ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2024, Volume 15, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

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13 Kazhymukan St
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EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 1 (2024), 75 – 90

POROSITY IN THE CONTEXT OF HYPERGROUPS

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Communicated by J.A. Tussupov

Key words: locally compact hypergroup, center of hypergroups, porosity, σ-lower porosity, spaceability, Lebesgue spaces, Hilbert spaces, convolution.

AMS Mathematics Subject Classification: 43A62, 46E30, 43A15, 54E52, 42A85, 44A35.

Abstract. In this paper we show that the set of all elements $g \in L^p(\mathcal{H})$ for which $(|g|*|g|)(x) < \infty$ for a center element $x \in B$, is σ-c-lower porous, where $p > 2$, $\mathcal{H}$ is a non-compact unimodular hypergroup and B is some special symmetric compact neighborhood of the identity element. As an application, we give some new equivalent condition for the finiteness of a discrete Hermitian hypergroup. Moreover, we give some sufficient conditions for the set of all pairs (f, g) in $L^p(\mathcal{H}) \times L^q(\mathcal{H})$ for which for a center element $x \in B$, $(|f| * |g|)(x) < \infty$, is a σ-c-lower porous, where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} < 1$. Also, we show that the complement of this set is spaceable in $L^p(\mathcal{H}) \times L^q(\mathcal{H})$.

DOI: https://doi.org/10.32523/2077-9879-2024-15-1-75-90

1 Introduction and preliminaries

σ-porous sets, as a category of small sets, were introduced by Dolženko in 1967 in order to study of singular points of holomorphic functions. In recent decades, the relationship between this concept and many topics has been discovered, including hypercyclicity, L^p-conjecture, spaceability etc; see [4, 8, 9, 10]. We refer the reader to survey papers [23, 24] for more information on porosity on the real line, metric spaces and normed spaces. If $c \in (0, 1)$ and X is a metric space, a subset $M \subseteq X$ is called *c-lower porous* if for each $x \in M$,

$$\liminf_{r>0} \frac{\gamma(x, r, M)}{r} \geq \frac{c}{2}, \tag{1.1}$$

where

$$\gamma(x, r, M) := \sup\{s \geq 0 : \text{ for some } z \in X,\ B(z; s) \subseteq B(z; r) \setminus M\}, \tag{1.2}$$

and $B(x; r)$ is the open ball with center x and radius r. We denote also $B(x; 0) := \varnothing$. We say that M is *σ-c-lower porous* if it can be represented as a countable union of c-lower porous subsets of X. In the following result which was proved in [25] one can find some equivalent condition for σ-lower porosity of subsets of normed spaces; see also [24, Proposition 2.2].

Theorem 1.1. *Let X be a normed linear space and $c \in (0, 1]$. Then, a subset $E \subseteq X$ is σ-c-lower porous if and only if $E = \bigcup_{m=1}^{\infty} P_m$, where for each $m \in \mathbb{N}$, $x \in X$ and $r > 0$ there exists some $y \in X$ such that $B(y; cr) \subseteq B(x; r) \setminus P_m$.*

In 2010, due to study of some classes of convolution Banach function algebras, S. Głąb and F. Strobin started to investigate the σ-lower porous subsets of Lebesgue spaces in the context of locally compact groups. They proved that there is a σ-porous set such that for each (f, g) in its complement, $f * g$ does not exist on a set of positive measure [8], and also proved the following statement.

Theorem 1.2. *If G is a non-compact unimodular group and $p > 2$, then for each compact subset $F \subset G$ there exists some $c > 0$ such that the set*

$$E := \{f \in L^p(G) : \exists\, x \in F \text{ such that } |f| * |f|(x) < \infty\}$$

is a σ-c-lower porous subset of $L^p(G)$.

After that, I. Akbarbaglu and S. Maghsoudi in [1] draw a similar picture for some Orlicz spaces which are generalization of Lebesgue ones. They and J.B. Seoane-Sepúlveda in [2] studied also this topic for Lebesgue spaces on discrete semigroups.

On the other hand, locally compact hypergroups which are important generalizations of locally compact groups were introduced in [6, 11, 19]; see [12, 13, 14, 15, 16, 17] as recent works on locally compact hypergroups. There exists some convolutions among the regular measures of a locally compact hypergroup, while in contrast to the group case, the convolution of two Dirac measures of a hypergroup is not necessarily a Dirac measure. In Section 2 of this paper we initiate investigations of porosity in the context of hypergroups and as a main result we give an extension of Theorem 1.2 to hypergroups. Indeed, we prove that whenever $\mathcal{H}$ is a non-compact unimodular hypergroup and $p > 2$, then for each symmetric compact neighborhood B of the identity element of $\mathcal{H}$ if there is a constant $L > 0$ such that for each $x_1, \ldots, x_n \in \mathcal{H}$ we have

$$\frac{\sum_{k=1}^{n} \lambda(x_k * B * B)}{\sum_{k=1}^{n} \lambda(B * \check{x_k})} \leq L, \tag{1.3}$$

then the set

$$E_B := \{g \in L^p(\mathcal{H}) : \text{ for some } x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|g| * |g|)(x) < \infty\}$$

is σ-c-lower porous in $L^p(\mathcal{H})$, where $\mathrm{Ma}(\mathcal{H})$ is the center of the hypergroup and the Lebesgue space is with respect to a left invariant measure on $\mathcal{H}$. This fact directly implies that for each infinite discrete Hermitian hypergroup $\mathcal{H}$ and $p > 2$, $L^2(\mathcal{H})$ is a σ-c-lower porous subset of $L^p(\mathcal{H})$.

In Section 3 among other results we give some equivalent condition for a hypergroup to be compact. In fact, in Corollary 3.2 we prove that if $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$, $\mathcal{H}$ is a unimodular hypergroup and B is a symmetric compact neighborhood of the identity e in $\mathcal{H}$ with L-property, then $\mathcal{H}$ is non-compact if and only if the set M_B is a σ-c-lower porous subset of $L^p(\mathcal{H}) \times L^q(\mathcal{H})$ for some $c \in (0, 1)$, and this holds if and only if the set $\big(L^p(\mathcal{H}) \times L^q(\mathcal{H})\big) \setminus M_B$ is spaceable in $L^p(\mathcal{H}) \times L^q(\mathcal{H})$, where

$$M_B := \{(f, g) \in L^p(\mathcal{H}) \times L^q(\mathcal{H}) : \exists\, x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|f| * |g|)(x) < \infty\}.$$

We will show that in several classes of hypergroups one can find such neighborhoods B with L-property.

