R}_+)} \le (1+\varepsilon)d_0^{-1}\rho_{\check{K}(T)}(\check{h}); \tag{4.17}$$

$$u = G * f \in H_E^G(\mathbb{R}^n), \quad \|u\|_{H_E^G} \le \|f\|_{E(\mathbb{R}^n)} \le (1+\varepsilon)d_0^{-1}\rho_{\check{K}(T)}(\check{h}). \tag{4.18}$$

Then, from (1.6), (1.7) it follows that

$$h(t) := u^*(t) \in M_E^G(T), \tag{4.19}$$

$$\rho_{M(T)}(h) \le \|u\|_{H_E^G} \le (1+\varepsilon)d_0^{-1}\rho_{\check{K}(T)}(\check{h}). \tag{4.20}$$

Let us estimate $h(t)$ from below. Estimate (4.11) and the decreasing of Φ imply

$$G(x-y) \ge d_0\Phi(|x-y|) \ge d_0\Phi(|x|+|y|), \qquad x, y \in \mathbb{R}^n,$$

so that

$$u(x) = \int_{\mathbb{R}^n} G(x-y)f(y)dy \ge d_0 \int_{\mathbb{R}^n} \Phi(|x|+|y|)f(y)dy.$$

Now we take into account equality (4.15) and take the spherical coordinates in the integral. Then

$$u(x) \ge c_n \int_0^\infty \Phi(|x|+\rho)g(V_n\rho^n)\rho^{n-1}d\rho, \tag{4.21}$$

where $c_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ (see for example, [4, p. 403]). The function in the right-hand side of (4.21) is radially symmetric, nonnegative and decreasing as a function of $|x|$. It coincides with its symmetric rearrangement. Thus, for $r \in \mathbb{R}_+$, (4.21) implies

$$u^\sharp(r) \ge c_n \int_0^\infty \Phi(r+\rho)g(V_n\rho^n)\rho^{n-1}d\rho.$$

Further,

$$r+\rho \le 2\max\{r, \ \rho\} \Rightarrow \Phi(r+\rho) \ge \Phi(2\max\{r, \ \rho\});$$

$$u^\sharp(r) \ge c_n \left[\int_0^r \Phi(2r)g(V_n\rho^n)\rho^{n-1}d\rho + \int_r^\infty \Phi(2\rho)g(V_n\rho^n)\rho^{n-1}d\rho\right] =$$

$$\frac{c_n}{nV_n}\left[\Phi(2r)\int\limits_0^{V_nr^n} g(\tau)d\tau + \int\limits_{V_nr^n}^{\infty} \Phi\left(2\left(\frac{\tau}{V_n}\right)^{\frac{1}{n}}\right) g(\tau)d\tau\right].$$

Here $c_n = nV_n$, so that for $t = V_nr^n$ we have

$$\Phi(2r) = \Phi\left(2\left(\frac{t}{V^n}\right)^{\frac{1}{n}}\right) = \varphi(2^n t); \quad u^\sharp(r) = u^*(t), \tag{4.22}$$

and arrived at the estimate

$$u^*(t) \geq \varphi(2^n t)\int\limits_0^t g(\tau)d\tau + \int\limits_t^\infty \varphi(2^n\tau)g(\tau)d\tau = \int\limits_0^\infty \check{f}_\varphi(t,\tau)g(\tau)d\tau \tag{4.23}$$

(see (2.3)), so that $h(t) \geq \check{h}(t),\ \ t \in \mathbb{R}_+$, (see (4.14), (4.19)). As a result, for every $\check{h} \in \check{K}(T)$ we find function $h \in M^G_E(T)$, such that estimate (4.20) holds with the constant $c_0 = (1+\varepsilon)d_0^{-1},\ \forall\epsilon > 0$ and $h(t) \geq \check{h}(t),\ \ t \in \mathbb{R}_+$. It gives covering (4.12) with covering constants (4.13). □

Theorem 4.2. *In the notation and assumptions (4.1) - (4.4), let* $R \in (0,\infty],\ \Phi \in I_n(R),\ \ d_1 \in \mathbb{R}_+$.

1. *If* $R = \infty$*, we assume that*

$$G^\#(r) \leq d_1\Phi(r),\ r \in \mathbb{R}_+, \tag{4.24}$$

and that condition (2.17) is satisfied.

