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KHARIN STANISLAV NIKOLAYEVICH

(to the 80th birthday)

Stanislav Nikolayevich Kharin was born on December 4, 1938 in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis “Heat phenomena in electrical contacts and associated singular integral equations”, and in 1990 his doctoral thesis “Mathematical models of thermo-physical processes in electrical contacts” in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. Using models based on the new original methods for solving free boundary problems he described mathematically the phenomena of arcing, contact welding, contact floating, dynamics of contact blow-open phenomena, electrochemical mechanism of electron emission, arc-to-glow transition, thermal theory of the bridge erosion. For these achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research and the evidence of this in the new monograph “Mathematical models of phenomena in electrical contacts” published last year in Novosibirsk.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Stanislav Nikolayevich on the occasion of his 80th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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HAHN-BANACH TYPE THEOREMS ON FUNCTIONAL SEPARATION FOR CONVEX ORDERED NORMED CONES

F.S. Stonyakin

Communicated by M.L. Goldman

Key words: abstract convex cone, Hahn-Banach separation theorem, strict convex normed cone, convex ordered normed cone, sublinear injective isometric embedding, R*ä*dstr*ö*m embedding theorem.

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Abstract. We consider a special class of convex ordered normed cones $CONC$. For such structures we obtain Hahn-Banach type theorems on functional separation for points. On the base of a Hahn-Banach type theorem on functional separation for points we prove a sublinear version of the R*ä*dstr*ö*m embedding theorem for the class $CONC$. Some analogues of Hahn-Banach separation theorem for some type of sets in $CONC$ are obtained.

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1 Introduction

The theory of so-called abstract convex normed cones is actively developed in recent decades (see, e.g., [7, 9, 13, 16, 17, 18, 23]). In particular, such known mathematicians as J. R*ä*dstr*ö*m, K. Keimel, W. Roth, R. Tix and others were engaged in this theory.

The special types of convex cones in some functional spaces were studied by M. L. Goldman, P. P. Zabreiko and E. G. Bakhtigareeva (see, e.g., [1, 2, 8]). Recently, in connection with some problems of nonsmooth analysis, the so-called *subnormed cones* were considered by I. V. Orlov (see, e.g., [10, 11, 12]). For the cone of convex compact subsets of a normed space E with a separable conjugate space E^* a new analogue of the Shauder fixed-point theorem was proved by us in [19]. Sublinear versions of the Banach-Alaoglu theorem and Banach-Mazur theorem in normed cones were proved by us recently [20, 21].

Generally, norm in a cone may not be a trace of the usual norm (or seminorm) in any linear space. In particular, it holds for each linear space with the so-called *asymmetric norm* (see, e.g. [3, 4]). We note some applications of asymmetric normed spaces to theoretical computer science [6, 15, 18] and approximation theory [5, 14].

In this paper we obtain new Hahn-Banach type theorems on functional separation for points and sets in a special class of abstract convex cones with a norm. Note that in [9, 16, 17, 18, 23] some Hahn-Banach type theorems on functional separation by linear bounded functionals were considered for different types of normed cones. However, in this aspect there is such a problem: in contrast to the class of normed spaces the boundedness of a linear functional $\ell : X \to \mathbb{R}$ does not imply the boundedness of the functional $-\ell : X \to \mathbb{R}$ in a cone. Corresponding examples (see, e.g. [3]) are known even in linear spaces with asymmetric norm, which can also be considered as convex normed cones. In connection with this problem known analogues of the Hahn-Banach separation theorem are obtained only in special classes of normed cones using *non-negative monotonic* linear functionals (see, e.g. [4, 16, 17, 18, 23]). However, this approach leads to some problems. In particular, such functionals may not separate points in a normed cone [16]. We suggest considering not only non-negative linear functionals and we introduce a conjugate cone as a set of linear upper bounded (or semi-bounded) functionals, which are non-negative at some non-zero points. This allows us proving the existence of a sublinear isometric embedding into a linear normed space for a wider class of normed cones without specific conditions of monotony (see Theorem 1).

Many results of the theory of convex normed cones are related to the possibility of endowing them with a metric structure (a metric or some analogue of it). Note the well-known J. Rädström theorem [13] on the linear isometric embedding of a cone in a normed space with homogeneous and shift invariant metric $d : X \times X \longrightarrow \mathbb{R}^+$:

$$d(\lambda x, \lambda y) = \lambda d(x, y) \quad d(x + z, y + z) = d(x, y) \text{ for all } x, y, z \in X, \lambda \geqslant 0.$$

In recent works [4, 7, 16] it was shown how one can introduce in cones the so-called *quasi-metric* $q : X \times X \longrightarrow \mathbb{R}^+$, which can be asymmetric (generally, $q(x, y) \neq q(y, x)$). It is possible that $q(x, y) = +\infty$ and $q(x, y) = 0$ for some $x \neq y$. Note that such a quasi-metric is homogeneous and *subinvariant* (see, e.g. [4, 7, 16]):

$$q(x + z, y + z) \leqslant q(x, y) \text{ for all } x, y, z \in X.$$

In this paper on the base of an analogue of the Hahn-Banach theorem on functional separation (see Theorem 3.1) we introduce a special finite homogeneous metric $d_* : X \times X \longrightarrow \mathbb{R}^+$ for a new class of normed cones X. Using this metric we prove an analogue of the J. Rädström embedding theorem on the existence of a *sublinear* isometric embedding of a normed cone into a linear normed space (see Theorem 1). Generally speaking, the linearity of such an embedding is impossible (see Remark 6).

Our paper is based on [22], where an analogue of the analytic version of the Hahn-Banach theorem in abstract convex cones was proved and some applications of this result were considered. In particular, in ([22], see Section 4) a special class of *strict convex normed cones* ($SCNC$) was considered and the existence of a sublinear injective isometric embedding of each $SCNC$ in a Banach space E was proved.

We are developing research [22] in the following directions. Instead of the standard property of monotony for norms in a cone (see, e.g. [16, 18, 22]) for all $x, y, z \in X$:

$$y = x + z \Longrightarrow \|x\| \leqslant \|y\| \tag{1.1}$$

we consider its new generalization

$$x \neq 0 \Longrightarrow \inf\{\|y\| \,|\, y = x + z \text{ for some } z \in X\} > 0 \tag{1.2}$$

and select a class of *convex ordered normed cone* $CONC$ as a corresponding modification of the class of $SCNC$ (see Definition 10). Property (1.2) can be fulfilled in cones with a non-monotonic norm (see Examples 4 and 5). Thus, our approach is new even for the case, when non-negative linear bounded functionals separate points of the cone.

We obtain an analogue of the Hahn-Banach separation theorem for points and introduce the second conjugate space X^{**} of $CONC$ X and prove the opportunity to consider X as a metric space (X, d_*) with some metric $d_* : X \times X \longrightarrow \mathbb{R}$. On the base of this result for $CONC$ we obtain an analogue of the Hahn-Banach separation theorem by linear semi-bounded functionals for special types of sets in X.

The paper consists of the introduction and four main sections.

In Section 2 first of all we give an overview of the main concepts and results of the paper [22], on which we base our reasoning: an analogue of the Hahn-Banach theorem on the extension of a linear functional from a subcone $Y \subset X$ to the whole cone X with preservation of an estimate by a convex functional (Theorem 2.1), along with the analogue of the Lemma on a support functional in the class of *convex normed cones* CNC (Corollary 2.1). Further, in Section 2 we construct some new examples of normed cones that are not linearly injectively isometrically embedded in any linear normed space (see Examples 4 and 5). Also we show that we need additional requirements for the theorem on functional separation in the class CNC (see Lemma 2.1 and Example 6).

Section 3 is devoted to the theorem on functional separation of points by linear semi-bounded functionals in $CONC\ X$ (see Theorem 3.1) and its applications to a sublinear analogue of the J. Rädström embedding theorem (see Theorem 1). Theorem 3.1 gives us the opportunity to define a homogeneous metric $d_* : X \times X \to \mathbb{R}$ and prove the existence of an injective sublinear d_*-continuous embedding of each $CONC\ X$ in X^{**} (see Theorem 1). Generally speaking, d_* loses the property of subinvariance with respect to shifts (see Remark 9).

Section 4 is devoted to the analogues of the Hahn-Banach theorem on functional separation of a point and a d_*-closed (d_*-open) set in $CONC$. We obtain an analogue of the Hahn-Banach separation theorem of a point and a d_*-closed (d_*-open) convex set containing 0 (see Theorems 4.1 and 4.2). We show the impossibility of strengthening of the obtained result for sets, which do not contain 0 (see Example 11).

In the last section on the base of Theorem 1 we prove an analogue of Theorem 4.1 for the case, where instead of a point and a convex set we consider two closed sets with special properties (see Theorem 5.1).

2 An analogue of the Hahn-Banach extension theorem and convex normed cones

In this section we consider some auxiliary concepts, results and examples. Recall that an *abstract convex cone* or *convex cone* is a collection X of elements with the operations of addition and non-negative scalar multiplication, where X is a commutative semigroup under addition, such that for arbitrary numbers $\lambda, \mu \geqslant 0$ and elements $x, y \in X$ the following relations hold:

$$1 \cdot x = x; \quad (\lambda\mu)x = \lambda(\mu x); \quad 0 \cdot x = 0; \quad \lambda(x+y) = \lambda x + \lambda y; \quad (\lambda+\mu)x = \lambda x + \mu x.$$

Note, that for many results in the theory of convex cones the following *cancellation law* is essential for all $x, y, z \in X$:

$$x + y = y + z \Longleftrightarrow x = z. \tag{2.1}$$

Definition 1. A mapping $p : X \to \mathbb{R}$ is called a *convex functional*, if for each $x, y \in X$ and $\lambda \geqslant 0$ the following conditions hold:

$$p(x) \geqslant 0,\ p(\lambda x) = \lambda p(x) \quad \text{and} \quad p(x+y) \leqslant p(x) + p(y).$$

We also recall the concept of a subcone in the class of convex cones, introduced by us in [22].

