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KHARIN STANISLAV NIKOLAYEVICH

(to the 80th birthday)

Stanislav Nikolayevich Kharin was born on December 4, 1938 in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis "Heat phenomena in electrical contacts and associated singular integral equations", and in 1990 his doctoral thesis "Mathematical models of thermo-physical processes in electrical contacts" in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. Using models based on the new original methods for solving free boundary problems he described mathematically the phenomena of arcing, contact welding, contact floating, dynamics of contact blow-open phenomena, electrochemical mechanism of electron emission, arc-to-glow transition, thermal theory of the bridge erosion. For these achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research and the evidence of this in the new monograph "Mathematical models of phenomena in electrical contacts" published last year in Novosibirsk.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Stanislav Nikolayevich on the occasion of his 80th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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A NEW RESULT ON MATRIX SUMMABILITY FACTORS OF FOURIER SERIES

Ş. Yıldız

Communicated by N.A. Bokayev

Key words: summability factors, absolute matrix summability, numerical series, Fourier series, Hölder's inequality, Minkowski's inequality, sequence space.

AMS Mathematics Subject Classification: 26D15, 40D15, 40F05, 40G99, 42A24, 46A45.

Abstract. Sulaiman [10] has investigated absolute weighted mean summability theorems for numerical and Fourier series. In the present paper, we have extended the result of Sulaiman to the $|A, p_n|_k$ summability method. Also some new and known results are obtained by using some basic summability methods.

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1 Introduction

Let $\sum a_n$ be a given numerical series with partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, that is (see [2])

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{1.2}$$

The series $\sum a_n$ is said to be $|C, \alpha|_k$ summable, $k \geq 1$, if (see [5],[8])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{1.3}$$

If we take $\alpha = 1$, then $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability.

Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1.4}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.5}$$

defines the sequence (t_n) of the Riesz means or simply the $(\bar{N},p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [6]).

The series $\sum a_n$ is said to be $\left|\bar{N},p_n\right|_k$ summable, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty. \tag{1.6}$$

In the special case when $p_n = 1$ for all values of n (respectively $k = 1$), $\left|\bar{N},p_n\right|_k$ summability is the same as $|C,1|_k$ (respectively $\mid \bar{N},p_n \mid$) summability.

Given a normal matrix $A = (a_{nv})$, i.e., a lower triangular matrix with nonzero diagonal entries. We associate with A two lower triangle matrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n,v = 0,1,... \quad \bar{\Delta}a_{nv} = a_{nv} - a_{n-1,v} \quad a_{-1,0} = 0 \tag{1.7}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta}\bar{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1,2,... \tag{1.8}$$

It should be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Next, let

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{1.9}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{1.10}$$

Let $A = (a_{nv})$ be a normal matrix. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v \quad n = 0,1,... \tag{1.11}$$

The series $\sum a_n$ is said to be $|A,p_n|_k$ summable, $k \geq 1$, if (see [11])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty, \tag{1.12}$$

Note that in the special case when A is the matrix of weighted means, i.e.,

$$a_{nv} = \begin{cases} \frac{p_v}{P_n}, & 0 \leq v \leq n \\ 0, & n > v, \end{cases}$$

then the $|A,p_n|_k$ summability reduces to the $\mid \bar{N},p_n \mid_k$ summability, and if we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n reduces to the $|C,1|_k$ summability. Also, if we take $p_n = 1$ for all values of n reduces to the $|A|_k$ (see [13]) summability.

For any sequence (λ_n) we write $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1} \geq 0$.

Let the formal expansion of a function f, periodic with period 2π, and integrable in the sense of Lebesgue over $[-\pi,\pi]$, in a Fourier trigonometric series be given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) = \sum_{n=0}^{\infty} C_n(x),$$