Next, we recall some basic information regarding hypergroups.

1.1 Hypergroups

We denote the space of all complex Radon measures on a locally compact Hausdorff space $\mathcal{X}$ by $\mathcal{M}(\mathcal{X})$. Also, the set of all non-negative measures in $\mathcal{M}(\mathcal{X})$ by $\mathcal{M}^+(\mathcal{X})$. The support of each $\mu \in \mathcal{M}(\mathcal{X})$ is denoted by $\mathrm{supp}(\mu)$. We denote a point-mass measure at $x \in \mathcal{X}$ by δ_x.

Definition 1. A locally compact Hausdorff space $\mathcal{H}$ equipped with a (*convolution*) product $*$ on $\mathcal{M}(\mathcal{H})$ and an *involution* map $x \mapsto \check{x}$ is called a *locally compact hypergroup* (or simply a *hypergroup*) if the following conditions hold.

1. $(\mathcal{M}(\mathcal{H}), +, *)$ is a Banach algebra.

2. For each $x, y \in \mathcal{H}$, $\delta_x * \delta_y$ is a compact supported probability measure.

3. The mappings $(x, y) \mapsto \delta_x * \delta_y$ from $\mathcal{H} \times \mathcal{H}$ into $\mathcal{M}^+(\mathcal{H})$ is continuous, where $\mathcal{M}^+(\mathcal{H})$ is equipped with the cone topology.

4. The mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $\mathcal{H} \times \mathcal{H}$ into the family of all nonempty compact subsets of $\mathcal{H}$, $\mathbf{C}(\mathcal{H})$, is continuous, where $\mathbf{C}(\mathcal{H})$ is equipped with the Michael topology.

5. The involution map is an involutive homeomorphism from $\mathcal{H}$ onto $\mathcal{H}$ such that for each $x, y \in \mathcal{H}$, $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$.

6. There exists an element $e \in \mathcal{H}$ (called *identity*) such that for each $x \in \mathcal{H}$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$. Moreover, for each $x, y \in \mathcal{H}$, $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $y = \check{x}$.

Any locally compact group, equipped with the usual convolution and the inverse mapping as involution, is a hypergroup. Contrary to the group case, for each x, y in a hypergroup $\mathcal{H}$, the convolution $\delta_x * \delta_y$ of two Dirac measures is not necessarily a Dirac measure. We refer to the book [5] for more information and examples. $\mathcal{H}$ is called *commutative* if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in \mathcal{H}$.

A nonzero nonnegative Radon measure λ on a hypergroup $\mathcal{H}$ is called *left-invariant* if for each $x \in \mathcal{H}$, $\delta_x * \lambda$ is defined and $\delta_x * \lambda = \lambda$. For each measurable set $E \subseteq \mathcal{H}$ we have

$$\|\chi_E\, \Delta^{-1}\|_1 = \lambda(\check{E}), \tag{1.4}$$

where Δ is the modular function. By [11, Theorem 4.3C], any hypergroup $\mathcal{H}$ admits a left sub-invariant measure λ with $\operatorname{supp}(\lambda) = \mathcal{H}$, while so far it has been remained as a conjecture that any hypergroup has a left-invariant measure.

In sequel, $\mathcal{H}$ is a hypergroup and λ is a left-invariant measure on $\mathcal{H}$. Also, for each $p \geq 1$, $L^p(\mathcal{H})$ is the Lebesgue space with the measure λ.

For each complex-valued Borel functions f and g on $\mathcal{H}$ and all $x, y \in \mathcal{H}$ we denote

$$f(x * y) := \int_{\mathcal{H}} f d(\delta_x * \delta_y) \qquad \text{and} \qquad (g * f)(x) := \int_{\mathcal{H}} g(y)\, f(\check{y} * x)\, d\lambda(y).$$

The convolution of two subsets $A, B \subseteq \mathcal{H}$ is defined by

$$A * B := \bigcup_{x \in A,\, y \in B} \operatorname{supp}(\delta_x * \delta_y).$$

The *center* of a hypergroup $\mathcal{H}$ is defined by

$$\operatorname{Ma}(\mathcal{H}) := \{x \in \mathcal{H} :\ \delta_x * \delta_{\check{x}} = \delta_{\check{x}} * \delta_x = \delta_e\}.$$

$\operatorname{Ma}(\mathcal{H})$ is the maximal subgroup of $\mathcal{H}$. Let $x \in \operatorname{Ma}(\mathcal{H})$ and $y \in \mathcal{H}$. Then, by [11, Section 10.4], $\delta_x * \delta_y$ is a Dirac measure; see also [18]. In this case, we denote the unique element in $\operatorname{supp}(\delta_x * \delta_y)$ by xy. Similarly, $\delta_y * \delta_x$ is the Dirac measure δ_{yx}. Note that xy and yx do not belong to the center in general. For each Borel measurable function $f : \mathcal{H} \to \mathbb{C}$, $x \in \operatorname{Ma}(\mathcal{H})$ and $y \in \mathcal{H}$ we have

$$\begin{aligned} |f(x * y)| &= \left| \int_{\mathcal{H}} f(t)\, d(\delta_x * \delta_y)(t) \right| \\ &= \left| \int_{\mathcal{H}} f(t)\, d\delta_{xy}(t) \right| \\ &= |f(xy)| = |f|(xy) \\ &= |f|(x * y). \end{aligned}$$

Example 1. Let G be a locally compact group such that the quotient space $G/Z(G)$ is compact, where $Z(G) := \{z \in G :$ for each $x \in G,\ zx = xz\}$. Let $I := \mathrm{Inn}(G)$ be the set of all inner automorphisms of G. Then, the orbit space $G^I := \{x^I : x \in G\}$ is a hypergroup where $x^I := \{g^{-1}xg : g \in G\}$ [11, Theorem 8.3A], and by [18] we have $\mathrm{Ma}(G^I) = \{z^I : z \in Z(G)\}$.

Remark 1. For each $A, B \subseteq \mathcal{H}$, $a \in \mathrm{Ma}(\mathcal{H})$ and $b \in \mathcal{H}$ we have:

1. $\lambda(A * \{\check{a}\}) = (\lambda * \delta_a)(A)$ and $(\lambda * \delta_b)(A) = \Delta(\check{b})\lambda(A)$.
2. $(A \cap B) * \{a\} = (A * \{a\}) \cap (B * \{a\})$ and $(A \cup B) * \{b\} = (A * \{b\}) \cup (B * \{b\})$.
3. for each $a \in \mathrm{Ma}(\mathcal{H})$ and $A, B \subseteq K$, we have
$$(A \cap B) * \{a\} = (A * \{a\}) \cap (B * \{a\}).$$

2 Porosity on $L^p(\mathcal{H})$

In this section, we study some porous subsets of Lebesgue spaces on hypergroups which helps us to give new equivalent conditions for a discrete Hermitian hypergroup to be infinite. The following lemma which was shortly proved in the proof of [21, Theorem 2.3] plays a key role in the main results of this paper.

