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KHARIN STANISLAV NIKOLAYEVICH

(to the 80th birthday)

Stanislav Nikolayevich Kharin was born on December 4, 1938 in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis "Heat phenomena in electrical contacts and associated singular integral equations", and in 1990 his doctoral thesis "Mathematical models of thermo-physical processes in electrical contacts" in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. Using models based on the new original methods for solving free boundary problems he described mathematically the phenomena of arcing, contact welding, contact floating, dynamics of contact blow-open phenomena, electrochemical mechanism of electron emission, arc-to-glow transition, thermal theory of the bridge erosion. For these achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research and the evidence of this in the new monograph "Mathematical models of phenomena in electrical contacts" published last year in Novosibirsk.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Stanislav Nikolayevich on the occasion of his 80th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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HARDY TYPE INEQUALITY WITH SHARP CONSTANT FOR $0 < p < 1$

A. Senouci, N. Azzouz

Communicated by T.V. Tararykova

Key words: Hardy operator, Hardy-type inequality, sharp constant.

AMS Mathematics Subject Classification: 35J20, 35J25.

Abstract. A power-weighted integral inequality with sharp constant for $0 < p < 1$ was established by V.I. Burenkov for the Hardy operator $\big(Hf\big)(x) = \dfrac{1}{x}\int_0^x f(t)\ dt$ for non-negative non-increasing functions f. In this work we consider a more general class of functions f and prove a new Hardy-type inequality with sharp constant for functions of this class.

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1 Introduction

The integral inequality

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^p x^\alpha\ dx \le C_1\ \int_0^\infty f^p(x)\ x^\alpha\ dx, \tag{1.1}$$

where $C_1 = \big(\frac{p}{p-1-\alpha}\big)^p$, known as Hardy's inequality, is satisfied for all functions f non-negative and measurable on $(0,\infty)$ with $p \ge 1$ and $\alpha < p-1$. The constant C_1 is sharp i.e. the smallest possible (for more details see [10]).

If $0 < p < 1$, inequality (1) with any $C_1 > 0$ is not satisfied for arbitrary non-negative measurable functions (see [11]).

In [4, 5] for $0 < p < 1$, inequality of type (1) with a sharp constant was proved for non-negative non-increasing functions. Namely the following statement was proved there.

Theorem 1.1. *Let $0 < p < 1$ and $-1 < \alpha < p-1$, then for all function $f \ge 0$ non-increasing on $(0,\infty)$,*

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^p x^\alpha\ dx \le C_2\ \int_0^\infty f^p(x)\ x^\alpha\ dx, \tag{1.2}$$

where $C_2 = \frac{p}{p-1-\alpha}$. Moreover, the constant C_2 is sharp.

Remark 1. 1) The assumption $\alpha > -1$ is natural because if $\alpha \le -1$ then for each function $f \ge 0$ which is non-increasing on $(0,\infty)$ and is not identically equal to zero we have $\int_0^\infty f^p(x)\ x^\alpha\ dx = \infty$.

2) If $\alpha \ge p-1$ then there exists a function f non-negative and non-increasing on $(0,\infty)$ such that $\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^p x^\alpha\ dx = \infty$ and $\int_0^\infty f^p(x)\ x^\alpha\ dx < \infty$ (for example, $f(x) = \frac{1}{x}\chi_{(0,1)}(x)$).

Later inequalities of type (1) were proved in [2, 3] for non-negative quasi-decreasing functions, also with sharp constants.

We also note the recent survey paper [9] dedicated to weighted integral inequalities on the cone of monotone functions and books [12, 11] which contain a lot of informations about weighted inequalities of Hardy type.

2 Main results

The aim of this paper is to change the monotonicity condition by a more general one and obtain an inequality of type (1.1) with a sharp constant which coincides with the constant C_2 in Theorem 1.1 for the class of all functions non-negative and f is non-increasing on $(0, +\infty)$. Moreover an example of non-monotonic functions satisfying this new condition is given.