2. *If* $R < \infty$*, then in the decomposition*

$$G = G^0_R + G^1_R, \quad G^0_R = G\chi_{B_R}, \quad G^1_R = G\chi_{\mathbb{R}^n\setminus B_R}, \tag{4.25}$$

we assume that

$$(G)^\#(r) \leq d_1\Phi(r),\ r \in (0,R);\ G^1_R \in E'(\mathbb{R}^n). \tag{4.26}$$

Then, there is a pointwise covering

$$(B) : \tilde{M}^G_E(T) \leq \tilde{K}_{\varphi,\tilde{E}}(T) \tag{4.27}$$

with covering constants:

$$c_0(B) = d_1(1+\varepsilon), \quad \forall\varepsilon > 0; \quad c_1(B) = 0, \tag{4.28}$$

if $T = \infty$*,*

and

$$c_0(B) = d_1(1+\varepsilon), \quad ; \quad c_1(B) = (1+\varepsilon)\parallel G^1_R \parallel_{E'(\mathbb{R}^n)},\ \forall\varepsilon > 0, \tag{4.29}$$

if $T < \infty$*.*

Proof. Let $h \in \tilde{M}(T)$. Then, for $\varepsilon > 0$ there exists $u = u_\varepsilon \in H^G_E(\mathbb{R}^n)$, such that

$$h(t) = u^{**}(t), t \in (0,T); \parallel u \parallel_{H^G_E} \leq \sqrt{1+\varepsilon}\rho_{\tilde{M}(T)}(h).$$

For $u \in H^G_E(\mathbb{R}^n)$ there exists $f = f_\varepsilon \in E(\mathbb{R}^n)$, such that

$$u = G * f; \quad \| f \|_{E(\mathbb{R}^n)} \leq \sqrt{1+\varepsilon} \, \| u \|_{H_E^G} \leq (1+\varepsilon)\rho_{\tilde{M}(T)}(h).$$

Further,

$$f \in E(\mathbb{R}^n) \Rightarrow f^* \in \tilde{E}^{\downarrow}(\mathbb{R}_+); \; \| f^* \|_{\tilde{E}(\mathbb{R}_+)} = \| f \|_{E(\mathbb{R}^n)} \leq (1+\varepsilon)\rho_{\tilde{M}(T)}(h). \tag{4.30}$$

Let $\tilde{g}(\tau) = f^*(\tau)\chi_{(0,T)}(\tau)$. Then,

$$\tilde{g} \in \tilde{E}^{\downarrow}(0,T); \quad \| \tilde{g} \|_{\tilde{E}(\mathbb{R}_+)} \leq \| f^* \|_{\tilde{E}(\mathbb{R}_+)} \leq (1+\varepsilon)\rho_{\tilde{M}(T)}(h).$$

We define $\tilde{h} = \tilde{h}_\varepsilon$ by formula

$$\tilde{h}(t) = d_1 \int_0^\infty \tilde{f}_\varphi(t,\tau)\tilde{g}(\tau)d\tau \in \tilde{K}(T). \tag{4.31}$$

Then,

$$\rho_{\tilde{K}(T)}(\tilde{h}) \leq \| d_1\tilde{g} \|_{\tilde{E}(\mathbb{R}_+)} = d_1 \, \| \tilde{g} \|_{\tilde{E}(\mathbb{R}_+)} \leq d_1(1+\varepsilon)\rho_{\tilde{M}(T)}(h). \tag{4.32}$$

To estimate the function $h(t) = u^{**}(t) = (G * f)^{**}(t)$ from above we apply the O'Neil inequality [10]

$$h(t) \leq \frac{1}{t} \left(\int_0^t G^* d\tau \right) \left(\int_0^t f^* d\tau \right) + \int_t^\infty G^* f^* d\tau, \quad t \in \mathbb{R}_+. \tag{4.33}$$

For $R = \infty$ (that is, $T = \infty$) it follows from (4.24) that for $\tau \in \mathbb{R}_+$

$$G^*(\tau) = G^{\#}\left(\left(\frac{\tau}{V_n} \right)^{\frac{1}{n}} \right) \leq d_1 \Phi \left(\left(\frac{\tau}{V_n} \right)^{\frac{1}{n}} \right) = d_1\varphi(\tau). \tag{4.34}$$