Definition 2. We say that Y is a *subcone* of X, if $Y \subset X$, Y is a convex cone and for all $x \in X$ and $y, z \in Y \subset X$ the condition $z = x + y$ implies that $x \in Y$.

Example 1. For each fixed elements $x_1, x_2, ..., x_n \in X$ the set

$$Y = \left\{ x \in X \;\middle|\; x + \sum_{k=1}^{n} \mu_k x_k = \sum_{k=1}^{n} \lambda_k x_k \text{ for some } \lambda_k, \mu_k \geqslant 0,\ k = \overline{1, n} \right\}$$

is a subcone of X. One more example is considered further in the proof of Theorem 3.1.

Now we recall the analogue of the Hahn-Banach theorem on the extension of a linear functional from a subcone $Y \subset X$ to the whole cone X for the class of convex cones X with the cancellation law [22].

Theorem 2.1. *Let X be a convex cone with the cancellation law, $p : X \to \mathbb{R}$ be a convex functional on X, Y be a subcone of X. Assume that $\ell : Y \to \mathbb{R}$ is a linear functional with the estimate $\ell(y) \leqslant p(y)$ for all $y \in Y$.*

Then there exists a linear functional $L : X \to \mathbb{R}$ such that $L(x) \leqslant p(x)$ for all $x \in X$ and $L(y) = \ell(y)$ for all $y \in Y$.

It is interesting to know conditions under which an abstract convex cone with a norm may be isometrically embedded in a linear normed space with some convenient properties of the corresponding embedding. It is clear, that such a norm in X should satisfy the following property for all $x, y \in X$:

$$x + y = 0 \Longrightarrow \|x\| = \|y\|. \tag{2.2}$$

Generally, the last property (2.2) does not follow from the standard axioms of a norm in the class of convex cones. Such an example was given in [22] (see Example 2 below).

Let us recall the concept of a *convex normed cone* CNC X, some examples of CNC and the analogue of the well-known Lemma on a support functional for such cones [22]. Recently, the so-called subnormed cones were considered by I. V. Orlov (see [11]). Let us recall this concept.

Definition 3. A subnorm $\|\cdot\| : X \to \mathbb{R}$, where X is a convex cone, is a function satisfying the following conditions: for all $x, y \in X$ and $\lambda \geqslant 0$:

$$\|x\| \geqslant 0; \quad \|x\| = 0 \Leftrightarrow x = 0; \quad \|\lambda x\| = \lambda \|x\|; \quad \|x + y\| \leqslant \|x\| + \|y\|.$$

X is called a subnormed cone.

Now we consider some modification of the previous concept, introduced by us in [22].

Definition 4. A norm $\|\cdot\| : X \to \mathbb{R}$, where X is a convex cone, is a function satisfying the above axioms of a subnorm and (2.2). X is called a convex normed cone CNC.

We start with some simple examples of convex normed cones.

Example 2. Let $X = (-\infty; +\infty)$. The non-negative scalar multiplication is standard. The addition $x_1 \oplus x_2$ is introduced in the following way: $x_1 \oplus x_2 := \min\{x_1, x_2\}$. We introduce the standard subnorm in the convex cone X: $\|x\| := |x| \geqslant 0$.

Note, that with the addition $x_1 \oplus x_2 := \max\{x_1, x_2\}$ the set $X = [0; +\infty)$ is a convex normed cone.

Example 3. Let X be the collection of all non-negative bounded real-valued functions $f : [0; 1] \to \mathbb{R}$ with the usual addition and scalar multiplication. In this case we can consider the usual sup-norm $\|f\|_X := \sup\limits_{0 \leqslant t \leqslant 1} |f(t)|$.

Next, let us consider some important examples. Assume that a convex cone X is linearly injectively embedded in a linear normed space E. Generally, *a norm on X may not be a trace of some seminorm in E.*

Example 4. Let

$$X = \{(0,0)\} \cup \{(a,b) \mid a > 0 \quad \text{and} \quad b > 0\}.$$

We introduce the norm in X in the following way:

$$\|(a,b)\| = \max\left\{a, \frac{b^2}{a}\right\} \quad \text{for} \quad a \neq 0 \quad \text{and} \quad \|(0,0)\| = 0.$$

It is clear that $\|(a,b)\| = p_W((a,b))$, where $p_W(\cdot)$ is the Minkowsky functional of the set W, bounded by the parabola $a = b^2$ and the straight line $a = 1$ (see Fig. 1).

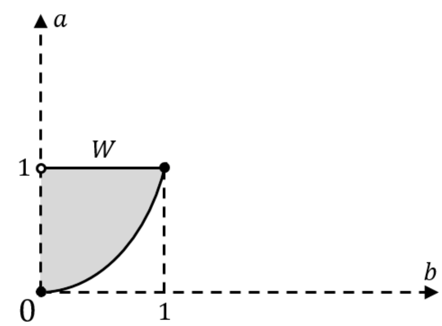

Figure 1.

Thus, $\|\cdot\|$ is a convex functional on X. Let us show that $\|\cdot\|$ is not be a seminorm in linear space $E \supset \varphi(X)$ for each linear injective embedding $\varphi : X \to E$. Indeed, $\left(\frac{1}{8}, \frac{7}{8}\right) + \left(\frac{7}{8}, \frac{1}{8}\right) = (1,1)$ and

$$\left\|\left(\frac{1}{8}, \frac{7}{8}\right)\right\| = \frac{49}{8}, \quad \left\|\left(\frac{7}{8}, \frac{1}{8}\right)\right\| = \frac{7}{8}, \quad \|(1,1)\| = 1,$$

$$\|(1,1)\| + \left\|\left(\frac{7}{8}, \frac{1}{8}\right)\right\| < \left\|\left(\frac{1}{8}, \frac{7}{8}\right)\right\|,$$

i.e. the inequality $\|(a_1,b_1)\| + \|(a_1,b_1) + (a_2,b_2)\| \geqslant \|(a_2,b_2)\|$ is not satisfied.

Analogously to the preceding example we give one more example of convex normed cone that is not linearly injectively isometrically embedded in any linear normed space.

Example 5. Let X be the same as in 4 We introduce the norm in X in the following way:

$$\|(a,b)\| = \max\left\{\frac{a^2}{b}, \frac{b^2}{a}\right\} \quad \text{for} \quad a, b \neq 0 \quad \text{and} \quad \|(0,0)\| = 0.$$

It is clear that $\|(a,b)\| = p_W((a,b))$, where $p_W(\cdot)$ is the Minkowsky functional of the set W (see Fig. 2).

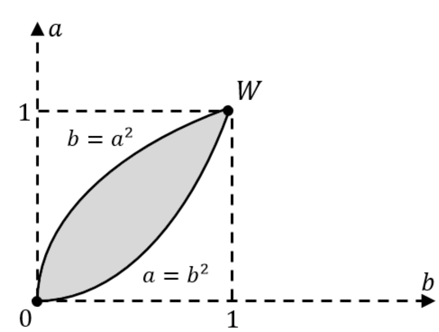

Figure 2.

Let us recall the following analogue of the well-known Lemma on a support functional in CNC ([22], Corollary 3.1). We start with auxiliary concepts of a semi-bounded linear functional and a conjugate cone.

Definition 5. We say that a linear functional ℓ is *semi-bounded* on CNC X, if for some $C > 0$ the following inequality holds: $\ell(x) \leqslant C\|x\|$ for all $x \in X$.

Note that the inequality $\ell(x) \leqslant C\|x\|$ does not imply $|\ell(x)| \leqslant C\|x\|$ in view of non-invertibility of elements of CNC X (see, e.g. [3]).

Clearly, the collection of all semi-bounded linear functionals on X is a convex cone if we introduce addition of functionals and scalar multiplication in the usual way. Denote by X^* the collection of all semi-bounded linear functionals $\ell : X \to \mathbb{R}$ such that $\ell(x_0) \geqslant 0$ for some $x_0 \in X$, $x_0 \neq 0$. We introduce the seminorm on X^* in the following way:

$$\|\ell\|_* := \sup_{x \neq 0} \left\{ \frac{\ell(x)}{\|x\|} \right\}.$$

Definition 6. We say that X^* is a *conjugate cone* of X.

An analogue of the known Lemma on a support functional for the class CNC immediately follows from Theorem 2.1.

Corollary 2.1. *(An analogue of the Lemma on a support functional in CNC) Let X be a CNC with the cancellation law. Then for all $x_0 \neq 0$, $x_0 \in X$ there exists $\ell \in X^* \setminus \{0\}$, such that $\|\ell\|_* = 1$ and $\ell(x_0) = \|x_0\|$.*

Remark 1. An analogous result is known in the special class of normed cones X for non-negative linear functionals $f : X \to \mathbb{R}$ (see [18], Theorem 2.14). However, in general, the equality $f(x_0) = \|x_0\|$ is not possible (see Remark after Theorem 2.14 in [18]). Note, that the condition $\ell(x_0) = \|x_0\|$ in Corollary 2.1 is essential for some further results of our paper (see Theorems 3.1 and 1).

Naturally, it is interesting to know whether functionals $\ell \in X^*$ separate elements of CNC. It turns out that it is possible to give an example of CNC, where $\ell(x_1) = \ell(x_2)$ for each $\ell \in X^*$ for different x_1, $x_2 \in X$. To give such an example we need an auxiliary statement.