Lemma 2.1. *Let $\mathcal{H}$ be a non-compact hypergroup and B be a compact symmetric neighborhood of the identity e in $\mathcal{H}$. Then, there exists a sequence $(a_n)_n$ in $\mathcal{H}$ with $\Delta(a_n) \leq 1$ for all $n \in \mathbb{N}$ such that for each distinct $m, n \in \mathbb{N}$,*
$$(\{a_n\} * B * B) \bigcap (\{a_m\} * B * B) = \varnothing \tag{2.1}$$
and
$$(B * \{\check{a_n}\}) \bigcap (B * \{\check{a_m}\}) = \varnothing. \tag{2.2}$$

Now, we give the main result of this paper which improves Theorem 1.2 proved by S. Głąb and F. Strobin. The method of the proof is similar to [8, Theprem 2] but its details and basics are different.

Theorem 2.1. *Let $\mathcal{H}$ be a non-compact unimodular hypergroup and $p > 2$. Let B be a symmetric compact neighborhood of the identity e in $\mathcal{H}$ and there is a constant $L > 0$ such that for each $x_1, \ldots, x_n \in \mathcal{H}$,*
$$\frac{\sum_{k=1}^{n} \lambda(x_k * B * B)}{\sum_{k=1}^{n} \lambda(B * \check{x}_k)} \leq L. \tag{2.3}$$
Then, there is a constant $c > 0$ such that the set
$$E_B := \{g \in L^p(\mathcal{H}) : \textit{for some } x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|g| * |g|)(x) < \infty\}$$
is σ-c-lower porous in $L^p(\mathcal{H})$.

Proof. Trivially we have $E_B = \bigcup_{m=1}^{\infty} P_m$, where
$$P_m := \{g \in L^p(\mathcal{H}) : \text{for some } x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|g| * |g|)(x) < m\}.$$

We show that the collection $\{P_m\}_m$ satisfies the equivalent condition in Theorem 1.1.

Step 1. By Lemma 2.1 one can find $a_1, a_2, \ldots$ in $\mathcal{H}$ satisfying
$$(\{a_n\} * B * B) \bigcap (\{a_m\} * B * B) = \varnothing \tag{2.4}$$

and
$$(B * \{\check{a_n}\}) \bigcap (B * \{\check{a_m}\}) = \varnothing \tag{2.5}$$
for all distinct $m, n \in \mathbb{N}$.

Step 2. Fix a number k with $0 < k < \frac{1}{1+L^{\frac{1}{p}}}$. For each $0 < x < 1$ we define
$$F(x) := 2\left(\frac{x}{k\,(1-x)}\right)^p. \tag{2.6}$$
Then, F is a continuous strictly increasing function on the interval $(0,1)$, $\lim_{x\to 0^+} F(x) = 0$ and $\lim_{x\to 1^-} F(x) = \infty$. This implies that there exists a number $0 < \gamma < 1$ such that $F(\gamma) = 1$, and so for each fixed number $0 < c < \gamma$, $0 < F(c) < 1$. Define
$$G(x) := 1 - 2\left(\frac{c}{kx}\right)^p. \tag{2.7}$$
By the continuity of G on $(0,1)$, since $G(1-c) = 1 - F(c) > 0$, there are $0 < \eta < 1-c$ and $0 < \alpha < 1$ such that
$$1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p > 0.$$

Step 3. Let $r > 0$ and $f \in L^p(\mathcal{H})$. Then, by disjointness properties (2.4) and (2.5) we have
$$\sum_{n=1}^{\infty} \|\chi_{B*\check{a_n}}\, f\|_p^p \leq \|f\|_p^p < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \|\chi_{a_n*B*B}\, g\|_q^q \leq \|g\|_q^q < \infty.$$
Hence, there is some $n_0 \in \mathbb{N}$ such that
$$\|\chi_I\, f\|_p^p = \sum_{n=n_0}^{\infty} \|\chi_{B*\check{a_n}}\, f\|_p^p < [\frac{1}{2}\,(1-c-\eta)r]^p \tag{2.8}$$
and
$$\|\chi_J\, f\|_p^p = \sum_{n=n_0}^{\infty} \|\chi_{a_n*B*B}\, f\|_p^p < [\frac{1}{2}\,(1-c-\eta)r]^p, \tag{2.9}$$
where
$$I := \bigcup_{n=n_0}^{\infty} B * \check{a_n} \quad \text{and} \quad J := \bigcup_{n=n_0}^{\infty} a_n * B * B.$$
Let $m \in \mathbb{N}$. Choose a natural number $n_1 > n_0$ such that
$$\alpha^2\eta^2\, r^2\, k^2 (n_1 - n_0 + 1)^{1-\frac{2}{p}}\, \lambda(B)^{1-\frac{2}{p}} \left(1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p\right) > m, \tag{2.10}$$
Set
$$A := \bigcup_{n=n_0}^{n_1} B * \check{a_n} \quad \text{and} \quad D := \bigcup_{n=n_0}^{n_1} a_n * B * B.$$
Then, by (2.8) and (2.9) we have
$$\|\chi_A\, f\|_p \leq \frac{1}{2}\,(1-c-\eta)r \quad \text{and} \quad \|\chi_D\, f\|_p \leq \frac{1}{2}\,(1-c-\eta)r. \tag{2.11}$$

We have

$$M\,\lambda(A)^{\frac{1}{p}} + M\,\lambda(D)^{\frac{1}{p}} \leq \eta r, \tag{2.12}$$

where

$$M := r\eta k\,\lambda(A)^{\frac{-1}{p}}. \tag{2.13}$$

Define $\tilde{f} := M\chi_{A\cup D} + f\chi_{(A\cup D)^c}$. Then, we have

$$\begin{aligned}
\|f-\tilde{f}\|_p &= \left\|\chi_{A\cup D}\,(f-\tilde{f}) + \chi_{(A\cup D)^c}\,(f-\tilde{f})\right\|_p \\
&= \left\|\chi_{A\cup D}\,(f-\tilde{f})\right\|_p \\
&= \left\|(\chi_A + \chi_{D-A})\,(f-\tilde{f})\right\|_p \\
&\leq \|\chi_A\, f\|_p + \|\chi_A\,\tilde{f}\|_p + \|\chi_{D-A}\, f\|_p + \|\chi_{D-A}\,\tilde{f}\|_p \\
&= \|\chi_A\, f\|_p + \|\chi_A\, M\|_p + \|\chi_{D-A}\, f\|_p + \|\chi_{D-A}\, M\|_p \\
&\leq \|\chi_A\, f\|_p + M\,\lambda(A)^{\frac{1}{p}} + \|\chi_D\, f\|_p + M\,\lambda(D)^{\frac{1}{p}} \\
&\leq (1-c-\eta)r + \eta r = (1-c)\,r.
\end{aligned}$$

This implies that $B(\tilde{f}, cr) \subseteq B(f; r)$.