Therefore, (4.33) implies that: for $t \in \mathbb{R}_+$

$$\begin{aligned} h(t) \leq d_1 &\left[\left(\frac{1}{t} \int_0^t \varphi d\tau \right) \left(\int_0^t f^* d\tau \right) + \int_t^\infty \varphi f^* d\tau \right] \\ &= d_1 \int_0^\infty \tilde{f}_\varphi(t,\tau)\tilde{g}(\tau)d\tau = \tilde{h}(t) \end{aligned} \tag{4.35}$$

(we took into account that $f^*(\tau) = \tilde{g}(\tau)$, $\tau \in \mathbb{R}_+$). From (4.32) and (4.35) follows (4.27) with covering constants (4.28).

Now let $R < \infty$, i.e. $T = V_n R^n = \mu(B_R) < \infty$. Then, for $t \in (0,T)$, estimate (4.33) gives

$$h(t) \leq \frac{1}{t} \left(\int_0^t G^* d\tau \right) \left(\int_0^t f^* d\tau \right) + \int_t^T G^* f^* d\tau + \int_T^\infty G^* f^* d\tau. \tag{4.36}$$

For $\tau \in (0,T)$ we have $r = ((\frac{\tau}{V_n})^{\frac{1}{n}}) \in (0,R)$, so the first estimate in (4.26) is applicable to the first two terms in (4.36), which gives inequality (4.34) for $\tau \in (0,T)$. As a result, for $t \in (0,T)$, instead of (4.35), we obtain

$$h(t) \leq d_1 \int_0^T \tilde{f}_\varphi(t,\tau)\tilde{g}(\tau)d\tau + \int_T^\infty G^* f^* d\tau = \tilde{h}(t) + \int_T^\infty G^*(\tau) f^*(\tau) d\tau. \tag{4.37}$$

Finally, we have to estimate the second term in the right-hand side of (4.37). We show that for $\tau > T$

$$G^*(\tau) \leq (G_R^1)^*(\tau - T). \tag{4.38}$$

Decomposition (4.25) implies that

$$\{x \in \mathbb{R}^n : \ |G(x)| > y\} = \{x \in B_R : |G_R^0(x)| > y\} \cup \{x \in \mathbb{R}^n \backslash B_R : |G_R^1| > y\},$$

so that for the distribution functions the following estimate holds

$$\lambda_G(y) = \mu\{x \in \mathbb{R}^n : |G(x)| > y\} \leq \lambda_{G_R^0}(y) + \lambda_{G_R^1}(y) \leq T + \lambda_{G_R^1}(y),$$

since $\mu(B_R) = V_n R^n = T$. Then, for $\tau > T$ we have

$$\lambda_{G_R^1}(y) \leq \tau - T \Rightarrow \lambda_G(y) \leq \tau \Rightarrow \{y : \lambda_{G_R^1}(y) \leq \tau - T\} \subset \{y : \lambda_G(y) \leq \tau\}.$$

Therefore,

$$G^*(\tau) = inf\{y > 0 : \lambda_G(y) \leq \tau\} \leq inf\{y > 0 : \lambda_{G_R^1}(y) \leq \tau - T\} = (G_R^1)^*(\tau - T),$$

which gives estimate (4.38). It follows that

$$\int_T^\infty G^*(\tau) f^*(\tau) d\tau \leq \int_T^\infty (G_R^1)^*(\tau - T) f^*(\tau) d\tau$$

$$= \int_0^\infty (G_R^1)^*(\xi) f^*(\xi + T) d\xi \leq \int_0^\infty (G_R^1)^*(\xi) f^*(\xi) d\xi.$$

Hence, by *Hölder* inequality we obtain

$$\int_T^\infty G^*(\tau) f^*(\tau) d\tau \leq \| G_R^1 \|_{E'(\mathbb{R}^n)} \| f \|_{E(\mathbb{R}^n)} \leq \| G_R^1 \|_{E'(\mathbb{R}^n)} (1+\varepsilon) \rho_{\tilde{M}(T)}(h)$$

(in the last inequality we apply estimate (4.30)). We substitute this estimate in (4.37):

$$h(t) \leq \tilde{h}(t) + (1+\varepsilon) \| G_R^1 \|_{E'(\mathbb{R}^n)} \rho_{\tilde{M}(T)}(h), \quad t \in (0, T). \tag{4.39}$$

Inequalities (4.32) and (4.39) prove covering (4.27) with covering constants (4.29).