In the following theorem we establish and prove an inequality of type (2) with sharp constant under assumptions weaker than monotonicity.

Theorem 2.1. *Let $k > 0$, $0 < p < 1$ and $-1 < \alpha < p-1$. If a function $f \geq 0$ measurable on $(0, \infty)$ satisfies for almost all $x > 0$ the inequality*

$$f(x) \leq \frac{k}{x}\Big(\int_0^x f^p(y)\, y^{p-1}\, dy\Big)^{\frac{1}{p}}, \tag{2.1}$$

then

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\, dt\Big)^p x^\alpha\, dx \leq C_3 \int_0^\infty f^p(x)\, x^\alpha\, dx, \tag{2.2}$$

where $C_3 = \frac{p^p\, k^{p(1-p)}}{p-\alpha-1}$. Moreover the constant C_3 is sharp.

For the proof of Theorem 2.1, we need the following lemmas.

Lemma 2.1. *Let $0 < p < 1$ and $\alpha < p-1$. If f is a non-negative measurable function on $(0, \infty)$ and satisfies condition (2.1) then*

$$\Big(\int_0^x f(t)\, dt\Big)^p \leq p^p\, k^{p(1-p)} \int_0^x f^p(y)\, y^{p-1}\, dy\ . \tag{2.3}$$

Proof Let $x > 0$ and $f \geq 0$ be a measurable function on $(0, \infty)$ such that condition (2.1) is satisified. Note that by (2.1)

$$\begin{aligned} f(t) &= \big(f(t)\, t\big)^{1-p}\, \big(f^p(t)\, t^{p-1}\big) \\ &\leq k^{1-p}\Big(\int_0^t f^p(y)\, y^{p-1}\, dy\Big)^{\frac{1-p}{p}}\, \big(f^p(t)\, t^{p-1}\big) \end{aligned}$$

and integrating over $(0, x)$ we get

$$\Big(\int_0^x f(t)\, dt\Big) \leq k^{1-p} \int_0^x \Big(\int_0^t f^p(y)\, y^{p-1}\, dy\Big)^{\frac{1}{p}-1} f^p(t)\, t^{p-1}\ dt.$$

Let

$$\phi(t) = \int_0^t f^p(y)\, y^{p-1}\, dy, \tag{2.4}$$

then

$$\begin{aligned} \Big(\int_0^x f(t)\, dt\Big) &\leq k^{1-p} \int_0^x \big(\phi(t)\big)^{\frac{1}{p}-1} \phi'(t)\, dt \\ &= p\, k^{1-p}\Big(\int_0^x f^p(y)\, y^{p-1}\, dy\Big)^{\frac{1}{p}} \end{aligned}$$

and inequality (2.3) follows.

□

Lemma 2.2. *Let $0 < p < 1$ and g be a continuous function defined on $(0, \infty)$. Then the equality*

$$g(x) = \frac{k}{x}\Big(\int_0^x g^p(y)\, y^{p-1}\, dy\Big)^{\frac{1}{p}}. \tag{2.5}$$

holds for all $x > 0$ if and only if for some $M > 0$, $k > 0$ and for all $x > 0$

$$g(x) = M\, x^{(\frac{k^p}{p}-1)}. \tag{2.6}$$

Proof Let be g a continuous function on $(0, \infty)$ such that equality (2.5) holds for all $x > 0$. Then

$$g^p(x)\; x^p = k^p \int_0^x g^p(y)\, y^{p-1}\, dy. \tag{2.7}$$

By differentiating (2.7) we obtain

$$p\, g'(x)\; x + p\, g(x) = k^p\; g(x).$$

This equality implies that g is continuously differentiable on $(0, \infty)$. Consequently

$$\frac{g'(x)}{g(x)} = \left(\frac{k^p}{p} - 1\right)\; \frac{1}{x}. \tag{2.8}$$

Integrating (2.8), we get

$$g(x) = M\; x^{\frac{k^p}{p}-1}.$$

□

Proof of Theorem 2.1 Let $0 < p < 1$ and $-1 < \alpha < p - 1$. Let $f \geq 0$ be a measurable function on $(0, \infty)$ such that condition (2.1) is satisfied. By Lemma 2.1 we get