We write

$$\begin{aligned}
&\phi(u) = f(x+u) + f(x-u) - 2f(x),\\
&\varphi(t) = \int_t^{\delta} \frac{|\phi(u)|}{u} du, \quad \Phi(t) = \int_0^t |\phi(u)| du, \quad 0 < \delta \le \pi,\\
&\mu_n = \left(\prod_{v=1}^{\ell-1} log^v n\right) (log^{\ell} n)^{1+\epsilon}, \quad log^{\ell} n_0 > 0, \quad \epsilon > 0,
\end{aligned}$$

where

$$log^{\ell} n = log(log^{\ell-1} n), ..., log^2 n = loglog\ n.$$

2 The known results

Theorem 2.1 (Chow [4], 1941) *If* $\{\lambda_n\}$ *is a convex sequence and the series* $\sum n^{-1}\lambda_n$ *is convergent, then the series* $\sum C_n(x)\lambda_n$ *is* $|C,1|$ *summable for almost all values of* x.
Theorem 2.2 (Cheng [3], 1948) *If*

$$\Phi(t) = O(t), \quad as \quad t \to 0,$$

then the series

$$\sum_{n=2}^{\infty} C_n(x)/(log\ n)^{1+\epsilon}, \quad \epsilon > 0,$$

is $|C,\alpha|$ *summable,* $\alpha > 1$.
Theorem 2.3 (Hsiang [7], 1970) *If*

$$\Phi(t) = O(t), \quad as \quad t \to +0,$$

then the series

$$\sum_{n=0}^{\infty} C_n(x)/n^{\alpha}$$

where $\alpha > 0$, *is* $|C,1|$ *summable.*
Theorem 2.4 (Pandey [9], 1978) *If*

$$\varphi(t) = O\left\{(log^{\ell}(1/t))^{\eta}\right\} \quad as \quad t \to +0,$$

then the series

$$\sum_{n=0}^{\infty} C_n(x)/\mu_n$$

is $|C,1|$ *summable for* $0 < \eta < \epsilon$.

Sulaiman has proved the more general theorem dealing with $|\bar{N}, p_n|_k$ summability in the following form, which includes the theorems of Chow and Pandey as special cases, and hence all the previous results.

Theorem 2.5 [10] *Let $\{|\lambda_n|\}$ be a non-increasing numerical sequence such that*

$$|\Delta\lambda_n| = O\left(\frac{|\lambda_n|}{n}\right).$$

Let

$$P_n = O(np_n)$$

and $\mid \Delta(\frac{p_n}{P_n}) \mid = O(\frac{p_n}{nP_n})$.

(A) If

$$\sum p_n P_n^{-1} |\lambda_n|^k < \infty, \tag{2.1}$$

then the series

$$\sum C_n(x)\lambda_n$$

is $|\bar{N}, p_n|_k$ summable, $1 \leq k < 2$, for almost all values of x.

(B) If $\{\eta_n\}$ is a sequence of positive numbers such that $n^{-\gamma}\eta_n \to 0$ as $n \to \infty$, for some γ, $0 < \gamma < 1$, and if

$$\varphi(t) = O\left\{\eta_{(1/t)}\right\}, \quad t \to 0 \tag{2.2}$$

$$\sum p_n P_n^{-1} |\lambda_n|^k \eta_n^k < \infty, \tag{2.3}$$

then the series $\sum C_n(x)\lambda_n$ is $|\bar{N}, p_n|_k$ summable, $1 \leq k < \infty$.

Theorem 2.6 [10] *If t_n is the n-th Cesàro mean of the first order of the sequence $\{na_n\}$ and $\{\epsilon_n\}$ is a numerical sequence such that*

$$\sum_{n=1}^{m} n^{-1} |\epsilon_n|^k |t_n|^k = O(1), \quad m \to \infty, \tag{2.4}$$

$$\sum_{n=1}^{m} p_n P_n^{-1} |\epsilon_n|^k |t_n|^k = O(1), \quad m \to \infty, \tag{2.5}$$

$$\sum_{n=1}^{m} n^{k-1} |\Delta\epsilon_n|^k |t_n|^k = O(1), \quad m \to \infty, \tag{2.6}$$

$$\sum_{n=1}^{m} n^{-1} P_n = O(P_m), \quad m \to \infty, \tag{2.7}$$

then the series $\sum a_n\epsilon_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Corollary 2.7 [10] *Let t_n be the n-th Cesàro mean of the first order of the sequence $\{na_n\}$ and*