Step 4. In this step we show that $B(\tilde{f}; cr) \bigcap P_m = \varnothing$. Let $g \in B(\tilde{f}; cr)$ and $x \in B \cap \mathrm{Ma}(\mathcal{H})$. Set

$$A_1 := \{x \in A : |g(x)| \leq \alpha M\} \quad \text{and} \quad D_1 := \{x \in D : |g(x)| \leq \alpha M\}. \tag{2.14}$$

Then,

$$(1-\alpha)M\lambda(A_1)^{\frac{1}{p}} \leq \|\chi_{A_1}\,(|\tilde{f}| - |g|)\|_p \leq \|\chi_{A_1}\,(\tilde{f} - g)\|_p \leq \|\tilde{f} - g\|_p < cr. \tag{2.15}$$

By a similar argument it follows that relation (2.15) holds for D_1 too. So, we can conclude that

$$\max\{\lambda(A_1), \lambda(D_1)\} \leq \left(\frac{cr}{(1-\alpha)M}\right)^p = \left(\frac{c}{\eta(1-\alpha)k}\right)^p \lambda(A). \tag{2.16}$$

Put $A_2 := A \setminus A_1$ and $D_2 := D \setminus D_1$. Also, we put $F := A_2 \cap (\{x\} * \check{D}_2)$.

Then, $F \subseteq A_2$ and $\check{F} * \{x\} \subseteq D_2$. This implies that

$$\begin{aligned}
(|g| * |g|)(x) &= \int_{\mathcal{H}} |g(t)|\; |g|(\check{t} * x)\, d\lambda(t) \\
&\geq \int_F |g(t)|\; |g(\check{t} * x)|\, d\lambda(t) \\
&\geq \alpha^2 M^2 \lambda(F).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lambda(F) &= \lambda(\check{A}_2 \cap (D_2 * \{\check{x}\})) \\
&= \lambda(\check{A}_2) - \lambda(\check{A}_2 \setminus (D_2 * \{\check{x}\})) \\
&\geq \lambda(A) - \lambda(A_1) - \lambda(D_1) \\
&\geq \lambda(A)\left(1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p\right).
\end{aligned}$$

Therefore, by (2.10) we have

$$\begin{aligned}(|g| * |g|)(x) &\geq \alpha^2 M^2 \lambda(A) \left(1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p\right) \\ &= \alpha^2 \eta^2 r^2 k^2 \lambda(A)^{1-\frac{2}{p}} \left(1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p\right) \\ &\geq \alpha^2 \eta^2 r^2 k^2 (n_1 - n_0 + 1)^{1-\frac{2}{p}} \lambda(B)^{1-\frac{2}{p}} \left(1 - 2\left(\frac{c}{\eta(1-\alpha)k}\right)^p\right) \\ &> m,\end{aligned}$$

since

$$\begin{aligned}\lambda(A) &= \sum_{n=n_0}^{n_1} \lambda(B * \{\check{a_n}\}) \\ &= \sum_{n=n_0}^{n_1} \lambda(\{a_n\} * B) \\ &\geq \sum_{n=n_0}^{n_1} \lambda(B) \\ &= (n_1 - n_0 + 1)\, \lambda(B).\end{aligned}$$

thanks to [11, Lemma 3.3C] and the assumption that $\mathcal{H}$ is unimodular. □

Note that Theorem 2.1 is a generalization of Theorem 1.2 because if $\mathcal{H}$ is a locally compact group, then $\mathrm{Ma}(\mathcal{H}) = \mathcal{H}$ and condition (2.3) holds with $L = \frac{\lambda(B^2)}{\lambda(B)}$.

For each function $f : \mathcal{H} \to \mathbb{C}$ we define $\check{f}(x) := f(\check{x})$ for all $x \in \mathcal{H}$. We mention that in each discrete commutative hypergroup, $B := \{e\}$ is a compact symmetric neighborhood of the identity element, and in this case $B \cap \mathrm{Ma}(\mathcal{H}) = \{e\}$. Also, condition (2.3) trivially holds (with $L = 1$) for this neighborhood in the case in which the discrete hypergroup $\mathcal{H}$ is unimodular too. So, we can conclude the following result.

Corollary 2.1. *Let $\mathcal{H}$ be an infinite discrete commutative hypergroup and $p > 2$. Then, there is a constant $c > 0$ such that the set*

$$E := \{f \in L^p(\mathcal{H}) :\, f\check{f} \in L^1(\mathcal{H})\}$$

is σ-c-lower porous.

Recall that a hypergroup $\mathcal{H}$ is called *Hermitian* if $\check{x} = x$ for all $x \in \mathcal{H}$. Clearly, any Hermitian hypergroup is commutative.

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be equipped with the discrete topology and let p be a fixed prime number. For any $k \in \mathbb{N}_0$ and distinct $m, n \in \mathbb{N}$ define $\delta_k * \delta_0 = \delta_0 * \delta_k := \delta_k$, $\delta_m * \delta_n := \delta_{\max\{m,n\}}$ and

$$\delta_n * \delta_n := \frac{1}{p^{n-1}(p-1)}\delta_0 + \sum_{k=1}^{n-1} p^{k-1}\delta_k + \frac{p-2}{p-1}\delta_n.$$

Then, $\mathbb{N}_0$ is a Hermitian hypergroup with the left invariant measure m defined by

$$m(\{k\}) := \begin{cases} 1, & \text{if } k = 0, \\ (p-1)p^{k-1}, & \text{if } k \geq 1. \end{cases}$$

This class of discrete Hermitian hypergroups was introduced by Dunkl and Ramirez in [7]. In the final result of this paper, we give some porosity property of discrete Hermitian hypergroups. Just note that thanks to [11, Theorem 7.1A], if $\mathcal{H}$ is a discrete hypergroup, the measure λ given by

$$\lambda(\{x\}) := \frac{1}{(\delta_{\check{x}} * \delta_x)(\{e\})} \qquad (x \in \mathcal{H}) \tag{2.17}$$

is a left-invariant measure on $\mathcal{H}$. So, since the convolution of any two Dirac measures is a probability measure, we have

$$\inf\{\lambda(A) : \lambda(A) > 0\} \geq 1. \tag{2.18}$$

Hence, by [22, Theorem 1], for each $p > 2$, $L^2(\mathcal{H}, \lambda) \subseteq L^p(\mathcal{H}, \lambda)$. Now, Corollary 2.1 implies the next fact.