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\, dt\Big)^p x^\alpha\; dx \leq p^p\; k^{p(1-p)} \int_0^\infty \Big(\int_0^x f^p(y)\, y^{p-1}\; dy\Big)\; x^{\alpha-p}\; dx\;.$$

Fubini's theorem yields

$$\begin{aligned}\int_0^\infty \Big(\frac{1}{x}\int_0^r f(t)\, dt\Big)^p x^\alpha\; dx &\leq p^p\; k^{p(1-p)} \int_0^\infty \Big(\int_y^\infty x^{\alpha-p}\, dx\Big)\, f^p(y)\, y^{p-1}\; dy \\ &= p^p\; k^{p(1-p)} \int_0^\infty \frac{1}{-\alpha+p-1}\; y^{\alpha-p+1}\, f^p(y)\, y^{p-1}\; dy\end{aligned}$$

and inequality (2.2) follows with $C_3 = \frac{p^p\, k^{p(1-p)}}{p-\alpha-1}$.

Now we prove that the constant C_3 is sharp. Let $0 < p < 1$, $\alpha < p - 1$ and $k > 0$. We consider three cases.

1. If $k^p > p - \alpha - 1$, for $y > 0$ we define $g(y) = y^{(\frac{k^p}{p}-1)}\chi_{(0,1)}(y)$. Then

$$I = \int_0^\infty g^p(y) y^\alpha dy = \int_0^1 y^{k^p-p+\alpha} dy = \frac{1}{k^p - p + \alpha + 1},$$

$$I_1 = \int_0^1 \left(\frac{1}{x}\int_0^x g(t)dt\right)^p x^\alpha dx = \left(\frac{p}{k^p}\right)^p \frac{1}{(k^p - p + \alpha + 1)}$$

and

$$I_2 = \int_1^\infty \left(\frac{1}{x}\int_0^1 g(t)dt\right)^p x^\alpha dx = \left(\frac{p}{k^p}\right)^p \frac{1}{(p-\alpha-1)},$$

hence

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)dt\right)^p x^\alpha dx = I_1 + I_2$$

$$= \left(\frac{p}{k^p}\right)^p \left(\frac{1}{k^p - p + \alpha + 1} + \frac{1}{p-\alpha-1}\right) = \left(\frac{p}{k^p}\right)^p \left(\frac{k^p}{(k^p-p+\alpha+1)(p-\alpha-1)}\right).$$

Since $C_3 \geq \dfrac{I_1+I_2}{I} = \left(\dfrac{p}{k^p}\right)^p \dfrac{k^p}{(p-\alpha-1)} = \dfrac{p^p k^{p(1-p)}}{p-\alpha-1}$, we conclude that the constant $C_3 = \dfrac{p^k k^{p(1-p)}}{p-\alpha-1}$ is sharp.

2. If $k^p < p-\alpha-1$, for $y > 0$ we define $h(y) = y^{\frac{k^p}{p}-1}\chi_{(1,\infty)}(y)$. Then

$$J = \int_0^\infty h(y)y^\alpha dy = \int_1^\infty y^{k^p-p+\alpha}dy = \frac{1}{p-1-k^p-\alpha},$$

$$J_1 = \int_0^\infty \left(\frac{1}{x}\int_0^x h(y)dy\right) x^\alpha dx = \int_1^\infty \left(\frac{1}{x}\int_1^x y^{\frac{k^p}{p}-1}dy\right)^p x^\alpha dx$$

$$= \left(\frac{p}{k^p}\right)^p \int_1^\infty x^{\alpha-p}\left(x^{\frac{k^p}{p}}-1\right)^p dx.$$