$$T_n^{(k)} = \sum_{v=1}^{n} |t_v|^k.$$

Let $\{|\epsilon_n|\}$ be a non-increasing numerical sequence such that $|\Delta\epsilon_n| = O\left(\frac{|\epsilon_n|}{n}\right)$. Let $P_n = O(np_n)$ and $\left|\Delta\left(\frac{p_n}{P_n}\right)\right| = O\left(\frac{p_n}{nP_n}\right)$. If

$$\sum_{n=1}^{m} \frac{p_n |\epsilon_n|^k T_n^{(k)}}{nP_n} = O(1), \quad \text{as} \quad m \to \infty, \tag{2.8}$$

then the series $\sum a_n\epsilon_n$ is $|\bar{N}, p_n|_k$ summable, $k \geq 1$.

3 Main results

The aim of this paper is to generalize Theorem 2.5 for $|A,p_n|_k$ summability in the following form.
Theorem 3.1 *Suppose that $A=(a_{nv})$ be a normal matrix with positive entries such that*

$$\overline{a}_{n0}=1,\ \ n=0,1,..., \tag{3.1}$$
$$a_{n-1,v}\geq a_{nv},\ \text{ for } \ n\geq v+1, \tag{3.2}$$
$$a_{nn}=O(\frac{p_n}{P_n}), \tag{3.3}$$
$$\sum_{v=1}^{n-1} a_{vv}\hat{a}_{n,v}=O(a_{nn}). \tag{3.4}$$

Let $\{|\lambda_n|\}$ be a non-increasing numerical sequence such that $|\Delta\lambda_n|=O\left(\frac{|\lambda_n|}{n}\right)$, and let $P_n=O(np_n)$ and $\mid \Delta(\frac{p_n}{P_n})\mid = O(\frac{p_n}{nP_n})$.
(A) If

$$\sum p_n P_n^{-1}|\lambda_n|^k<\infty, \tag{3.5}$$

then the series $\sum C_n(x)\lambda_n$ is $|A,p_n|_k$ summable, $1\leq k<2$, for almost all values of x.
(B) If $\{\eta_n\}$ is a sequence of positive numbers such that $n^{-\gamma}\eta_n\to 0$ as $n\to\infty$, for some γ, $0<\gamma<1$, and if

$$\varphi(t)=O\left\{\eta_{(1/t)}\right\},\quad t\to 0 \tag{3.6}$$
$$\sum p_n P_n^{-1}|\lambda_n|^k\eta_n^k<\infty, \tag{3.7}$$

then the series $\sum C_n(x)\lambda_n$ is $|A,p_n|_k$ summable, $1\leq k<\infty$.

The following lemmas and theorem are necessary for our aim.

Lemma 3.2 [12] *Under the assumptions of Theorem 3.1, we have*

$$\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}|\leq a_{nn}, \tag{3.8}$$
$$\hat{a}_{n,v}\geq 0 \tag{3.9}$$
$$\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}=O(1). \tag{3.10}$$

Lemma 3.3 [10] *Let $r_n(x)$ be the n-th Cesàro mean of the first order of the sequence $\{nC_n(x)\}$ and $R_n^{(k)}=\sum\limits_{v=1}^{n}|r_v(x)|^k$. Then*

$$R_n^{(k)}(x)=O\left\{n\eta_n^k\right\},\quad 1\leq k\leq\infty, \tag{3.11}$$

provided $\{\eta_n\}$ is a sequence of positive numbers such that $n^{-\gamma}\eta_n\to 0$ as $n\to\infty$, for some γ, $0<\gamma<1$, such that

$$\varphi(t)=O\left\{\eta_{(1/t)}\right\},\quad t\to 0,$$
$$R_n^{(k)}(x)=O(n),\quad 1\leq k<2, \tag{3.12}$$

for almost all values of x.

For the proof of Theorem 3.1 firstly, we generalize Theorem 2.6 for $|A, p_n|_k$ summability method in the following manner.