Lemma 2.1. *Each linear functional $\ell : X_2 \to \mathbb{R}$, where*

$$X_2 = \{(a, b) \,|\, a, b \geqslant 0 \text{ and } a = 0 \Rightarrow b = 0\}$$

is given by

$$\ell((a, b)) = \lambda a + \mu b. \tag{2.3}$$

for some constants λ and μ.

If we consider the norm $\|(a, b)\| := a$ in X_2 then the conjugate cone

$$X_2^* = \{\ell((a; b)) = \lambda a + \mu b \,|\, \lambda \geqslant 0, \mu \leqslant 0\}. \tag{2.4}$$

Proof. Firstly, each functional ℓ is homogeneous:

$$\ell(\alpha(a, b)) = \alpha \ell((a, b)) \; \forall \alpha \geqslant 0. \tag{2.5}$$

If we take some pair $x_0 = (a_0, b_0) \in X_2$ then in the line of the ray $\{\alpha x_0\}_{\alpha \geqslant 0}$ we have for $x_0 \neq (0, 0)$:

$$\ell(\alpha x_0) < 0 \text{ for each } \alpha > 0, \text{ or } \quad \ell(\alpha x_0) > 0 \text{ for each } \alpha > 0,$$

$$\text{or } \ell(\alpha x_0) = 0 \text{ for each } \alpha \geqslant 0, \tag{2.6}$$

If (2.6) is true then we call $\{\alpha x_0\}_{\alpha \geqslant 0}$ a *neutral ray* for ℓ.

a) If $\ell \neq 0$, then there does not exist more than one neutral ray. Indeed, if there exist two different neutral rays p_1 and p_2, then for each pair $x = (a, b)$ we have $\ell(x) = 0$ in the cone K_0 (see Fig. 3), bounded by the rays p_1 and p_2.

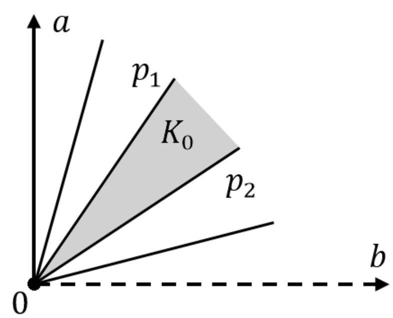

Figure 3. Two neutral rays.

From the linearity of ℓ we have that ℓ takes a zero value in all points of the cones, symmetric to K_0 with respect to p_1 and p_2. It is possible to cover the cone X_2 by a finite set of such cones as K_0, i.e. $\ell = 0$.

b) If one neutral ray exists for ℓ then due to the linearity of ℓ it can be shown that on rays parallel to it ℓ takes constant values (see Fig. 4), i.e. ℓ is given by (2.3).

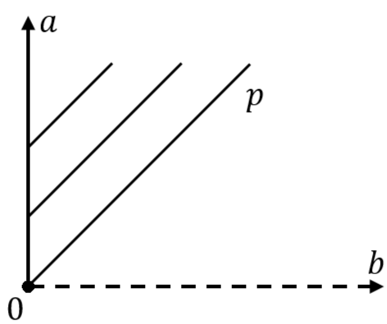

Figure 4. One neutral ray.

c) If there is no neutral rays for ℓ в X_2 then on all rays given by $\{\alpha x_0\}_{\alpha \geqslant 0}$ $(x_0 \neq (0,0))$ ℓ takes either positive, or negative values.

Let us take such three rays p_1, p_2 and p_3 (see Fig. 5) and choose such points x, y and z on them that $\ell(x) = \ell(y) = \ell(z) \neq 0$, which can be done due to the homogeneity of ℓ.

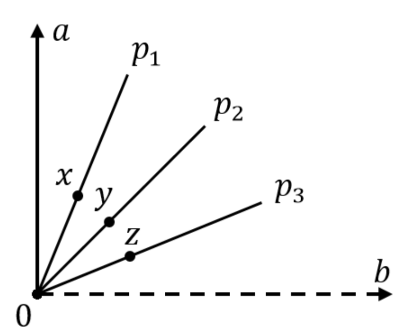

Figure 5. Three rays.

It is clear that $y = \alpha_1 x + \alpha_2 z$ for some $\alpha_1, \alpha_2 > 0$. Then

$$\ell(y) = \alpha_1 \ell(x) + \alpha_2 \ell(z) = (\alpha_1 + \alpha_2)\ell(y),$$

whence $\alpha_1 + \alpha_2 = 1$ owing to $\ell(y) \neq 0$. It means that x, y and z lie on the same line. Consequently, for each $C \neq 0$ the sets given by $\{x \in X_2 \,|\, \ell(x) = C\}$ are the parts of parallel lines, whence ℓ is given by (2.3).

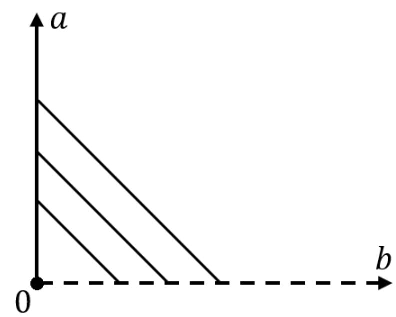

Figure 6. There is no neutral ray.

The case $\ell \equiv 0$ is trivial.

The condition $\ell \in X_2^*$ means that $\lambda a + \mu b \leqslant Ca$ for all $(a, b) \in X$ and for some number $C > 0$. It is clear that for all $\mu > 0$ one may choose sufficiently large $b > 0$, for which the inequality $\lambda a + \mu b \leqslant Ca$ does not hold. Analogously, for $\lambda < 0$ $\ell \notin X_2^*$. Therefore, (2.4) holds. □

Let us return to the example of CNC, where linear semi-bound functionals may not separate points.

Example 6. Let $X = X_2' = \{(a, b) \,|\, a \geqslant 0, b \in \mathbb{R}; a = 0 \Rightarrow b = 0\}$. The norm in X_2' is introduced by $\|(a, b)\| = a$.

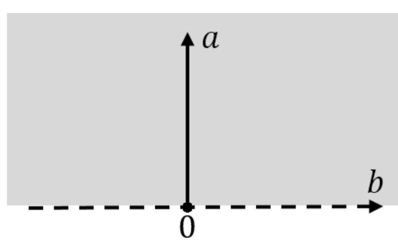

Figure 7. The cone X_2'.

By Lemma 2.1 for each linear functional $\ell : X_2' \to \mathbb{R}$ $\ell((a, b)) = \lambda a + \mu b$ for some constants λ and μ. Since $a \neq 0$ and b can be an arbitrary number, the condition of the boundedness

$$\ell((a, b)) = \lambda a + \mu b \leqslant Ca$$

for some $C > 0$ implies $\mu = 0$ (otherwise the inequality does not hold at the corresponding choice of b). Obviously, functionals given by $\ell((a, b)) = \lambda a$ do not separate the points of X_2'.

Remark 2. The cone X_2' is equivalent to the following cone

$$X_2' = \{[\alpha; \beta] \,|\, \alpha < \beta;\ \alpha, \beta \in \mathbb{R}\} \bigcup \{0\}.$$

with the norm $\|[\alpha; \beta]\| = \beta - \alpha$ in X_2'.

Previous examples show that for the validity of the theorem on functional separation it is necessary to introduce some additional requirements and to consider the corresponding subclass of CNC. The next section is devoted to our approach to this problem.

3 A Hahn-Banach type theorem on functional separation for points and its application to a sublinear version of J. Rädström theorem for $CONC$

Naturally, it is interesting to get conditions of isometrical embedding of an abstract convex normed cone in a normed space only using the norm. In [22] on the base of Theorem 2.1 and Corollary 2.1 this question was studied for $SCNC$ X. An existence of convex isometrical embedding $\varphi : X \to E$ for each $SCNC$ X in the second conjugate normed space X^{**} was proved in [22]. In the work [13] it was proved by J. Rädström that if convex cone X has a metric $d : X \times X \longrightarrow \mathbb{R}$ with special properties then X is linearly injectively and isometrically (with respect to the metric) embedded in some linear normed space.

This section of our paper is devoted to sublinear generalization of J. Rädström Theorem on injective isometrical embedding in Banach space for $CONC$ (see Definition 10). Note that a convex normed cone from this class may not be linearly injectively isometrically embedded in any Banach space (see Remark 6).

We prove a Hahn-Banach type theorem on functional separation of points by linear semi-bounded functionals in $CONC$ X (see Theorem 3.1). On the base of Theorem 3.1 we construct a metric $d_* : X \times X \longrightarrow \mathbb{R}$ for each $CONC$ X and prove the existence of injective sublinear d_*-continuous embedding of X in the second conjugate Banach space (see Theorem 1). We start with auxiliary definitions, notations, results and examples.

Definition 7. We say that X is a strict cone, if the following property holds:

$$x + y = 0 \Longrightarrow x = y = 0 \text{ for all } x, y \in X. \tag{3.1}$$

Note that strict convex cones were considered earlier, for example, in [18]. In each strict convex cone X we can consider the following standard partial order [18]:

$$x \preceq y \text{ if } y = x + z \text{ for some } z \in X. \tag{3.2}$$

Now we recall the property of *order separability* for strict convex cones and the concept of strict convex normed cone $SCNC$ and some examples of $SCNC$, considered by us in [22].

Definition 8. We say that X is order separable if for all $x, y \in X$:

$$\alpha x \preceq y \preceq \beta x \quad \text{for each} \quad \alpha < 1 < \beta \Longrightarrow y = x. \tag{3.3}$$

Definition 9. An abstract convex cone X is called a strict convex normed cone $SCNC$, if X is a strict order separable convex normed cone with the cancellation law and for all $x \in X$:

$$x \preceq y \Longrightarrow \|x\| \leqslant \|y\|. \tag{3.4}$$

Now we consider some examples of $SCNC$.