By using the inequality $(a-b)^p \geq a^p - b^p$, $a \geq b$, $0 < p < 1$, we obtain

$$J_1 \geq \left(\frac{p}{k^p}\right)^p \int_1^\infty x^{\alpha-p}(x^{k^p}-1)dx = \left(\frac{p}{k^p}\right)^p \left(\int_1^\infty x^{\alpha-p+k^p}dx - \int_1^\infty x^{\alpha-p}dx\right)$$

$$= \frac{p^p k^{p(1-p)}}{(p-1-k^p-\alpha)(p-\alpha-1)},$$

hence $C_3 \geq \dfrac{p^p k^{p(1-p)}}{p-\alpha-1}$ and the constant C_3 is sharp.

3. If $k^p = p-\alpha-1$, then for $R > 1$ and for $y > 0$ we define $\varphi_R(y) = y^{\frac{k^p}{p}-1}\chi_{(1,R)}(y)$
$L = \int_1^R y^{k^p-p+\alpha}dy = \int_1^R \dfrac{dy}{y} = lnR$,

$$L_1 = \int_0^R \left(\frac{1}{x}\int_0^x \varphi_R(y)dy\right) x^\alpha dx = \int_1^R \left(\frac{1}{x}\int_1^x y^{\frac{k^p}{p}-1}dy\right)^p x^\alpha dx$$

$$= \left(\frac{p}{k^p}\right)^p \int_1^R \left(x^{\frac{k^p}{p}}-1\right)^p x^{\alpha-p}dx \geq \left(\frac{p}{k^p}\right)^p \int_1^R \left(x^{k^p}-1\right) x^{\alpha-p}dx$$

$$= \left(\frac{p}{k^p}\right)^p \int_1^R \left(x^{k^p+\alpha-1} - x^{\alpha-p}\right) dx = \left(\frac{p}{k^p}\right)^p \left(\int_1^R \frac{dx}{x} - \int_1^R x^{\alpha-p}dx\right)$$

$$= \left(\frac{p}{k^p}\right)^p \left[LnR - \frac{R^{\alpha-p+1}}{\alpha-p+1} - \frac{1}{p-\alpha-1}\right],$$

$$L_2 = \int_R^\infty \left(\frac{1}{x}\int_0^x \varphi_R(y)dy\right) x^\alpha dx = \int_R^\infty \left(\frac{1}{x}\int_1^R y^{\frac{k^p}{p}-1}dy\right)^p x^\alpha dx$$

$$= \left(\frac{p}{k^p}\right)^p \left(R^{\frac{k^p}{p}}-1\right)^p \int_R^\infty x^{\alpha-p}dx$$

$$\geq \left(\frac{p}{k^p}\right)^p \int_R^{\infty} \left(R^{k^p}-1\right) x^{\alpha-p} dx = \left(\frac{p}{k^p}\right)^p \left(R^{k^p}-1\right) \frac{R^{\alpha-p+1}}{p-\alpha-1}$$

$$= \left(\frac{p}{k^p}\right)^p \frac{1}{p-\alpha-1} \left(1-R^{\alpha-p+1}\right),$$

hence $L_1 + L_2 \geq \left(\frac{p}{k^p}\right)^p Ln\ R$, consequently

$$C_3 \geq \frac{L_1+L_2}{L} \geq \left(\frac{p}{k^p}\right)^p = \left(\frac{p}{k^p}\right)^p \frac{k^p}{(p-\alpha-1)} = \frac{p^p k^{p(1-p)}}{p-\alpha-1},$$

hence the constant C_3 is sharp.

□

Remark 2. In the cone of non-increasing functions on $(0,\infty)$ $k = p^{\frac{1}{p}}$ and Theorem 2.1 reduces to Theorem 1.1 with $C_3 = \frac{p}{p-\alpha-1} = C_2$.