Theorem 3.4 *Let A be a positive normal matrix satisfying conditions (3.2)-(3.5) of Theorem 3.1. If t_n is the n-th Cesàro mean of the first order of the sequence $\{na_n\}$ and $\{\epsilon_n\}$ is a numerical sequence satisfying conditions (2.5)-(2.8) of Theorem 2.6, then the series $\sum a_n\epsilon_n$ is $|A, p_n|_k$ summable, $k \geq 1$.*

Remark If we take $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.4, we have Theorem 2.6 dealing with $|\bar{N}, p_n|_k$ summability.

4 Proofs of Theorems 3.4 and 3.1

Proof of Theorem 3.4. Let (I_n) denote the A-transform of the series $\sum_{n=1}^{\infty} a_n\epsilon_n$, then

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{n,v} a_v \epsilon_v = \sum_{v=1}^{n} v^{-1} \hat{a}_{n,v} v a_v \epsilon_v.$$

Applying Abel's transformation to this sum, we have that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{n,v}\epsilon_v v^{-1}) \sum_{r=1}^{v} r a_r + a_{nn}\epsilon_n n^{-1} \sum_{v=1}^{n} v a_v \\
&= \sum_{v=1}^{n-1} (v+1) t_v (v^{-1}(v+1)^{-1}\hat{a}_{n,v}\epsilon_v + (v+1)^{-1}\bar{\Delta} a_{nv}\epsilon_v + (v+1)^{-1}\hat{a}_{n,v+1}\Delta\epsilon_v) + \frac{n+1}{n} a_{nn}\epsilon_n t_n \\
&= \sum_{v=1}^{n-1} t_v \bar{\Delta} a_{nv}\epsilon_v + \sum_{v=1}^{n-1} t_v \hat{a}_{n,v+1}\Delta\epsilon_v + \sum_{v=1}^{n-1} v^{-1} t_v \hat{a}_{n,v}\epsilon_v + \frac{n+1}{n} a_{nn}\epsilon_n t_n \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 3.4, by Minkowski's inequality, it suffices to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid I_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{4.1}$$

First, by applying Hölder's inequality with the exponents k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid I_{n,1} \mid^k &\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left\{ \sum_{v=1}^{n-1} \left|\bar{\Delta} a_{nv}\right| |\epsilon_v||t_v| \right\}^k \\
&\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \sum_{v=1}^{n-1} \left|\bar{\Delta} a_{nv}\right| |\epsilon_v|^k |t_v|^k \times \left\{ \sum_{v=1}^{n-1} \left|\bar{\Delta} a_{nv}\right| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1} \left\{ \sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}||\epsilon_v|^k |t_v|^k \right\} \\
&= O(1) \sum_{v=1}^{m} |\epsilon_v|^k |t_v|^k \sum_{n=v+1}^{m+1} |\bar{\Delta} a_{nv}| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{p_v}{P_v}\right) |\epsilon_v|^k |t_v|^k = O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.4 and Lemma 3.2. Now, using Hölder's inequality we have that

$$\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid I_{n,2} \mid^k \leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\epsilon_v||t_v|\right\}^k \\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\epsilon_v|^k|t_v|^k a_{vv}^{1-k} \times \left\{\sum_{v=1}^{n-1}a_{vv}\hat{a}_{n,v+1}\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\epsilon_v|^k|t_v|^k a_{vv}^{1-k} \\
&= O(1)\sum_{v=1}^{m}|\Delta\epsilon_v|^k|t_v|^k a_{vv}^{1-k}\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1} \\
&= O(1)\sum_{v=1}^{m}|\Delta\epsilon_v|^k|t_v|^k\left(\frac{p_v}{P_v}\right)^{1-k} \\
&= O(1)\sum_{v=1}^{m}\frac{|\epsilon_v|^k}{v^k}|t_v|^k\left(\frac{p_v}{P_v}\right)\left(\frac{P_v}{p_v}\right)^k \\
&= O(1)\sum_{v=1}^{m}\left(\frac{p_v}{P_v}\right)|\epsilon_v|^k|t_v|^k \\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.4 and Lemma 3.2. Next, we have that