Corollary 4.1 *Suppose that in the assumptions of Theorem 4.2, $R < \infty$, $T = V_n R^n$, and in (4.25)-(4.26) $G = G_R^0$ is the kernel with support in the ball B_R. Then, covering (4.27) is valid with covering constants (4.28).* □

5 Criteria for embeddings of potentials in RIS

5.1 Criteria of embedding for generalized Riesz potentials

Let us describe the application of the above obtained results to the generalized Riesz potentials. We keep the notation and the conditions (4.1) - (4.5), given in Section 4, assuming that $R = \infty$, condition (2.17) is satisfied, and the following two-sided estimate holds with constant $0 < d_0 < d_1 < \infty$:

$$d_0 \Phi(r) \leq G(x) \leq d_1 \Phi(r), \quad r = |x| \in \mathbb{R}_+. \tag{5.1}$$

Then, Theorems 4.1 and 4.2 imply the pointwise coverings of the cones (2.9), (1.6), (1.8), and (2.10) with $T = \infty$, namely

$$\check{K}(\infty) \leq M_E^G(\infty) \leq \tilde{M}_E^G(\infty) \leq \tilde{K}(\infty). \tag{5.2}$$

If additionally the condition $A_\varphi < \infty$ is fulfilled, then the chain of coverings (5.2) is completed by means of the cone $K(\infty)$ (see (2.8), (2.28), (2.29) with $T = \infty$ and Remark 2.6):

$$\tilde{K}(\infty) \leq K(\infty) \leq \check{K}(\infty), \tag{5.3}$$

so that all these cones are pointwise equivalent. In particular, when $A_\varphi < \infty$

$$M_E^G(\infty) \cong \tilde{M}_E^G(\infty) \cong K(\infty). \tag{5.4}$$

From (5.4) and (1.10) we obtain the criterion for embedding of the space of generalized Riesz potentials in the RIS $X(\mathbb{R}^n)$:

$$H_E^G(\mathbb{R}^n) \subset X(\mathbb{R}^n) \Leftrightarrow K(\infty) \mapsto \tilde{X}(\mathbb{R}_+). \tag{5.5}$$

Here $\tilde{X}(\mathbb{R}_+)$ is the Luxemburg representation for RIS $X(\mathbb{R}^n)$ (see (4.6)).

We show that the results of Sections 2-4 allow us to substantially weaken the requirement $A_\varphi < \infty$ for obtaining criterion (5.5). Now let the condition $A_\varphi < \infty$ with $T = \infty$ be replaced by the condition $B_\varphi < \infty$ (2.34). Then, by Theorem 2.7, the order coverings of the cones hold

$$\tilde{K}(\infty) \prec K(\infty) \prec \check{K}(\infty), \tag{5.6}$$

with respect to the order relation (2.23). In addition, the chain of pointwise coverings (5.2) implies the corresponding chain of order coverings

$$\check{K}(\infty) \prec M_E^G \prec \tilde{M}_E^G \prec \tilde{K}(\infty).$$

It is completed because of covering (5.6). So, when $B_\varphi < \infty$ (with $T = \infty$) there is order equivalence

$$M_E^G(\infty) \approx \tilde{M}_E^G(\infty) \approx K(\infty) \tag{5.7}$$

with respect to order relation (2.23). Since the norm in the RIS $X(\mathbb{R}^n)$ is correlated with order relation (2.23), (see, for example, [1], Ch.2, Theorem 4.6), the embeddings of cones (5.7) in the RIS $X(\mathbb{R}^n)$ are equivalent to each other, so that under condition $B_\varphi < \infty$ the validity of criterion (5.5) follows from (1.10). These considerations yield the following result.