Corollary 2.2. *Let $\mathcal{H}$ be a discrete Hermitian hypergroup. Then, the following conditions are equivalent.*

1. *$\mathcal{H}$ is infinite.*

2. *There exists some $p > 2$ and a constant $c > 0$ such that the set $L^2(\mathcal{H})$ is a σ-c-lower porous subset of $L^p(\mathcal{H})$.*

3. *For each $p > 2$ there exists a constant $c > 0$ such that the set $L^2(\mathcal{H})$ is a σ-c-lower porous subset of $L^p(\mathcal{H})$.*

3 Porosity and spaceability on hypergroups

In this section we intend to give some equivalent conditions by porosity and spaceability for a hypergroup to be compact.

Remark 2. We say that a neighborhood B has *L-property* for some constant $L > 0$, if there exists a sequence $(a_n)_n$ satisfying the conditions of Lemma 2.1 such that

$$\sup\{\lambda(\{a_n\} * B * B) : n \in \mathbb{N}\} \leq L. \tag{3.1}$$

We will use this condition in the assumptions of some results in this paper. Next, we show that any locally compact group has this condition and also we present some classes of hypergroups which are not groups, but have L-property.

Example 2. 1. Let a hypergroup $\mathcal{H}$ have a non-compact open center. Then, there exists a compact symmetric neighborhood B of e such that $B \subset \mathrm{Ma}(\mathcal{H})$. In this case, B has L-property for some $L > 0$, because by the proof of Lemma 2.1 one can choose a sequence $(a_n)_n \subseteq \mathrm{Ma}(\mathcal{H})$ satisfying condition (3.1). In particular, if $\mathcal{H}$ is a non-compact group, then we have $\mathcal{H} = \mathrm{Ma}(\mathcal{H})$ and so any compact symmetric neighborhood of e in $\mathcal{H}$ has L-property for some $L > 0$.

2. Let G is a non-compact group with a left Haar measure $\lambda = dx$ and let H be a compact non-normal subgroup of G with normalized Haar measure dh. Let $\mathcal{H} = H\backslash G/H := \{HxH : x \in G\}$ be the double coset hypergroup with convolution $\delta_{\dot{x}} * \delta_{\dot{y}} = \int_H \delta(xhy)^{\cdot} dh$ and left Haar measure $\dot{\lambda} = \int_G \delta_{\dot{x}} dx$, where $\dot{x} := HxH$ is the image of x in $H\backslash G/H$. Let B be a compact symmetric neighborhood of the identity element HeH in $\mathcal{H}$. Then, there exists a compact subset $E \subseteq G$ such that $B = \dot{E}$ and $\dot{x} * \dot{E} = (HxHEH)^{\cdot}$. Now, thanks to Lemma 2.1, this implies that if H is connected, compact and open, or if H is finite, then there is a constant $L > 0$ such that B has L-property.

In this paper, we consider the maximum norm on the product of two Banach spaces.

Theorem 3.1. *Let $\mathcal{H}$ be a non-compact hypergroup and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$. Let B be a symmetric compact neighborhood of e in $\mathcal{H}$ with L-property. For each $m \in \mathbb{N}$, put*

$$M_{B,m} := \{(f, g) \in L^p(\mathcal{H}) \times L^q(\mathcal{H}) : \exists x \in B \cap \mathrm{Ma}(\mathcal{H}), \ (|f| * |g|)(x) < m\}.$$

Then, there exists some $c \in (0, 1)$ such that for each $(f, g) \in L^p(\mathcal{H}) \times L^q(\mathcal{H})$ and $r > 0$ there exists an element $(\tilde{f}, \tilde{g}) \in L^p(\mathcal{H}) \times L^q(\mathcal{H})$ such that

$$B((\tilde{f}, \tilde{g}); cr) \subseteq B((f, g); r) \setminus M_{B,m}.$$

Proof. Put $S := \sup_{x \in B} \Delta(x)$. For each $0 < x < 1$ we define

$$F(x) := \left(\frac{x}{1-x}\right)^p + \left(\frac{x}{1-x}\right)^q \frac{SL}{\lambda(B)}.$$

Then, F is a continuous strictly increasing function on the interval $(0, 1)$, $\lim_{x\to 0^+} F(x) = 0$ and $\lim_{x\to 1^-} F(x) = \infty$. This implies that there exists a number $0 < \gamma < 1$ such that $F(\gamma) = 1$, and so for each fixed number $0 < c < \gamma$, $0 < F(c) < 1$. Define

$$G(x) := 1 - \left(\frac{c}{x}\right)^p - \left(\frac{c}{x}\right)^q \frac{SL}{\lambda(B)}.$$

Since G is continuous on $(0, 1)$, there are $0 < \eta < 1-c$ and $0 < \alpha < 1$ such that $P := G((1-\alpha)\,\eta) > 0$.

Assume that $m \in \mathbb{N}$. Let $(f, g) \in L^p(\mathcal{H}) \times L^q(\mathcal{H})$ and $r > 0$. Assume that $(a_n)_n$ is the sequence given in Remark 2 corresponding to the symmetric compact neighborhood B. Then, by disjointness properties (2.4) and (2.5) there are some $n_0 \in \mathbb{N}$ and a natural number $n_1 > n_0$ such that

$$\alpha^2\eta^2 r^2 \left(\frac{L}{\lambda(B)}\right)^{\frac{-1}{q}} S^{\frac{1}{p}-1} P \,\lambda(B)^{1-\frac{1}{p}-\frac{1}{q}} (n_1 - n_0 + 1)^{1-\frac{1}{p}-\frac{1}{q}} > m, \tag{3.2}$$

$$\sum_{n=n_0}^{n_1} \|\chi_{B*\{\check{a_n}\}}\, f\|_p^p < [(1-c-\eta)r]^p \tag{3.3}$$

and

$$\sum_{n=n_0}^{n_1} \|\chi_{\{a_n\}*B*B}\, g\|_q^q < [(1-c-\eta)r]^p. \tag{3.4}$$