Example 7. Let X be the collection of all non-negative numbers with the usual addition and scalar multiplication. For each $a, b \in X$ $a + b = 0$ means $a = b = 0$ and $a \leqslant b \Longleftrightarrow b = a + c$ for some $c \geqslant 0$. Clearly, $\alpha b \preceq a \preceq \beta b$ for all $\alpha < 1 < \beta$ means that $b \leqslant a \leqslant b$, i.e. $a = b$.

Example 8. Let X be the collection of all non-negative bounded real-valued functions $f : [0; 1] \to \mathbb{R}_+$ with the usual addition and scalar multiplication. In this case for each $f, g \in X$ $f \preceq g \Longleftrightarrow f(t) \leqslant g(t)$ for all $t \in [0; 1]$.

Example 9. Let X be the collection of all segments $A = [a; b] \subset \mathbb{R}^+$ with the Minkowsky addition and usual scalar multiplication. In this case we can consider the following order: for each $A, B \in X$

$$A \preceq B \Longleftrightarrow \quad \text{there exists} \quad C \in X \quad \text{such that} \quad B = A + C.$$

Let us introduce a class of convex normed cones with some generalization of property (3.4).

Definition 10. We say that an abstract convex cone X is a convex ordered normed cone $CONC$, if X is a strict order separable convex normed cone with the cancellation law and for all $x \in X$:

$$x \neq 0 \Longrightarrow \inf \{\|y\| \mid x \preceq y\} > 0. \tag{3.5}$$

Remark 3. Clearly, (3.5) follows from (3.4). Therefore, each $SCNC$ is $CONC$, but there are $CONC$ without property (3.4). Examples of such cones has been considered in Examples 4 and 5.

Let us formulate and prove a Hahn-Banach type theorem on functional separation of points for $CONC$.

Theorem 3.1. *Let X be a $CONC$. Then for all $e_1 \neq e_2$ $(e_1, e_2 \in X)$ there exists a functional $\ell \in X^* \backslash \{0\}$, such that $\ell(e_1) \neq \ell(e_2)$ and $\ell(e_1) > 0$ or $\ell(e_2) > 0$.*

Proof. 1) If $\|e_1\| \neq \|e_2\|$ (for example, $e_1 = 0$ or $e_2 = 0$) then for $i = 1, 2$ by Corollary 2.1 there exists $\ell_i \in X^* \backslash \{0\}$, such that $\ell_i(e_i) = \|e_i\|$. If $\|e_1\| < \|e_2\|$ then by $\|\ell_2\|_* = 1$ we have $\ell_2(e_1) \leqslant \|e_1\| < \|e_2\| = \ell_2(e_2)$ and $\ell_2(e_1) \neq \ell_2(e_2) > 0$. The case of $\|e_2\| < \|e_1\|$ is considered analogously.

2) Now we assume $\|e_1\| = \|e_2\|$, $e_1 \neq 0$ and $e_2 \neq 0$. Set $Y := \{\lambda e_1 | \lambda \geqslant 0\}$. Note, that Y is a subcone of X. Indeed, in view of the cancellation law of X for each $\lambda \geqslant \mu \geqslant 0$ from $x + \mu e_1 = \lambda e_1$ we have $x = (\lambda - \mu)e_1$. The case of $\lambda < \mu$ is impossible in view of strictness of X $(x + (\mu - \lambda)e_1 = 0$ means $x = e_1 = 0)$.

For each $x = \lambda e_1 \in Y$ we can define $\ell \in X^*$, such that $\ell(x) := \lambda \|e_1\|$. Clearly, for $e_2 \in Y \backslash \{e_1\}$ we have $\ell(e_2) \neq \|e_1\| = \ell(e_1) > 0$.

If $e_2 \notin Y$ we consider the set

$$Y_1 = \{x \in X \,|\, x + \lambda_2 e_2 + y_2 = \lambda_1 e_2 + y_1 \text{ for some } \lambda_1, \lambda_2 \geqslant 0,\ y_1, y_2 \in Y\}.$$

By virtue of the cancellation law of X for each $y_1 = \mu_1 e_1$ and $y_2 = \mu_2 e_1 \in Y$ from $x + \lambda_2 e_2 + \mu_2 e_1 = \lambda_1 e_2 + \mu_1 e_1$ we have

$$x = (\lambda_1 - \lambda_2)e_2 + (\mu_1 - \mu_2)e_1, \text{ for } \lambda_1 \geqslant \lambda_2 \text{ and } \mu_1 \geqslant \mu_2,$$

$$x + (\lambda_2 - \lambda_1)e_2 = (\mu_1 - \mu_2)e_1, \text{ for } \lambda_1 < \lambda_2 \text{ and } \mu_1 \geqslant \mu_2,$$

$$x + (\mu_2 - \mu_1)e_1 = (\lambda_1 - \lambda_2)e_2, \text{ for } \lambda_1 \geqslant \lambda_2 \text{ and } \mu_1 < \mu_2.$$

The equality $x + (\mu_2 - \mu_1)e_1 + (\lambda_2 - \lambda_1)e_2 = 0$ means $x = (\mu_2 - \mu_1)e_1 = (\lambda_2 - \lambda_1)e_2 = 0$ in view of (3.1). Hence, we can consider only three types of elements $x_1, x_2, x_3 \in Y_1$:

a) $x_1 = \lambda_1 e_2 + \alpha_1 \lambda_1 e_1$ for some $\alpha_1, \lambda_1 \geqslant 0$;

b) $x_2 + \lambda_2 e_2 = \alpha_2 \lambda_2 e_1$ in case of existing $x_2 \in X$ for some $\alpha_2, \lambda_2 \geqslant 0$;

c) $x_3 + \alpha_3 \lambda_3 e_1 = \lambda_3 e_2$ in case of existing x_3 for some $\alpha_3, \lambda_3 \geqslant 0$.

For $x_i \in X$ the conditions $\ell(x_i) \leqslant \|x_i\|$ $(i = 1, 2, 3)$ can be written in the following way:

$$\lambda_1\ell(e_2) + \alpha_1\lambda_1\ell(e_1) \leqslant \|x_1\|,\ -\lambda_2\ell(e_2) + \alpha_2\lambda_2\ell(e_1) \leqslant \|x_2\|,\ \lambda_3\ell(e_2) - \alpha_3\lambda_3\ell(e_1) \leqslant \|x_3\|,$$

or equivalently, for all possible $\alpha_i > 0$ and $\lambda_i > 0$ $(i = 1, 2, 3)$

$$\ell(e_2) \leqslant \left\|\frac{x_1}{\lambda_1}\right\| - \alpha_1\ell(e_1),\ \ell(e_2) \geqslant \alpha_2\ell(e_1) - \left\|\frac{x_2}{\lambda_2}\right\|,\ \ell(e_2) \leqslant \left\|\frac{x_3}{\lambda_3}\right\| + \alpha_3\ell(e_1).$$

According to the proof of Theorem 2.1 (see [22], the proof of Theorem 2.1) we can choose $\ell(e_2) \in [\alpha_{e_2}; \beta_{e_2}]$, where for all possible $\alpha_i > 0$ and $\lambda_i > 0$ $(i = 1, 2, 3)$

$$\alpha_{e_2} = \sup_{\alpha_2,\lambda_2>0}\left\{\alpha_2\ell(e_1) - \left\|\frac{x_2}{\lambda_2}\right\|\right\}$$

and

$$\beta_{e_2} = \inf_{\alpha_1,\alpha_3,\lambda_1,\lambda_3>0}\left\{\left\|\frac{x_1}{\lambda_1}\right\| - \alpha_1\ell(e_1);\ \left\|\frac{x_3}{\lambda_3}\right\| + \alpha_3\ell(e_1)\right\}.$$

Since

$$\alpha_2 e_1 = \frac{x_2}{\lambda_2} + e_2 \implies e_2 \preceq \alpha_2 e_1 \text{ and } \alpha_3 e_1 + \frac{x_3}{\lambda_3} + e_2 \implies \alpha_3 e_1 \preceq e_2,$$

then $\alpha_3 e_1 \preceq e_2 \preceq \alpha_2 e_1$ for some $\alpha_2 \geqslant \alpha_3 \geqslant 0$.

If $\inf\{\alpha_2 - \alpha_3\} = 0$ then we have $\sup\alpha_3 = \inf\alpha_2 = \alpha > 0$. Then by (3.3) $e_2 = \alpha \cdot e_1$. From $\|e_1\| = \|e_2\|$ we have $e_2 = e_1$, that is impossible.