Theorem 5.1. *Let $R = \infty$ in the conditions and notations (4.1)-(4.5), and assumptions (2.17), (5.1) and (2.34) with $T = \infty$ be satisfied. Then criterion (5.5) holds for the embedding where $K(\infty)$ is cone (2.8) with $T = \infty$.*

5.2 The embedding criterion for generalized Bessel potentials

Let $E(\mathbb{R}^n)$ be an RIS, G be an admissible kernel, $H_E^G(\mathbb{R}^n)$ be the space of potentials (1.1) - (1.3). Moreover, we now assume that for $R < \infty$

$$G_R^0 \equiv G\chi_{B_R} \in L_1(\mathbb{R}^n) \tag{5.8}$$

$$G_R^1 \equiv G\chi_{\mathbb{R}^n \setminus B_R} \in L_1(\mathbb{R}^n) \cap E'(\mathbb{R}^n), \tag{5.9}$$

where $E'(\mathbb{R}^n)$ is the associated RIS for $E(\mathbb{R}^n)$. Along with $H_E^G(\mathbb{R}^n)$ we also consider the potential space with "truncated kernels":

$$\dot{H}_E^G(R^n) \equiv H_E^{G_R^0}(R^n) = \{u = G_R^0 * f : f \in E(\mathbb{R}^n)\}, \tag{5.10}$$

$$\|u\|_{\dot{H}_E^G(R^n)} = \inf\{\|f\|_E : \quad f \in E(\mathbb{R}^n); \quad G_R^0 * f = u\}. \tag{5.11}$$

For RIS $X \equiv X(\mathbb{R}^n)$, we examine the problem of the criteria for embedding

$$H_E^G(\mathbb{R}^n) \subset X(\mathbb{R}^n). \tag{5.12}$$

Remark 6. In [5, Section 1.2] it is shown that for embedding (5.12) it is necessary that

$$E(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n) \subset X(\mathbb{R}^n). \tag{5.13}$$

Theorem 5.2. *Suppose that conditions (5.8), (5.9) and (5.13) are satisfied. Then, the embedding (5.12) is equivalent to embedding*

$$\dot{H}_E^G(R^n) \subset X(\mathbb{R}^n). \tag{5.14}$$

Proof. **1.** We show that (5.14) $\Rightarrow$ (5.12). For $u \in H_E^G(\mathbb{R}^n)$ for any $\varepsilon > 0$ there exits $f = f_\varepsilon \in E(\mathbb{R}^n)$, such that

$$u = G * f, \quad \|f\|_E \leq (1+\varepsilon)\|u\|_{H_E^G(R^n)}. \tag{5.15}$$

We put

$$u_0 = G_R^0 * f, \quad u_1 = G_R^1 * f. \tag{5.16}$$

Then, $u_0 \in \dot{H}_E^G(R^n)$,

$$\| u_0 \|_{\dot{H}_E^G} \leq \| f \|_E \leq (1+\varepsilon) \| u \|_{H_E^G} < \infty. \tag{5.17}$$

From embedding (5.14) it follows that $u_0 \in X(\mathbb{R}^n)$

$$\| u_0 \|_X \leq c_1 \| u_0 \|_{\dot{H}_E^G} \leq (1+\varepsilon) c_1 \| u \|_{H_E^G}, \tag{5.18}$$

where c_1 is the norm of the embedding operator (5.14). Further, condition (5.9) for the kernel G_R^1 implies the estimate for $u_1 = G_R^1 * f$:

$$\|u_1\|_{E\cap L_\infty} \leq \|G_R^1\|_{L_1\cap E'} \| f \|_E \leq (1+\varepsilon) \| G_R^1 \|_{L_1 \cap E'} \|u\|_{H_E^G} < \infty.$$

Hence embedding (5.13) implies that $u_1 \in X(\mathbb{R}^n)$ and

$$\|u_1\|_X \leq c_2 \|u_1\|_{E\cap L_\infty} \leq (1+\varepsilon) c_2 \|G_R^1\|_{L_1\cap E'} \| u \|_{H_E^G}, \tag{5.19}$$

where c_2 is the norm of the embedding operator (5.13). Further,

$$u = G * f = G_R^0 * f + G_R^1 * f = u_0 + u_1 \in X(\mathbb{R}^n)$$

and by virtue of (5.18), (5.19)

$$\|u\|_X \leq (1+\varepsilon)[c_1 + c_2\|G_R^1\|_{L_1\cap E'}]\|u\|_{H_E^G}, \quad \forall \varepsilon > 0.$$

Here u is independent of $\varepsilon > 0$, so that for $\varepsilon \to +0$ we get

$$\|u\|_X \leq [c_1 + c_2\|G_R^1\|_{L_1\cap E'}]\|u\|_{H_E^G}, \quad \forall u \in H_E^G(\mathbb{R}^n). \tag{5.20}$$

The embedding (5.12) with estimate (5.20) for the norm of the embedding operator is proved.