Set

$$A := \bigcup_{n=n_0}^{n_1} \{a_n\} * B \quad \text{and} \quad D := \bigcup_{n=n_0}^{n_1} \{a_n\} * B * B.$$

Then, by (3.3) and (3.4) we have

$$\|\chi_{\check{A}}\, f\|_p \leq (1-c-\eta)r \quad \text{and} \quad \|\chi_D\, g\|_q \leq (1-c-\eta)r. \tag{3.5}$$

Also, by [11, Lemma 3.3C] and property (3.1),

$$\lambda(B)\,(n_1 - n_0 + 1) \leq \lambda(A) \leq L\,(n_1 - n_0 + 1) \tag{3.6}$$

and

$$\lambda(B * B)\,(n_1 - n_0 + 1) \leq \lambda(D) \leq L\,(n_1 - n_0 + 1). \tag{3.7}$$

Put

$$M_1 := \eta\, r \lambda(A)^{\frac{-1}{p}} \quad \text{and} \quad M_2 := \eta\, r \lambda(D)^{\frac{-1}{q}}. \tag{3.8}$$

We define $\tilde{f} := M_1 \Delta^{\frac{-1}{p}} \chi_{\check{A}} + f\chi_{(\check{A})^c}$ and $\tilde{g} := M_2\, \chi_D + g\chi_{D^c}$. Then,

$$\begin{aligned}
\|\tilde{f} - f\|_p &= \left\| \chi_{\check{A}}\, M_1\, \Delta^{\frac{-1}{p}} - \chi_{\check{A}}\, f \right\|_p \\
&\leq \left\| \chi_{\check{A}}\, M_1\, \Delta^{\frac{-1}{p}} \right\|_p + \|\chi_{\check{A}}\, f\|_p \\
&\leq M_1\, \|\chi_{\check{A}} \Delta^{-1}\|_1^{\frac{1}{p}} + (1 - c - \eta) r \\
&= M_1\, \lambda(A)^{\frac{1}{p}} + (1 - c - \eta) r \\
&= \eta\, r + (1 - c - \eta) r = (1 - c) r
\end{aligned}$$

thanks to (3.5), (3.8) and (1.4). Similarly, $\|\tilde{g} - g\|_q \leq (1-c)r$. Therefore, $B((\tilde{f}, \tilde{g}); cr) \subseteq B((f, g); r)$. Now, let $(h, s) \in B((\tilde{f}, \tilde{g}); c\, r)$. Setting

$$A_1 := \{x \in \check{A} :\ |h(x)| \leq \alpha\, \tilde{f}(x)\},$$

we have

$$\begin{aligned}
c\, r &> \|h - \tilde{f}\|_p \\
&\geq \|\, |h| - |\tilde{f}|\, \|_p \\
&\geq \|\chi_{A_1}\, \big(|h| - |\tilde{f}|\big)\|_p \\
&\geq \big(1 - \alpha\big)\ \|\chi_{A_1}\, \tilde{f}\|_p \\
&= \big(1 - \alpha\big)\, M_1\ \|\chi_{A_1}\, \Delta^{-1}\|_1^{\frac{1}{p}} \\
&= \big(1 - \alpha\big)\, M_1\, \lambda(\check{A}_1)^{\frac{1}{p}},
\end{aligned}$$

and so,

$$\lambda(\check{A}_1) < \left(\frac{c\, r}{(1 - \alpha)\, M_1} \right)^p = \left(\frac{c}{(1 - \alpha)\, \eta} \right)^p \lambda(A). \tag{3.9}$$

Similarly, setting

$$D_1 := \{x \in D :\ |s(x)| \leq \alpha\, \tilde{g}(x)\}.$$

we have

$$\begin{aligned}
c\, r &> \|s - \tilde{g}\|_q \\
&\geq \|\, |s| - |\tilde{g}|\, \|_q \\
&\geq \|\chi_{D_1}\, \big(|s| - |\tilde{g}|\big)\|_q \\
&\geq \big(1 - \alpha\big)\ \|\chi_{D_1}\, \tilde{g}\|_q \\
&= \big(1 - \alpha\big)\, M_2\ \|\chi_{D_1}\|_q \\
&= \big(1 - \alpha\big)\, M_2\, \lambda(D_1)^{\frac{1}{q}},
\end{aligned}$$

and therefore by inequalities (3.6) and (3.7),

$$\lambda(D_1) < \left(\frac{c\, r}{(1 - \alpha)\, M_2} \right)^q = \left(\frac{c}{(1 - \alpha)\, \eta} \right)^q \lambda(D) \leq \left(\frac{c}{(1 - \alpha)\, \eta} \right)^q \frac{L\, \lambda(A)}{\lambda(B)}. \tag{3.10}$$

Let $x \in B \cap \mathrm{Ma}(\mathcal{H})$. Put $H := \{x\} * \big[(\{\check{x}\} * A_2) \cap \check{D}_2\big]$, where $D_2 := D \setminus D_1$ and $A_2 := \check{A} \setminus A_1$. Then, since x is a center element,

$$H \subseteq \{x\} * \{\check{x}\} * A_2 = A_2 \subseteq \check{A},$$

and $\check{H} * \{x\} \subseteq D_2 \subseteq D$. In fact, for each $t \in H$, we have $\check{t}x \in D_2$, and since $\check{H} \subseteq A$, for each $t \in \check{H}$, there is some $j \in \{n_0, \ldots, n_1\}$ such that $t \in \{a_j\} * B$. This means that there exists $y \in B$ such that $t \in \{a_j\} * \{y\}$. Now, by [11, Theorem 5.3C], we have $\Delta(t) = \Delta(a_j)\Delta(y) \leq S$ because $\Delta(a_j) \leq 1$ and $y \in B$. For each $t \in \mathcal{H}$ we have $|\phi|(t * x) = |\phi(t * x)|$ for all complex-valued measurable function ϕ on $\mathcal{H}$. This implies that