We put $\inf\{\alpha_2 - \alpha_3\} = \delta > 0$. By virtue of the cancellation law of X we have

$$\frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3} + \alpha_3 e_1 = \alpha_2 e_1 \text{ and } \frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3} = (\alpha_2 - \alpha_3)e_1 \in Y.$$

Hence,

$$\left\|\frac{x_3}{\lambda_3}\right\| + \alpha_3\ell(e_1) - \left(\alpha_2\ell(e_1) - \left\|\frac{x_2}{\lambda_2}\right\|\right) = \left\|\frac{x_2}{\lambda_2}\right\| + \left\|\frac{x_3}{\lambda_3}\right\| + \alpha_3\ell(e_1) - \ell\left(\frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3} + \alpha_3\ell(e_1)\right) =$$

$$= \left\|\frac{x_2}{\lambda_2}\right\| + \left\|\frac{x_3}{\lambda_3}\right\| - \ell\left(\frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3}\right) = \left\|\frac{x_2}{\lambda_2}\right\| + \left\|\frac{x_3}{\lambda_3}\right\| - \frac{1}{2}\left\|\frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3}\right\| \geqslant \left\|\frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3}\right\| - \frac{1}{2}\left\|\frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3}\right\| =$$

$$= \frac{1}{2}\left\|\frac{x_2}{\lambda_2} + \frac{x_3}{\lambda_3}\right\| = \frac{1}{2}\|(\alpha_2 - \alpha_3)e_1\| \geqslant \frac{1}{2}\delta\|e_1\| > 0 \text{ and}$$

$$\left\|\frac{x_1}{\lambda_1}\right\| - \alpha_1\ell(e_1) - \left(\alpha_2\ell(e_1) - \left\|\frac{x_2}{\lambda_2}\right\|\right) = \left\|\frac{x_1}{\lambda_1}\right\| + \left\|\frac{x_2}{\lambda_2}\right\| - \frac{1}{2}\ell\left(\alpha_1 e_1 + \alpha_2 e_1\right) =$$

$$= \|\alpha_1 e_1 + e_2\| + \left\|\frac{x_2}{\lambda_2}\right\| - \frac{1}{2}\|\alpha_1 e_1 + \alpha_2 e_1\| \geqslant \frac{1}{2}\|\alpha_1 e_1 + e_2\| + \frac{1}{2}\left\|\frac{x_2}{\lambda_2}\right\| + \frac{1}{2}\left\|\alpha_1 e_1 + e_2 + \frac{x_2}{\lambda_2}\right\| -$$

$$-\frac{1}{2}\|\alpha_1 e_1 + \alpha_2 e_1\| = \frac{1}{2}\|\alpha_1 e_1 + e_2\| + \frac{1}{2}\left\|\frac{x_2}{\lambda_2}\right\| \geqslant \frac{1}{2}\left\|\alpha_1 e_1 + \frac{x_2}{\lambda_2} + e_2\right\| = r > 0$$

by (3.5), since $e_2 \neq 0$.

Consequently, $\beta_{e_2} \geqslant \alpha_{e_2} + \min\left\{\frac{1}{2}\delta\|e_1\|; r\right\}$, i.e. $\beta_{e_2} > \alpha_{e_2}$ and we can choose $\ell \in X^*$ such that $\ell(e_2) \neq \ell(e_1) = \frac{1}{2}\|e_1\| > 0$. $\square$

By Corollary 2.1 and Theorem 3.1 the next fact for each *CONC* immediately follows.

Corollary 3.1. *Let X be a CONC. Then for all $x_0, x_1, x_2 \in X$:*

(i) *if* $x_0 \neq 0$ *then there exists* $\ell \in X^* \setminus \{0\}$ *such that* $\|\ell\|_* = 1$ *and* $\max(0, \ell(x_0)) = \|x_0\|$;

(ii) *if* $x_1 \neq x_2$ *then there exists a functional* $\ell \in X^* \setminus \{0\}$ *such that* $\max(0, \ell(x_1)) \neq \max(0, \ell(x_2))$.

Remark 4. *Note, that the condition of* $\ell(e_1) > 0$ *or* $\ell(e_2) > 0$ *in Theorem 3.1 is essential for this result.*

Remark 5. Note that in the convex normed cone X_2 from Example 6 all axioms of $CONC$ are fulfilled, besides (3.3). We can consider the pairs $x = (1, 1)$ and $y = (1, 0)$ to check this fact. Obviously, the assertion of Corollary 3.1(ii) does not hold for the cone X_2 from Example 6.

For each linear functional $\ell \in X^*$ we consider the following bounded functional:

$$p_\ell(x) = \max\{0, \ell(x)\}. \tag{3.6}$$

Corollary 3.1 means that functionals (3.6) separate points of each $CONC$ X. Denote by X^*_{sub} the minimal convex cone including all functionals (3.6):

$$X^*_{sub} := \left\{ \sum_{k=1}^{n} p_{\ell_k}(x) = \sum_{k=1}^{n} \max\{0, \ell_k(x)\} \mid \ell_k \in X^* \text{ for all } k = \overline{1, n}, n \in \mathbb{N} \right\}.$$

Definition 11. We say that X^*_{sub} is a subconjugate cone of X.

The norm on the convex cone X^*_{sub} is introduced in a natural way:

$$\|p\|_* := \sup_{x \in X \setminus \{0\}} \left\{ \frac{p(x)}{\|x\|} \right\} \text{ for all } p \in X^*_{sub}. \tag{3.7}$$

Now we consider the collection of all linear functionals $\psi : X^*_{sub} \to \mathbb{R}$ with natural operations of addition and scalar multiplication:

$$[\psi_1 + \psi_2](p) := \psi_1(p) + \psi_2(p); \; [\lambda\psi](p) := \lambda\psi(p)$$

for all linear functionals $\psi, \psi_{1,2} : X^*_{sub} \to \mathbb{R}$, for each number $\lambda \in \mathbb{R}$ and $p \in X^*_{sub}$.

We say that the conjugate space $(X^*_{sub})^*$ of X^*_{sub} is the collection of all bounded linear functionals $\psi : X^*_{sub} \to \mathbb{R}$ with the following norm:

$$\|\psi\|_{**} := \sup_{p \in X^*_{sub} \setminus \{0\}} \left\{ \frac{|\psi(p)|}{\|p\|_*} \right\} = \sup_{p : \|p\|_* = 1} |\psi(p)|.$$

Definition 12. We say that $(X^*_{sub})^* =: X^{**}$ is the second conjugate space of $CONC$ X.

The partial order $\preccurlyeq$ can be introduced in X^{**} in the following way:

$$\forall \psi_{1,2} \in X^{**} \; \psi_1 \preccurlyeq \psi_2 \iff \psi_1(p) \leqslant \psi_2(p) \; p \in X^*_{sub}. \tag{3.8}$$

Let us formulate the following sublinear analogue of the J. Rädström embedding theorem.

Theorem 3.2. *Let* X *be a* $CONC$. *Then there is such a metric* $d_* : X \times X \longrightarrow \mathbb{R}^+$ *that* X *is sublinearly injectively isometrically and* d_**-continuously embedded into the second conjugate space* X^{**}.

Proof. It is natural to consider the embedding $\varphi : X \to X^{**}$:

$$\varphi(x) = \psi_x(\cdot), \text{ where } \psi_x(p) = p(x) \text{ for all } p \in X^*_{sub}. \tag{3.9}$$

Indeed, for all $x \in X$, $p_1, p_2 \in X^*_{sub}$ and $\lambda_1, \lambda_2 \geqslant 0$ we have

$$\psi_x(\lambda_1 p_1 + \lambda_2 p_2) = [\lambda_1 p_1 + \lambda_2 p_2](x) = \lambda_1 p_1(x) + \lambda_2 p_2(x) = \lambda_1 \psi_x(p_1) + \lambda_2 \psi_x(p_2),$$

i.e. $\psi_x(\cdot) \in X^{**}$.

Clearly, for all $x_1, x_2 \in X$, $\lambda_1, \lambda_2 \geqslant 0$ and $p \in X^*_{sub}$

$$\psi_{\lambda_1 x_1 + \lambda_2 x_2}(p) = p(\lambda_1 x_1 + \lambda_2 x_2) \leqslant \lambda_1 p(x_1) + \lambda_2 p(x_2) = [\lambda_1 \psi_{x_1} + \lambda_2 \psi_{x_2}](p),$$

i.e.

$$\psi_{\lambda_1 x_1 + \lambda_2 x_2} \preccurlyeq \lambda_1 \psi_{x_1} + \lambda_2 \psi_{x_2}.$$

Assume that

$$d_*(x, y) := \|\varphi(x) - \varphi(y)\|_{X^{**}} \tag{3.10}$$

By Corollary 3.1 $d_*(x, y) = 0 \iff x = y$. Hence, $d_* : X \times X \longrightarrow \mathbb{R}^+$ is a metric. Further, for all $x, y \in X$

$$\|\varphi(x)\|_{X^{**}} = \sup_{p : \|p\|_* = 1} p(x) = \sup_{\|\ell\|_* = 1} \max\{0, \ell(x)\} = \sup_{\|\ell\|_* = 1} \ell(x) = \|x\|$$

by Corollary 2.1, i.e. φ is an isometrical embedding.

So, $\varphi : X \to X^{**}$ is an injective sublinear isometric embedding of X into the linear normed space X^{**}; d_*-continuity of φ is obvious. □

Remark 6. The linearity of embedding $\varphi : X \to X^{**}$ is impossible if d_* is not invariant under shifts (see Example 10 and Remark 9).

Remark 7. Note that on the base of Corollary 3.1 we can introduce one more metric $d_\circ$: $X \times X \longrightarrow \mathbb{R}$:

$$d_\circ(x, y) = \sup_{\ell \in X^* : \|\ell\|_* = 1} |\max\{0, \ell(x)\} - \max\{0, \ell(y)\}| \leqslant d_*(x, y),$$

where X^* is the conjugate cone of $CONC\ X$ (see Definition 6).

Remark 8. The system of closed d_*-neighbourhoods (and $d_\circ$-neighbourhoods) of points in a $CONC\ X$ is defined in the following natural way ($x \in X$, $\varepsilon > 0$):

$$O_\varepsilon(x) = \{y \in X \,|\, d_*(x, y) \leqslant \varepsilon\}; \tag{3.11}$$

$$O^\circ_\varepsilon(x) = \{y \in X \,|\, d_\circ(x, y) \leqslant \varepsilon\}. \tag{3.12}$$

Obviously, for each $x \in X$ and $\varepsilon > 0$,

$$O_\varepsilon(x) \subset O^\circ_\varepsilon(x), \quad O_\varepsilon(0) = O^\circ_\varepsilon(0) = \{x \in X \,|\, \|x\| \leqslant \varepsilon\}. \tag{3.13}$$

Now we illustrate $d_\circ$-neighbourhoods of points $x \in X$ in $CONC\ X$.