2. We show that (5.12) $\Rightarrow$ (5.14). The reasoning is analogous to that in step 1. For $u_0 \in \dot{H}^G_E(\mathbb{R}^n)$, $\forall\varepsilon > 0$ there exists $f_0 \in E(\mathbb{R}^n)$:

$$u_0 = G^0_R * f_0, \quad \|f_0\|_E \leq (1+\varepsilon)\|u_0\|_{\dot{H}^G_E}.$$

We put

$$u = G * f_0, \qquad u_1 = G^1_R * f_0.$$

Then, $u \in H^G_E$,

$$\|u\|_{H^G_E} \leq \|f_0\|_E \leq (1+\varepsilon)\|u_0\|_{\dot{H}^G_E} < \infty.$$

From embedding (5.12) it follows that $u \in X(\mathbb{R}^n)$, and

$$\|u\|_X \leq c_3\|u\|_{H^G_E} \leq (1+\varepsilon)c_3\|u_0\|_{\dot{H}^G_E}, \tag{5.21}$$

where c_3 is the norm of the embedding operator (5.12). Further, similarly to (5.19), we get that $u_1 \in X(\mathbb{R}^n)$ and

$$\|u_1\|_X \leq (1+\varepsilon)c_2\|G^1_R\|_{L_1\cap E'}\|u_0\|_{\dot{H}^G_E}. \tag{5.22}$$

From (5.21) and (5.22) we obtain for $u_0 = u - u_1$ the estimate analogous to (5.20):

$$\|u_0\|_X \leq \|u\|_X + \|u_1\|_X \leq [c_3 + c_2\|G^1_R\|_{L_1\cap E'}]\|u_0\|_{\dot{H}^G_E}. \tag{5.23}$$

Thus, embedding (5.14) is obtained with the estimate of the norm of the embedding operator (5.23). □

Theorem 5.3. *1. Let $R \in (0,\infty)$, conditions (5.8), (5.9) and (5.13) be satisfied. Then, embedding (5.12) is equivalent to embedding*

$$M_0(T) \mapsto \tilde{X}(0,T). \tag{5.24}$$

Here $T = V_n R^n$,

$$M_0(T) = \left\{h(t) = u^*(t),\ u \in \dot{H}^G_E, \quad t \in (0,T)\right\}, \tag{5.25}$$

$$\rho_{M_0(T)}(h) = \inf\left\{\|u\|_{\dot{H}^G_E} : \quad u \in \dot{H}^G_E;\ u^*(t) = h(t), \quad t \in (0,T)\right\}; \tag{5.26}$$

$\tilde{X}(\mathbb{R}_+)$ is the Luxemburg representation for RIS $X(\mathbb{R}^n)$; $\tilde{X}(0,T)$ is the restriction of $\tilde{X}(\mathbb{R}_+)$ on $(0,T)$.

Proof. Theorem 5.2 is applicable here. Therefore, it suffices to show that (5.14) $\Leftrightarrow$ (5.24).

1. First we show that (5.14) $\Rightarrow$ (5.24). Let $h \in M_0(T)$. For any $\varepsilon > 0$ there is $u = u_\varepsilon \in \dot{H}^G_E$, such that

$$h(t) = u^*(t),\ t \in (0,T); \quad \|u\|_{\dot{H}^G_E} \leq (1+\varepsilon)\rho_{M_0(T)}(h). \tag{5.27}$$

Consider the function $\tilde{h}(t) = u^*(t), \quad t \in \mathbb{R}_+$. Then, $\tilde{h} \in M_0(\infty)$ and

$$\rho_{M_0(\infty)}(\tilde{h}) \leq \|u\|_{\dot{H}^G_E} \leq (1+\varepsilon)\rho_{M_0(T)}(h).$$

Moreover, by virtue of (5.14), there exists $c_0 \in \mathbb{R}_+$ such that

$$\|\tilde{h}\|_{\tilde{X}(\mathbb{R}_+)} = \|u\|_{X(\mathbb{R}^n)} \leq c_0\|u\|_{\dot{H}^G_E} \leq (1+\varepsilon)c_0\rho_{M_0(T)}(h). \tag{5.28}$$

Here c_0 is the norm of the embedding operator (5.14). Then $h \in \tilde{X}(0,T)$ and

$$\|h\|_{\tilde{X}(0,T)} = \|\tilde{h}\chi_{(0,T)}\|_{\tilde{X}(\mathbb{R}_+)} \le (1+\varepsilon)c_0\rho_{M_0(T)}(h).$$

Here h does not depend on $\varepsilon > 0$, so that when $\varepsilon \to +0$ we get

$$\|h\|_{\tilde{X}(0,T)} \le c_0\rho_{M_0(T)}(h), \quad \forall h \in M_0(T). \tag{5.29}$$

This proves embedding (5.24).