$$\begin{aligned}
(|h| * |s|)(x) &= \int_{\mathcal{H}} |h(t)|\ |s|(\check{t} * x)\, d\lambda(t) \\
&\geq \int_{H} |h(t)|\ |s(\check{t}x)|\, d\lambda(t) \\
&\geq \alpha^2 \int_{H} \tilde{f}(t)\, \tilde{g}(\check{t}x)\, d\lambda(t) \\
&= \alpha^2 M_1 M_2 \int_{H} \Delta(t)^{\frac{-1}{p}}\, d\lambda(t) \\
&= \alpha^2 M_1 M_2 \int_{\check{H}} \Delta(t)^{\frac{1}{p}-1}\, d\lambda(t) \\
&\geq \alpha^2 M_1 M_2\, S^{\frac{1}{p}-1} \int_{\check{H}} d\lambda(t) \\
&= \alpha^2 M_1 M_2\, S^{\frac{1}{p}-1}\, \lambda(\check{H}),
\end{aligned}$$

thanks to [11, Theorem 5.3B]. On the other hand,

$$\begin{aligned}
\check{H} &= [(\check{A}_2 * \{x\}) \bigcap D_2] * \{\check{x}\} \\
&= \big((\check{A}_2 * \{x\} * \{\check{x}\})\big) \bigcap \big(D_2 * \{\check{x}\}\big) \\
&= \check{A}_2 \bigcap \big(D_2 * \{\check{x}\}\big) \\
&= \check{A}_2 - [\check{A}_2 - \big(D_2 * \{\check{x}\}\big)],
\end{aligned}$$

since $x \in \mathrm{Ma}(\mathcal{H})$. Also, we have

$$\check{A}_2 * \{x\} \subseteq A * \{x\} = \bigcup_{n=n_0}^{n_1} \big(\{a_n\} * B * \{x\}\big) \subseteq \bigcup_{n=n_0}^{n_1} \big(\{a_n\} * B * B\big) = D,$$

and so $\check{A}_2 \subseteq (D * \{\check{x}\})$. This implies that

$$\begin{aligned}
\lambda(\check{H}) &= \lambda(\check{A}_2) - \lambda\big(\check{A}_2 - (D_2 * \{\check{x}\})\big) \\
&\geq \lambda(\check{A}_2) - \lambda\big((D * \{\check{x}\}) - \big(D_2 * \{\check{x}\}\big)\big) \\
&= \lambda(\check{A}_2) - \lambda\big((D - D_2) * \{\check{x}\}\big) \\
&= \lambda(\check{A}_2) - \lambda\big(D_1 * \{\check{x}\}\big) \\
&= \lambda(A) - \lambda(\check{A}_1) - \lambda\big(D_1 * \{\check{x}\}\big) \\
&= \lambda(A) - \lambda(\check{A}_1) - (\lambda * \delta_x)\big(D_1\big) \\
&= \lambda(A) - \lambda(\check{A}_1) - \Delta(\check{x})\lambda\big(D_1\big).
\end{aligned}$$

Therefore, by inequalities (3.9) and (3.10)

$$\begin{aligned}\lambda(\check{H}) &\geq \lambda(A) - \left(\frac{c}{(1-\alpha)\,\eta}\right)^{p} \lambda(A) - \Delta(\check{x}) \left(\frac{c}{(1-\alpha)\,\eta}\right)^{q} \frac{L\,\lambda(A)}{\lambda(B)} \\ &\geq \left(1 - \left(\frac{c}{(1-\alpha)\,\eta}\right)^{p} - \left(\frac{c}{(1-\alpha)\,\eta}\right)^{q} \frac{SL}{\lambda(B)}\right) \lambda(A) \\ &= P\,\lambda(A).\end{aligned}$$

Therefore,

$$\begin{aligned}(|h| * |s|)(x) &\geq \alpha^2 M_1 M_2\, S^{\frac{1}{p}-1}\, P\,\lambda(A) \\ &= \alpha^2\eta^2\, r^2 \lambda(A)^{\frac{-1}{p}}\, \lambda(D)^{\frac{-1}{q}}\, S^{\frac{1}{p}-1}\, P\,\lambda(A) \\ &\geq \alpha^2\eta^2\, r^2 \left(\frac{L}{\lambda(B)}\right)^{\frac{-1}{q}} S^{\frac{1}{p}-1}\, P\,\lambda(A)^{1-\frac{1}{p}-\frac{1}{q}} \\ &\geq \alpha^2\eta^2\, r^2 \left(\frac{L}{\lambda(B)}\right)^{\frac{-1}{q}} S^{\frac{1}{p}-1}\, P\,\lambda(B)^{1-\frac{1}{p}-\frac{1}{q}}\, (n_1 - n_0 + 1)^{1-\frac{1}{p}-\frac{1}{q}} \\ &> m,\end{aligned}$$

thanks to inequality (3.2). This shows that $(h, s) \notin M_{B,m}$ and the proof is complete. □

Corollary 3.1. *Let $\mathcal{H}$ be a non-compact hypergroup and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} < 1$. Let B be a symmetric compact neighborhood of e in $\mathcal{H}$ with L-property. Then, there exists some $c \in (0, 1)$ such that the set*

$$M_B := \{(f, g) \in L^p(\mathcal{H}) \times L^q(\mathcal{H}) : \exists x \in B \cap \mathrm{Ma}(\mathcal{H}),\ (|f| * |g|)(x) < \infty\} \tag{3.11}$$

is a σ-c-lower porous.

Proof. Note that $M_B = \bigcup_{n=1}^{\infty} M_{B,n}$, and directly apply Theorem 1.1 and Theorem 3.1. □

In the sequel, we intend to give some extension of [10, Theorem 13]. In the proof of this fact, we use a recent result regarding spaceability subsets of Banach spaces from [3]. Recall that a subset S of a topological vector space $\mathcal{E}$ is called *spaceable* if $S \cup \{0\}$ contains a closed infinite-dimensional linear subspace of $\mathcal{E}$. We need the next definition given in [3] for proving our main theorem.

Definition 2. Let $\mathcal{E}$ be a topological vector space. We say that a relation $\sim$ on $\mathcal{E}$ has property (D) if the following conditions hold.

1. If (x_n) is a sequence in $\mathcal{E}$ such that $x_n \sim x_m$ for all distinct index m, n, then for each disjoint finite subsets A, B of $\mathbb{N}$ we have

$$\sum_{n\in A} \alpha_n x_n \sim \sum_{m\in B} \beta_m x_m,$$

where α_n and β_m's are arbitrary scalars.

2. If a sequence (x_n) converges to x in $\mathcal{E}$ and for some $y \in \mathcal{E}$, $x_n \sim y$ for all $n \in \mathbb{N}$, then $x \sim y$.

We say that a subset B of a vector space is a *cone* if for each scalar c, $cB \subseteq B$.