Example 10. We consider the *CONC*

$$X_2 = \{(a,b) \,|\, a, b \geqslant 0 \text{ and } a = 0 \Rightarrow b = 0\}$$

with the norm $\|(a,b)\| := a$. In ([22], Example 8) it was shown, that X_2 does not allow linear isometric injective embedding into any linear normed space.

By Lemma 2.1 each linear functional $f : X \to \mathbb{R}$ has the form $f((a,b)) = \lambda a + \mu b$, where $\lambda \geqslant 0$ and $\mu \leqslant 0$ are fixed constants:

$$X^* = \{\ell((a;b)) = \lambda a + \mu b \,|\, \lambda \geqslant 0, \mu \leqslant 0\}. \tag{3.14}$$

Hence

$$\|\ell\|_* = \|\max\{0, \ell(\cdot)\}\|_{X^*_{sub}} = \sup_{b\geqslant 0} \ell((1;b)) = \sup_{b\geqslant 0}(\lambda + \mu b) = \lambda,$$

i.e. $\|\ell\|_* = 1 \Longleftrightarrow \ell((a;b)) = a + \mu b$ for some $\mu \leqslant 0$. Thus, the following equality holds

$$d_\circ(A,B) = \sup_{\mu\leqslant 0} |\max\{0, a_1 + \mu b_1\} - \max\{0, a_2 + \mu b_2\}|, \tag{3.15}$$

where $A = (a_1, b_1)$, $B = (a_2, b_2) \in X = X_2$. Set

$$g(A,B,\mu) = \max\{0, a_1 + \mu b_1\} - \max\{0, a_2 + \mu b_2\}.$$

If $a_{1,2} \neq 0$ and $b_{1,2} \neq 0$, then the following cases are possible:

1) $g(A,B,\mu) = a_1 - a_2 + \mu(b_1 - b_2)$ for $\mu \geqslant \max\left\{-\frac{a_1}{b_1}; -\frac{a_2}{b_2}\right\}$;

2) $g(A,B,\mu) = a_1 + \mu b_1$ for $-\frac{a_1}{b_1} \leqslant \mu \leqslant -\frac{a_2}{b_2}$;

3) $g(A,B,\mu) = -(a_2 + \mu b_2)$ for $-\frac{a_2}{b_2} \leqslant \mu \leqslant -\frac{a_1}{b_1}$;

4) $g(A,B,\mu) = 0$ otherwise.

It is easy to see, that for fixed A and B the function $|g(A,B,\mu)|$ attains its maximax value on $\alpha \leqslant \mu \leqslant \beta$ only for $\mu = \alpha$ or for $\mu = \beta$. Hence, in the case $-\frac{a_1}{b_1} \leqslant -\frac{a_2}{b_2}$ the largest possible value $|g(A,B,\mu)|$ can be one of the following numbers:

$$|a_1 - a_2|, \; a_2 - \frac{a_2}{b_2} b_1 = \frac{a_1 b_2 - a_2 b_1}{b_2},$$

$$\left| a_1 - a_2 - \frac{a_2}{b_2}(b_1 - b_2)\right| = \left| a_1 - a_2 - \frac{a_2}{b_2} b_1 + a_2 \right| = \left| \frac{a_1 b_2 - a_2 b_1}{b_2} \right| = \frac{a_1 b_2 - a_2 b_1}{b_2},$$

and for $\frac{a_1}{b_1} \geqslant \frac{a_2}{b_2}$ $(b_1, b_2 \neq 0)$ we have

$$d_\circ(A,B) = \max\left\{|a_1 - a_2|, \frac{a_1 b_2 - a_2 b_1}{b_2}\right\}.$$

Analogously, for $\frac{a_2}{b_2} \geqslant \frac{a_1}{b_1}$ $(b_1, b_2 \neq 0)$ we have

$$d_\circ(A,B) = \max\left\{|a_1 - a_2|, \frac{a_2 b_1 - a_1 b_2}{b_1}\right\}.$$

Thus, for $b_1, b_2 > 0$ (and hence $a_1, a_2 > 0$)

$$d_\circ(A,B) = \max\left\{|a_1 - a_2|, \frac{a_2 b_1 - a_1 b_2}{b_1}, \frac{a_1 b_2 - a_2 b_1}{b_2}\right\}. \tag{3.16}$$

For $b_1 = 0$ we have $d_\circ(A,B) = \sup_{\mu \leqslant 0} |\max\{0, a_2 + \mu b_2\} - a_1|$. For $b_2 > 0$ and $\mu \leqslant -\frac{a_2}{b_2}$ one has $|\max\{0, a_2 + \mu b_2\} - a_1| = a_1$; for $\mu \geqslant -\frac{a_2}{b_2}$ the following relation holds:

$$|\max\{0, a_2 + \mu b_2\} - a_1| = |a_2 - a_1 + \mu b_2| \leqslant \max\{a_1, |a_1 - a_2|\}.$$

In the case $b_1 = b_2 = 0$ $d_\circ(A,B) = |a_1 - a_2|$. So, the following relation holds:

$$d_\circ(A,B) = \begin{cases} \max\{a_1, |a_1 - a_2|\}, & \text{for } b_1 = 0, b_2 > 0; \\ \max\{a_2, |a_1 - a_2|\}, & \text{for } b_1 > 0, b_2 = 0; \\ |a_1 - a_2|, & \text{for } b_1 = b_2 = 0. \end{cases} \tag{3.17}$$

Relations (3.16) and (3.17) stated above allow us to clearly describe the $d_\circ$-neighbourhoods of any $A_0 = (a_0, b_0) \in X_2$ of the form $\{O_\varepsilon(A_0)\}_{\varepsilon > 0}$, where

$$O_\varepsilon^\circ(A_0) = \{A = (a,b) \in X \mid d_\circ(A, A_0) \leqslant \varepsilon\}. \tag{3.18}$$

If $b_0 > 0$ then $A = (a,b) \in O_\varepsilon^\circ(A_0)$ for $b > 0$, if the following inequalities hold

$$\begin{cases} a_0 - \varepsilon \leqslant a \leqslant a_0 + \varepsilon, \\ a - \frac{a_0}{b_0} b \leqslant \varepsilon, \\ a_0 - \frac{a}{b} b_0 \leqslant \varepsilon, \end{cases}$$

that for a sufficiently small $\varepsilon > 0$ determine a trapezium (see Fig. 8).

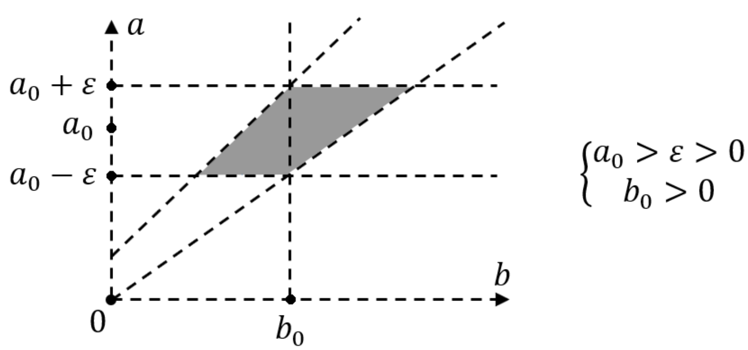

Figure 8.

Moreover, the $d_\circ$-neighbourhood also includes points of the form

$$A = \{(a, 0) : \max\{a_0, |a - a_0|\} \leqslant \varepsilon\},$$

that is, for $a_0 \leqslant \varepsilon$ this relation will satisfy all $a : |a - a_0| \leqslant \varepsilon$. Thus, if $a_0 \leqslant \varepsilon$ then the $d_\circ$-neighbourhood $A_0 = (a_0, b_0)$ will have the form shown in Fig. 9. As one can see, this neighbourhood is a non-convex set.

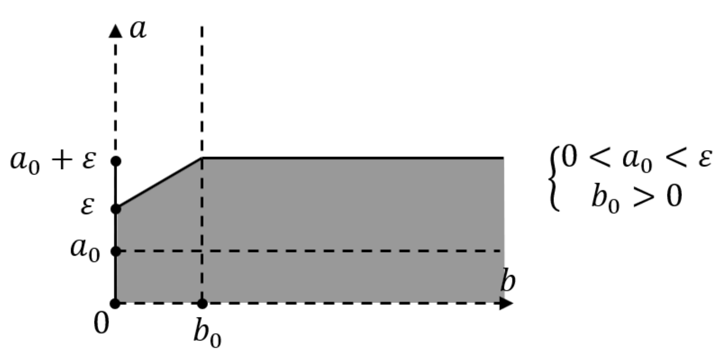

Figure 9.

Relations (3.17) allow us to determine the $d_\circ$-neighbourhood of $A_0 = (a_0, 0)$. If $0 < a_0 \leqslant \varepsilon$, then such a neighbourhood will have the form of a strip (see Fig. 10), as in $a_0 = 0$ (see Fig. 11).