2. Now we prove that (5.24) $\Rightarrow$ (5.14).

Let $u \in \dot{H}^G_E$. For any $\varepsilon > 0$, there exists $f = f_\varepsilon \in E(\mathbb{R}^n)$ such that

$$G^0_R * f = u; \quad \|f\|_E \le (1+\varepsilon)\|u\|_{\dot{H}^G_E}. \tag{5.30}$$

We set $h(t) = u^*(t), \quad t \in \mathbb{R}_+$. Then, $h \in M_0(\infty), \quad h|_{(0,T)} \in M_0(T)$ and $\rho_{M_0(T)}(h) \le \|u\|_{\dot{H}^G_E}$ (see (5.25), (5.26)). We denote by

$$h_0(t) = u^*(t)\chi_{(0,T)}(t), \quad h_1(t) = u^*(t)\chi_{[T,\infty)}(t), \quad t \in \mathbb{R}_+. \tag{5.31}$$

According to (5.24), denoting by $c_1 \in \mathbb{R}_+$ the embedding constant, we have $h|_{(0,T)} \in M_0(T) \Rightarrow h|_{(0,T)} \in \tilde{X}(0,T)$; $\|\tilde{h}\|_{\tilde{X}(0,T)} \le c_1\rho_{M_0(T)}(h)$. Then $h_0 \in \tilde{X}(\mathbb{R}_+)$,

$$\|h_0\|_{\tilde{X}(\mathbb{R}_+)} = \|h\|_{\tilde{X}(0,T)} \le c_1\rho_{M_0(T)}(h) \le c_1\|u\|_{\dot{H}^G_E}$$

(in the last step, we take into account relation (5.26)). Further, embedding (5.13) is accompanied by the estimate

$$\theta_{E\,X}(T) := \sup\left\{\|u^*\chi_{[T,\infty)}\|_{\tilde{X}(\mathbb{R}_+)} : \quad u^* \in \tilde{E}(\mathbb{R}_+); \quad \|u^*\|_{\tilde{E}(\mathbb{R}_+)} \le 1\right\} < \infty \tag{5.32}$$

(see [3], and also [5, Section 4.2]). Therefore, for h_1 (5.31) we have $h_1 \in \tilde{X}(\mathbb{R}_+)$;

$$\|h_1\|_{\tilde{X}(\mathbb{R}_+)} \le \theta_{E\,X}(T)\|u^*\|_{\tilde{E}(\mathbb{R}_+)} = \theta_{E\,X}(T)\|u\|_{E(\mathbb{R}^n)}. \tag{5.33}$$

For $u \in \dot{H}^G_E$ from equality $u = G^0_R * f$, where $G^0_R \in L_1(\mathbb{R}^n)$, it follows that

$$\|u\|_E \le \|G^0_R\|_{L_1}\|f\|_E \le \|G^0_R\|_{L_1}(1+\varepsilon)\|u\|_{\dot{H}^G_E}.$$

Substituting this estimate in (5.33), we obtain

$$\|h_1\|_{\tilde{X}(\mathbb{R}_+)} \le (1+\varepsilon)\theta_{E\,X}(T)\|G^0_R\|_{L_1}\|u\|_{\dot{H}^G_E}.$$

Here h_1 and u do not depend on ε. As a result, when $\varepsilon \to +0$, for $h = h_0 + h_1$ we get $h \in \tilde{X}(\mathbb{R}_+)$ and

$$\|h\|_{\tilde{X}(\mathbb{R}_+)} \le \|h_0\|_{\tilde{X}(\mathbb{R}_+)} + \|h_1\|_{\tilde{X}(\mathbb{R}_+)} \le [c_1 + \theta_{E\,X}(T)\|G^0_R\|_{L_1}]\|u\|_{\dot{H}^G_E}.$$