$$\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid I_{n,3} \mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v}\epsilon_v\frac{t_v}{v}\right|^k \\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=1}^{n-1}\hat{a}_{nv}|\epsilon_v|^k\frac{|t_v|^k}{v^k}a_{vv}^{1-k} \times \left\{\sum_{v=1}^{n-1}a_{vv}\hat{a}_{nv}\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}\hat{a}_{n,v}|\epsilon_v|^k\frac{|t_v|^k}{v^k}\left(\frac{p_v}{P_v}\right)\left(\frac{P_v}{p_v}\right)^k = O(1)\sum_{v=1}^{m}\left(\frac{p_v}{P_v}\right)|\epsilon_v|^k|t_v|^k\sum_{n=v+1}^{m+1}\hat{a}_{n,v} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{p_v}{P_v}\right)|\epsilon_v|^k|t_v|^k = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.4 and Lemma 3.2. Finally, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{p_n}{P_n}\right)^k|\epsilon_n|^k|t_n|^k \\
&= O(1)\sum_{n=1}^{m}\left(\frac{p_n}{P_n}\right)|\epsilon_n|^k|t_n|^k = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.4. So, this completes the proof of Theorem 3.4. □

Corollary 3.5 *Suppose that A is a normal matrix satisfying conditions (3.2)-(3.5) of Theorem 3.1. Let $\{\epsilon_n\}$ be a non-increasing sequence of positive numbers such that $|\Delta\epsilon_n| = O\left(\frac{|\epsilon_n|}{n}\right)$. Let*

$$P_n = O(np_n) \text{ and } \left|\Delta\left(\frac{p_n}{P_n}\right)\right| = O\left(\frac{p_n}{nP_n}\right).$$

If

$$\sum_{n=1}^{m} \frac{p_n|\epsilon_n|^k T_n^{(k)}}{nP_n} = O(1), \quad as \quad m \to \infty, \tag{4.2}$$

where t_n is the n-th Cesàro mean of the first order of the sequence $\{na_n\}$, i.e.,

$$t_n = \frac{1}{n+1}\sum_{r=1}^{n} ra_r, \text{ and } \quad T_n^{(k)} = \sum_{v=1}^{n} |t_v|^k,$$

then the series $\sum a_n\epsilon_n$ is $|A, p_n|_k$ summable, $k \geq 1$.
Proof. By Abel's transformation and Corollary 2.7 (see [10]) it follows that,

$$\sum_{v=1}^{m}\left(\frac{p_v}{P_v}\right)|\epsilon_v|^k|t_v|^k = \sum_{v=1}^{m-1}\Delta\left(\left(\frac{p_v}{P_v}\right)|\epsilon_v|^k\right)\sum_{r=1}^{v}|t_r|^k + \frac{p_m}{P_m}|\epsilon_m|^k\sum_{r=1}^{m}|t_r|^k$$
$$= O(1)\sum_{v=1}^{m-1}\Delta\left(\left(\frac{p_v}{P_v}\right)|\epsilon_v|^k\right)T_v^{(k)} + \frac{p_m}{P_m}|\epsilon_m|^k T_m^{(k)}$$
$$= O(1)\left\{\sum_{v=1}^{m}\frac{p_v|\epsilon_v|^k T_v^{(k)}}{vP_v}\right\} + T_m^{(k)}\frac{p_m}{P_m}|\lambda_m|^k.$$

Since

$$\Delta|\epsilon_v|^k \leq k|\epsilon_v|^{k-1}\Delta|\epsilon_v| \leq k|\epsilon_v|^{k-1}|\Delta\epsilon_v| = O\left(\frac{|\epsilon_v|^k}{v}\right)$$
$$= O\left\{\sum_{v=1}^{m}\frac{p_v|\epsilon_v|^k T_v^{(k)}}{vP_v}\right\},$$

and

$$T_m^{(k)}\frac{p_m}{P_m}|\epsilon_m|^k < \sum_{v=1}^{m}\frac{p_v|\epsilon_v|^k T_v^{(k)}}{vP_v},$$

it follows that

$$\sum_{v=1}^{m}\left(\frac{p_v}{P_v}\right)|\epsilon_v|^k|t_v|^k = O(1), \quad \text{as} \quad m \to \infty. \qquad \square$$

Proof of Theorem 3.1 The proof of Theorem 3.1 follows immediately from Corollary 3.5 and Lemma 3.3. $\square$

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