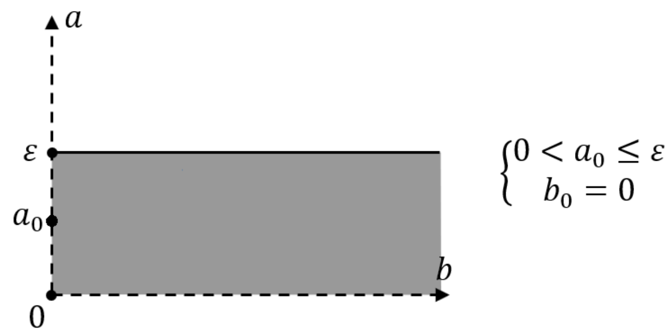

Figure 10.

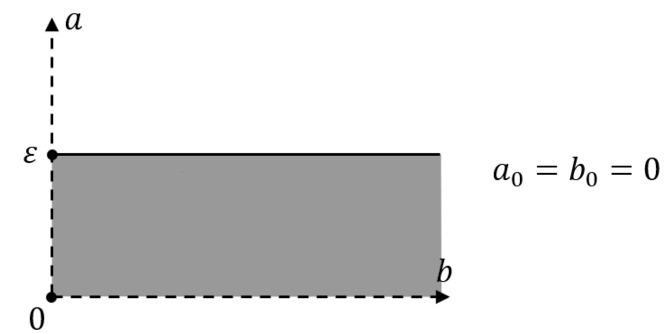

Figure 11.

If $a_0 > \varepsilon$ then $d_\circ$-neighbourhood has the form of the following segment (see Fig. 12)

$$\{(a, 0) \,|\, a_0 - \varepsilon \leqslant a \leqslant a_0 + \varepsilon\}.$$

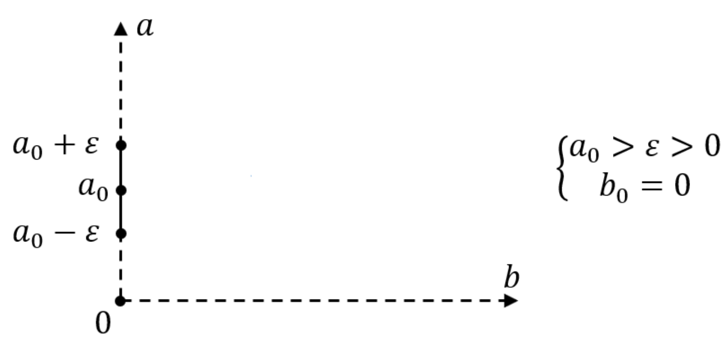

Figure 12.

Remark 9. Generally, the metrics d_* and $d_\circ$ are not subinvariant. Indeed, for some $x, y \in X_2$ $O_\varepsilon(y) \subset y + O_\varepsilon(0)$ (see Example 10, Fig. 8). If $x \in O_\varepsilon(0)$ (i.e. $\|x\| < \varepsilon$) such that $x + y \notin O_\varepsilon(y)$ then

$$\|x\| = d_*(x, 0) = d_\circ(x, 0) < d_\circ(x + y, y) \leqslant d_*(x + y, y).$$

4 An analogue of the Hahn-Banach theorem on separation of a point and a convex set for *CONC*

In Section 2 we have shown that each *CONC* X can be sublinearly injectively isometrically and d_*-continuously embedded in a linear normed space. This indicates the possibility of transferring classical results of analysis in linear normed spaces to the class $CONC$. In this section we obtain an analogue of the Hahn-Banach theorem on separability of a point and a d_*-closed (or d_*-open) convex set for convex ordered normed cones (see Theorems 4.1 and 4.2). We start with an example showing the existence of d_*-closed convex set $U \subset X$, such that for some point $x_0 \in X \setminus \{A\}$:

$$\inf_{a \in A} \ell(a) \leqslant \ell(x_0) \leqslant \sup_{a \in A} \ell(a) \ \text{ for all } \ell \in X^*. \tag{4.1}$$

Example 11. We consider the $CONC$

$$X_2 = \{(a,b)\,|\,a,b \geqslant 0 \text{ and } a = 0 \Rightarrow b = 0\}$$

with the norm $\|(a,b)\| := a$ (see Example 10).

By Lemma 2.1 each linear functional $f : X \to \mathbb{R}$ has the form $f((a,b)) = \lambda a + \mu b$, where $\lambda \geqslant 0$ and $\mu \leqslant 0$ are fixed constants:

$$X^* = \{\ell((a;b)) = \lambda a + \mu b\,|\,\lambda \geqslant 0, \mu \leqslant 0\}. \tag{4.2}$$

Let U has the form of a non-closed interval (see Fig. 13). If for $x \in X$ and some sequence $\{x_n\}_{n=1}^{\infty} \subset X$: $\lim\limits_{n\to\infty} d_*(x,x_n) = \lim\limits_{n\to\infty} d_\circ(x,x_n) = 0$ then for all $\ell \in X^*$ $\lim\limits_{n\to\infty}(\max\{0,\ell(x_n)\} - \max\{0,\ell(x)\}) = 0$, which implies that the set U in Fig. 13 is d_*-closed in X_2.

If we select a pair $x_0 = (a_0,b_0)$ (see Fig. 13), then $\ell(x_0) < \ell(x_1)$ and $\ell(x_2) < \ell(x_0)$, where $x_1 = (a_1,b_0)$ and $x_2 = (a_0,b_2)$ for some $a_1 > a_0$ and $b_2 > b_0$, that implies (4.1).

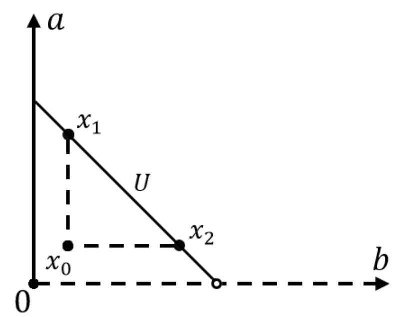

Figure 13. "Interval and point".

The above example clearly shows that we cannot transfer the Hahn-Banach theorem on separation of a point and a closed convex set for linear semi-bounded functionals to the class $CONC$ without additional conditions on U. Nevertheless, it is possible to obtain an analogue of the Hahn-Banach theorem on separation of a point and a d_*-closed convex set U with $0 \in U$. We use the notation:

$$O_\varepsilon(0) = \{x \in X\,|\,d_*(0,x) \leqslant \varepsilon\} = \{x \in X\,|\,\|x\| < \varepsilon\}.$$

The following theorem holds.

Theorem 4.1. *Let U be a d_*-closed convex set in X, $O_\varepsilon(0) \subset U$ for some $\varepsilon > 0$. If $x_0 \notin U$ then there exists $\ell \in X^* \setminus \{0\}$ such that $\ell(x_0) > \sup \ell(U)$.*

Proof. We consider the Minkowsky functional $p_U(x) = \inf\{\lambda \geqslant 0\,|\,x \in \lambda^{-1}U\}$. It is bounded for all $x \in X$ since $O_\varepsilon(0) \subset U$. Clearly, that $p_U(x) \leqslant 1$ for all $x \in U$ and $p_U(x_0) > 1$. Also it follows from $O_\varepsilon(0) \subset U$ that $p_U(x) \leqslant \frac{1}{\varepsilon}\|x\|_X$ for all $x \in X$. If X is a $CONC$ then $X_0 = \{\lambda x_0\,|\,\lambda \geqslant 0\}$ is a subcone X, p_U is a linear functional on X_0. Hence, by Theorem 2.1 there exists a linear functional $\ell : X \to \mathbb{R}$:

$$\ell(x_0) = p_U(x_0) \text{ and } \ell(x) \leqslant p_U(x) \leqslant \frac{1}{\varepsilon}\|x\|_X \text{ for all } x \in X.$$

It is clear that $\ell \in X^*$ and for all $x \in U$ $\ell(x) \leqslant p_U(x) \leqslant 1$ and $\ell(x_0) = p_U(x_0) > 1$, i.e. ℓ is a required functional. □

Remark 10. Note that the condition $U \subset X$ is d_*-closed can be replaced by the following one ($U \subset X$ is a *closed set on the rays*):

$$\text{for each} \quad x \in X \; \lambda x \in U \quad \text{for all} \quad \lambda < \lambda_0 \Longrightarrow \lambda_0 \cdot x \in U. \tag{4.3}$$

Remark 11. The condition $O_\varepsilon(0) \subset U$ for some $\varepsilon > 0$ is essential for the validity Theorem 4.1. Indeed, if we consider $X = X_2$ (see Example 11) then for $U = \{(a,0) \,|\, 0 \leqslant a \leqslant 2\}$ and $x_0 = (1,1)$:

$$\sup\{\ell((a,0)) \,|\, 0 \leqslant a \leqslant 2\}\} = 2\lambda \geqslant \ell((1,1)) = \lambda + \mu$$

for all $\lambda \geqslant 0$, $\mu \leqslant 0$ (by Lemma 2.1 for each $\ell \in X_2^*$ $\ell((a,b)) = \lambda a + \mu b$, where $\lambda \geqslant 0$, $\mu \leqslant 0$).

Analogously to the previous result it can be proved the following fact for d_*-open convex sets U. In this case the condition $x_0 \notin O_\varepsilon(0)$ for some $\varepsilon > 0$ follows from $0 \in U$.

Theorem 4.2. *Let U be a d_*-open convex set in X, $0 \in U$. If $x_0 \notin U$ then there exists $\ell \in X^*$ such that $\ell(x_0) \geqslant \sup \ell(U)$.*

5 An analogue of the Hahn-Banach separation theorem for closed sets in $CONC$

In this section we generalize Theorem 4.1 to the case, in which instead of a point and a convex set we consider two closed sets A and B. We assume that one of them A is bounded (i.e. $\|a\| \leqslant C$ for some $C > 0$) and another set B is d_*-compact. The following result holds.