So for any $u \in \dot{H}^G_E$ we get $u^* = h \in \tilde{X}(\mathbb{R}_+)$ i.e.

$$\|u\|_{X(\mathbb{R}^n)} = \|h\|_{\tilde{X}(\mathbb{R}_+)} \le [c_1 + \theta_{E\,X}(T)\|G^0_R\|_{L_1}]\|u\|_{\dot{H}^G_E} < \infty. \tag{5.34}$$

This proves embedding (5.14) and gives an estimate of the norm of the embedding operator. Thus, we established, under the conditions of Theorem 5.3 that there is an equivalence of embeddings:

$$(5.12) \Leftrightarrow (5.14) \Leftrightarrow (5.24).$$

Theorem 5.4. *Let $R \in (0,\infty)$, the conditions (5.9) and (5.13) be satisfied, $\Phi \in I_n(R)$ (see (4.1), (4.2)) and the estimate*

$$d_0\Phi(r) \le G_R^0(x) \le d_1\Phi(r), \quad r = |x| \in (0,R) \tag{5.35}$$

hold for some $0 < d_0 \le d_1 < \infty$. Suppose further that $B_\varphi(T) < \infty$ (see (2.34)) for $T = V_n R^n$. Then, embedding (5.12) is equivalent to embedding

$$K_{\varphi,\tilde{E}}(T) \mapsto \tilde{X}(0,T), \tag{5.36}$$

where $K_{\varphi,\tilde{E}}(T)$ is cone (2.8); $\tilde{E}(\mathbb{R}_+)$, $\tilde{X}(\mathbb{R}_+)$ are Luxemburg representations for RIS $E(\mathbb{R}^n)$, and $X(\mathbb{R}^n)$, respectively.

Proof. From estimate (5.35) it follows for $\Phi \in I_n(R)$ that $G_R^0(x) \in L_1(B_R)$, so that by Theorem 5.3 (5.12) $\Leftrightarrow$ (5.24). We have to prove that (5.24) $\Leftrightarrow$ (5.36) under condition $B_\varphi(T) < \infty$. For this, we apply Corollaries 2.10 and 2.11 of Theorem 2.7 Relations (2.39) and (2.40) prove the order equivalence of cones

$$\check{K}(T) \approx \tilde{K}(T) \approx K_{\varphi,\tilde{E}}(T) \tag{5.37}$$

with respect to order relation (2.23).

Next, inequality (5.35) is extended to any values of $r \in \mathbb{R}_+$, since for $r \in [R,\infty)$ which sides are equal to 0. Then, the left-inequality in (5.35) coincides with estimate (4.11) for $G = G_R^0$, so that Theorem 4.1 gives a pointwise covering of cones (4.12), which for $G = G_R^0$ coincides with

$$\check{K}(T) \le M_0(T). \tag{5.38}$$

The right inequality (as left-inequality above) in (5.35) leads to an estimate for the symmetric rearrangement $(G_R^0)^\#(r) \le d_1\Phi(r)$, $r \in \mathbb{R}_+$, so that relations (4.26) hold with $G = G_R^0$, $G_R^1 = 0$. By Theorem 4.2, this implies covering (4.27), which for $G = G_R^0$ gives

$$\tilde{M}_0(T) \le \tilde{K}(T). \tag{5.39}$$

Together with the obvious coverings $M_0(T) \le \tilde{M}_0(T)$. This gives a chain of pointwise covering for the cones

$$\check{K}(T) \le M_0(T) \le \tilde{M}_0(T) \le \tilde{K}(T), \tag{5.40}$$

from which follow the order coverings

$$\check{K}(T) \prec M_0(T) \prec \tilde{M}_0(T) \prec \tilde{K}(T) \tag{5.41}$$

with respect to the order relation (2.23). Together with (5.37), they show that all cones in chain (5.41) are order-equivalent to the cone $K_{\varphi,\tilde{E}}(T)$.

Thus,

$$\left\{M_0(T) \approx K_{\varphi,\tilde{E}}(T)\right\} \Rightarrow \{(5.24) \Leftrightarrow (5.36)\}.$$

As a result, applying Theorem 5.3, we obtain (5.12) $\Leftrightarrow$ (5.36). □

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