Theorem 3.2. *Let $(\mathcal{E}, \|\cdot\|)$ be a Banach space, $\sim$ be a relation on $\mathcal{E}$ with property (D), and K be a nonempty subset of $\mathcal{E}$. Assume that:*

1. *there is a constant $k > 0$ such that $\|x + y\| \geq k\|x\|$ for all $x, y \in \mathcal{E}$ with $x \sim y$;*

2. *K is a cone;*

3. *if $x, y \in \mathcal{E}$ such that $x + y \in K$ and $x \sim y$ then $x, y \in K$;*

4. *there is an infinite sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{E} \setminus K$ such that for each distinct $m, n \in \mathbb{N}$, $x_m \sim x_n$.*

Then, $\mathcal{E} \setminus K$ is spaceable in $\mathcal{E}$.

Proof. See [3, Theorem 4.2]. □

Now, the next result which is a generalization of [10, Theorem 13] can be obtained with some different proof.

Theorem 3.3. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$. If $\mathcal{H}$ is a non-compact unimodular hypergroup and B is a fixed symmetric compact neighborhood of e in $\mathcal{H}$ with L-property, then the set $\big(L^p(\mathcal{H}) \times L^q(\mathcal{H})\big) \setminus M_B$ is spaceable in $L^p(\mathcal{H}) \times L^q(\mathcal{H})$, where M_B is given by (3.11).*

Proof. Trivially M_B is a cone in the space $L^p(\mathcal{H}) \times L^q(\mathcal{H})$. We define the relation $\sim$ on $L^p(\mathcal{H}) \times L^q(\mathcal{H})$ by

$$(f_1, g_1) \sim (f_2, g_2) \text{ if and only if } \sigma(f_1) \cap \sigma(f_2) = \sigma(g_1) \cap \sigma(g_2) = \varnothing$$

up to a null set, for all $f_1, f_2 \in L^p(\mathcal{H})$ and $g_1, g_2 \in L^q(\mathcal{H})$, where $\sigma(f) := \{x \in G : f(x) \neq 0\}$. One can easily see that this relation satisfies condition (D) because convergence with respect to the L^p-norm implies almost everywhere subsequence convergence. This relation also satisfies conditions (1) (with $k = 1$) and (3) in Theorem 3.2. Indeed, if $(f_1, g_1) \sim (f_2, g_2)$, then we have $(|f_1| + |f_2|) * (|g_1| + |g_2|) = |f_1 + f_2| * |g_1 + g_2|$. In the sequel, we will show that condition (4) holds too. In this case the proof is complete. Assume that $(a_n)_n$ is the sequence in $\mathcal{H}$ obtained in Remark 2 regarding the neighborhood B. Define

$$f(x) := \sum_{n=1}^{\infty} n^{\frac{-q}{p+q}} \chi_{B * \{\check{a_n}\}} \qquad \text{and} \qquad g(x) := \sum_{n=1}^{\infty} n^{\frac{-p}{p+q}} \chi_{\{a_n\} * B * B} \tag{3.12}$$

for all $x \in \mathcal{H}$. Then, since $\mathcal{H}$ is unimodular we have

$$\begin{aligned} \int_{\mathcal{H}} |f|^p \, d\lambda &= \int_{\mathcal{H}} \sum_{n=1}^{\infty} n^{\frac{-pq}{p+q}} \chi_{B * \{\check{a_n}\}} \, d\lambda \\ &= \sum_{n=1}^{\infty} n^{\frac{-pq}{p+q}} \lambda(B * \{\check{a_n}\}) \\ &= \sum_{n=1}^{\infty} n^{\frac{-pq}{p+q}} \lambda(\{a_n\} * B) \\ &\leq L \sum_{n=1}^{\infty} n^{\frac{-pq}{p+q}} < \infty, \end{aligned}$$

because $\frac{pq}{p+q} > 1$. So $f \in L^p(\mathcal{H})$. Similarly, $g \in L^q(\mathcal{H})$. For each $N \subseteq \mathbb{N}$ we set

$$A_N := \bigcup_{n \in N} B * \{\check{a_n}\} \quad \text{and} \quad B_N := \bigcup_{n \in N} \{a_n\} * B * B.$$

Then, $f_N := \chi_{A_N} f \in L^p(\mathcal{H})$ and $g_N := \chi_{B_N} g \in L^q(\mathcal{H})$. Let $(N_k)_{k\in\mathbb{N}}$ be a partition of $\mathbb{N}$ with $\sum_{n\in N_k} \frac{1}{n} = \infty$ for all $k \in \mathbb{N}$. We denote $f_k := f_{A_{N_k}}$ and $g_k := g_{B_{N_k}}$. Then for each $k \in \mathbb{N}$ we have $(f_k, g_k) \in \big(L^p(\mathcal{H}) \times L^q(\mathcal{H})\big) \setminus M_B$ because

$$\begin{aligned}(f_k * g_k)(x) &= \int_{\mathcal{H}} f_k(y) g_k(\check{y} * x)\, d\lambda(y)\\ &= \int_{A_{N_k}} f_k(y) g_k(\check{y} * x)\, d\lambda(y)\\ &= \sum_{n\in N_k} \frac{1}{n}\, \lambda(B * \{\check{a_n}\})\\ &= \sum_{n\in N_k} \frac{1}{n}\, \lambda(\{a_n\} * B)\\ &\geq \lambda(B) \sum_{n\in N_k} \frac{1}{n} = \infty\end{aligned}$$

for all $x \in B \cap \mathrm{Ma}(\mathcal{H})$, thanks to [11, Lemma 3.3C]. Finally, it is easy to see that for each distinct numbers $k, m \in \mathbb{N}$, $(f_k, g_k) \sim (f_m, g_m)$. □

Corollary 3.2. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$. Let $\mathcal{H}$ be a unimodular hypergroup and B be a symmetric compact neighborhood of e in $\mathcal{H}$ with L-property. Then, the following conditions are equivalent.*

1. *$\mathcal{H}$ is non-compact.*
2. *$\big(L^p(\mathcal{H}) \times L^q(\mathcal{H})\big) \setminus M_B \neq \varnothing$.*
3. *M_B is a σ-c-lower porous subset of $L^p(\mathcal{H}) \times L^q(\mathcal{H})$ for some $c \in (0, 1)$*
4. *The set $\big(L^p(\mathcal{H}) \times L^q(\mathcal{H})\big) \setminus M_B$ is spaceable in $L^p(\mathcal{H}) \times L^q(\mathcal{H})$.*

Acknowledgments

The authors would like to thank the referee of this paper for reviewing this article and helpful comments.

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Received: 10.12.2022
Revised: 05.01.2024