Theorem 5.1. *Let A be a bounded convex set closed on the rays in $CONC$ X, $O_\varepsilon(0) \subset A$ for some $\varepsilon > 0$. Let B be a d_*-compact set in X, $A \cap B = \varnothing$. Then there exists a finite set of linear semi-bounded functionals $\{\ell_k\}_{k=1}^n \subset X^*$, with $\|\ell_k\|_* = 1$ such that*

$$\inf_{b \in B} \max\{\ell_1(b), \ell_2(b), \ldots, \ell_n(b)\} > \sup_{a \in A} \max\{\ell_1(a), \ell_2(a), \ldots, \ell_n(a)\} \tag{5.1}$$

Proof. 1) Since A is a bounded closed set on the rays then we can consider the equivalent norm $p_A(\cdot)$ in X. Therefore, we assume that

$$\|x\| = p_A(x) = \inf\left\{\lambda > 0 \,\middle|\, \frac{x}{\lambda} \in A\right\}.$$

Since $A \cap B = \varnothing$ and A is bounded and closed on rays, then $\inf\limits_{b \in B} \|b\| > 1 = \max\limits_{a \in A} \|a\|$. It means that we can choose such $\delta > 0$ that $\inf\limits_{b \in A} \|b\| > 1 + 2\delta$.

2) In virtue of d_*-compactness of B and d_*-continuity of the embedding $\varphi : X \to X^{**}$ by Theorem 1 we have compactness of the image $\varphi(B) \subset X^{**}$. This means that $\varphi(B)$ can be covered by a finite set of δ-neighbourhoods $O_\delta(\varphi(x_k))$ for $k = \overline{1,n}$, i.e. B is covered by δ-neighbourhoods $O_\delta(x_k) \subset X$ for $k = \overline{1,n}$. Let $y \in O_\delta(x_{k_0})$ for some $k_0 \in \{1, 2, \ldots, n\}$. Then $\|\varphi(y) - \varphi(x_{k_0})\|_{X^{**}} \leqslant \delta$, i.e.

$$\sup_{\ell \in X^* : \|\ell\|_* = 1} |\max\{0, \ell(y)\} - \max\{0, \ell(x_{k_0})\}| \leqslant \delta. \tag{5.2}$$

3) By Corollary 2.1 for all $k \in \{1, 2, \ldots, n\}$ there exists $\ell_k \in X^*$ such that $\|\ell_k\|_* = 1$ and

$$\ell_k(x_k) = \|x_k\| = \max\{0, \ell(x_k)\} > 1 + 2\delta. \tag{5.3}$$

It follows from (5.2) that $\max\{0, \ell_{k_0}(y)\} \geqslant \max\{0, \ell_{k_0}(x_0)\} - \delta > 1 + \delta > 0$, i.e. $\ell_{k_0}(y) > 1+\delta$ for all $y \in O_\delta(x_{k_0})$.

Since B is covered by δ-neighbourhoods of points $\{x_k\}_{k=1}^n O_\delta(x_k)$, then for all $b \in B$ there exists $\ell_k \in X^*$ satisfying (5.3), for which $\ell_k(b) > 1 + \delta$. Therefore,

$$\inf_{b \in B} \max\{\ell_1(b), \ell_2(b), \dots, \ell_n(b)\} > 1 + \delta.$$

On the other hand, in view of (5.3) ($\|\ell_k\|_* = 1$) $\ell_k(a) \leqslant \|a\| \leqslant 1$ for each $a \in A$ and (5.1) holds. ☐

Remark 12. Generally, one cannot replace a system of functionals in (5.1) by some functional $\ell \in X^*$. To illustrate this fact we consider the convex normed cone

$$X = \{(0,0)\} \cup \{(a,b) \,|\, a > 0 \text{ and } b > 0\}$$

with the norm $\|(a,b)\| = \sqrt{a^2 + b^2}$. Clearly, X is a $CONC$.

Assume that

$$A = \{(a,b) \in X \,|\, a^2 + b^2 \leqslant 1\}$$

and

$$B = \left\{(a,b) \in X \mid a^2 + b^2 = (1+\varepsilon)^2\right\}$$

for some $\varepsilon > 0$ (see Fig. 14).

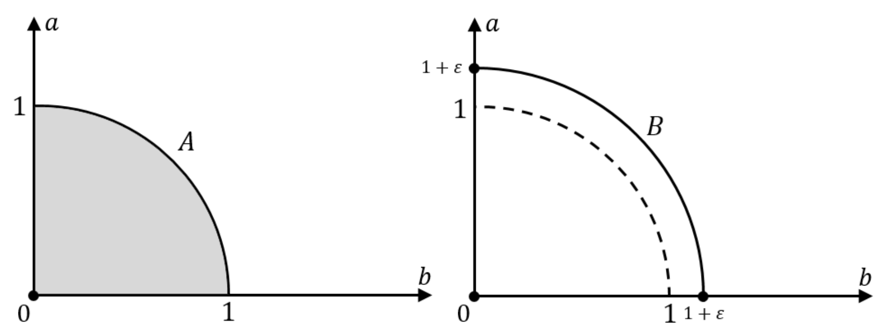

Figure 14.

The sets A and B satisfy the conditions of Theorem 5.1 (see Fig. 14), but for sufficiently small $\varepsilon > 0$ $A \bigcap coB \neq \varnothing$ (coB is a convex hull of B) and there is no functional $\ell \in X^*$ such that

$$\inf_{b \in B} \ell(b) > \sup_{a \in A} \ell(a).$$

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References

[1] E. G. Bakhtigareeva, M. L. Goldman, *Associate norms and optimal embeddings for a class of two-weighted integral quasi-norms.* Fund. Appl. Math. 19 (2014), no. 5, 3 – 33 (*in Russian*).

[2] E. G. Bakhtigareeva, *Optimal Banach function space for a given cone of decreasing functions in a weighted L_p-space.* Eurasian Math. J. 6 (2015), no. 1, 6 – 25.

[3] P. A. Borodin, *The Banach-Mazur theorem for spaces with asymmetric norm.* Math. Notes. 69 (2001), no. 3, 298 – 305.

[4] S. Cobzas, *Functional analysis in asymmetric normed spaces*, Basel: Birkhauser/Springer, 2013, 219 p.

[5] C. Mustÿata, *On the extremal semi-Lipschitz functions*, Ann. Numer. Theory Approx., 31 (2002), 103 – 108.

[6] L. M. Garcia-Raffi, S. Romaguera, E. A. Sanchez-Perez, *Sequence spaces and asymmetric norms in the theory of computational complexity*, Math. Comput. Modelling, 36 (2002), 1 – 11.

[7] L. M. Garcia-Raffi, S. Romaguera, E. A. Sanchez-Perez, O. Valero, *Metrizability of the unit ball of the dual of a quasi-normed cone*, Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. 7-B (2004), n.2, 483 – 492.

[8] M. L. Goldman, P. P. Zabreiko, *Optimal Banach function spaces generalized with the cone of nonnegative increasing functions.* Trans. Inst. Math. Nat. Akad. Sci. Belarus. 22 (2014), no. 1, 24 – 34 (*in Russian*).

[9] K. Keimel, *Topological cones: functional analysis in a T_0-setting.* Semigroup Forum. 77 (2008), 109 – 142.

[10] I. V. Orlov, *Inverse and implicat function theorems in the class of subsmooth maps.* Math. Notes. 99 (2016), no. 4, 619 – 622.

[11] I. V. Orlov, *Introduction to sublinear analysis.* Contemp. Math. Fundam. Direct. 53 (2014), 64 – 132 (*in Russian*).

[12] I. V. Orlov., S. I. Smirnova, *Invertibility of multivalued sublinear operators.* Eurasian Math. J. 6 (2015), no. 4, 44 – 58.

[13] J. H. Rädström, *An embedding theorem for space of convex sets.* Proc. Amer. Math. Soc. 3 (1952), 165 – 169.

[14] S. Romaguera, M. Sanchis, *Semi-Lipschitz functions and best approximation in quasi-metric spaces*, J. Approx. Theory, 103 (2000), 292 – 301.

[15] Romaguera S., Sanchez-Perez E. A., Valero O. The dual complexity space as the dual of a normed cone, *Electronic Notes in Theoretical Computer Science*, 161 (2006), 165 – 174.

[16] Romaguera S., Sanchez-Perez E. A., Valero O. A characterization of generalized monotone normed cones, *Acta Mathematica Sinica, English Series*, 23 (2007), No. 6, 1067 – 1074.

[17] W. Roth, *Hahn-Banach type theorems for locally convex cones.* Journal of the Australian Math. Soc. (Ser. A). 68 (2000), no. 1, 104 – 125.

[18] P. Selinger, *Towards a semantics for higher-order quantum computation.* Proceedings of the 2nd International Workshop on Quantum Programming Languages. Turku Centre for Computer Science General Publication. 33 (2004), 127 – 143.

[19] F. S. Stonyakin, *Analogues of the Shauder theorem that use anticompacta.* Math. Notes. 99 (2016), no. 6, 954 – 958.

[20] F. S. Stonyakin, *On Sublinear analogues of weak topologies in normed cones.* Math. Notes. 103 (2018), no. 5, 859 – 864.

[21] F. S. Stonyakin, *A sublinear analogue of the Banach–Mazur Theorem in separated convex cones with norm.* Math. Notes. 104 (2018), no. 1, 111 – 120.

[22] F. S. Stonyakin, *An analogue of the Hahn-Banach Theorem for functionals on abstract convex cone*, Eurasian. Math. J., 7 (2016), no. 3, 89 – 99.

[23] R. Tix, *Some results on Hahn–Banach-type theorems for continuous D-cones*, Theoretical Comput. Sci., 264 (2001), 205 – 218.

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