3 Example of non-monotonic functions satisfying condition (2.1)

In [7] was stutied embeddings and the growth envelope of Besov spaces involving only slowly varying smoothness. Here we show that slowly varying functions satisfy (2.1), therefore we need the next notations and definition (for details see [7], pp. 1375-1376).

Notation 1. For two non-negative functions f and g, defined on $(0,\infty)$, we write $f \lesssim g$ if there is a constant $k > 0$ such that $f(t) \leq kg(t)$ for all $t \in (0,\infty)$. Analogously, we define $f \gtrsim g$. If $f \lesssim g$ and $f \gtrsim g$, we say that f and g are equivalent and write $f \sim g$.

Definition 1. A non-negative measurable function f on $(0,\infty)$ is said to be slowly varying if for each $\varepsilon > 0$, there are non-negative measurable functions g_ε and $g_{-\varepsilon}$ such that

g_ε is non-decreasing and $g_\varepsilon(t) \sim t^\varepsilon f(t)$.

$g_{-\varepsilon}$ is non-increasing and $g_{-\varepsilon}(t) \sim t^{-\varepsilon} f(t)$.

Lemma 3.1. *Let be $p > 0$. If f is a slowly varying function then*

1. *f^p is also a slowly varying function.*

2. *For all $\varepsilon > 0$*

$$\int_0^x t^{\varepsilon-1} f(t)\, dt \ \gtrsim\ x^\varepsilon f(x)\,,\, \forall\, x > 0.$$

Proof The first statement is clear.

For the second one, consider $x > 0$ and h a slowly varying function. For any $\varepsilon > 0$ there exists measurable functions $g_\varepsilon \geq 0$ non-decreasing and $g_{-\varepsilon} \geq 0$ then

$$\int_0^x t^{\varepsilon-1} f(t)\, dt \gtrsim \int_0^x t^\varepsilon\ g_{-1}(t)\, dt \geq g_{-1}(x) \int_0^x t^\varepsilon\, dt \gtrsim x^{-1}\ f(x) \int_0^x t^\varepsilon\, dt \gtrsim x^\varepsilon\ f(x)\ ,$$

hence

$$\int_0^x t^{\varepsilon-1} f(t)\, dt\ \gtrsim\ x^\varepsilon\ f(x).$$

□

Lemma 3.2. *Slowly varying functions are examples of non-monotonic functions satisfying condition (2.1).*

Proof Let f be a slowly varying function. We apply Lemma 3.1 for f replaced by f^p and $\varepsilon = p$, then there exists a constant $k > 0$ such that $\forall\, x > 0$

$$\int_0^x t^{p-1}\, f^p(t)\, dt \geq \; k\, x^p\, f^p(x)\ ;$$

so $\forall\, x > 0$

$$f(x) \;\leq \frac{k}{x} \Big(\int_0^x f^p(y)\, y^{p-1}\, dy \Big)^{\frac{1}{p}}\ .$$

□

Corollary 3.1. *If a positive function f is slowly varying on $(0,\infty)$ then there exists a constant $C_4 > 0$ such taht*

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\, dt\Big)^p x^\alpha\ dx \leq C_4\ \int_0^\infty f^p(x)\, x^\alpha\ dx.$$

Remark 3. The class of slowly varying functions on $(0,\infty)$ contains power functions, $\prod_{k=1}^n \Big(\ln_k t\Big)^{\alpha_k}$ where $\ln_k t = \ln\big(\ln_{k-1} t\big)$ and $\alpha_k \in \mathbb{R}$, $2+\sin(\ln_2 t)$, $\exp\big(|\ln t|^\alpha\big)$ with $\alpha \in (0,1)$, $\frac{1}{t}\int_a^t \frac{1}{\ln s} ds$ with $a > 0$ etc. (for details see [13]).

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