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KHARIN STANISLAV NIKOLAYEVICH

(to the 80th birthday)

Stanislav Nikolayevich Kharin was born on December 4, 1938 in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis "Heat phenomena in electrical contacts and associated singular integral equations", and in 1990 his doctoral thesis "Mathematical models of thermo-physical processes in electrical contacts" in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. Using models based on the new original methods for solving free boundary problems he described mathematically the phenomena of arcing, contact welding, contact floating, dynamics of contact blow-open phenomena, electrochemical mechanism of electron emission, arc-to-glow transition, thermal theory of the bridge erosion. For these achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research and the evidence of this in the new monograph "Mathematical models of phenomena in electrical contacts" published last year in Novosibirsk.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Stanislav Nikolayevich on the occasion of his 80th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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ON THE INCOMPLETE GAMMA FUNCTION AND ITS NEUTRIX CONVOLUTION FOR NEGATIVE INTEGERS

M. Lin, B. Fisher, S. Orankitjaroen

Communicated by V.I. Burenkov

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AMS Mathematics Subject Classification: 33B10; 46F10.

Abstract. We define the distributions $\gamma^+(-r,x)$ and $\gamma^-(-r,x)$ from the incomplete Gamma function $\gamma(-r,x)$ for negative integers. We then evaluate some convolutions and neutrix convolutions of these distributions and the functions $(x^s)_+, (x^s)_-$ and x^s.

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1 Introduction

The classical *Gamma function*, see for example, Sneddon [10], $\Gamma(\lambda)$ is usually defined by

$$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-t}\,dt, \qquad (\lambda > 0).$$

It is easily seen that

$$\Gamma(\lambda+1) = \lambda\Gamma(\lambda), \qquad (\lambda > 0).$$

This equality is then used to define $\Gamma(\lambda)$ for $\lambda < 0$ and $\lambda \neq -1, -2, \ldots$.

It follows by induction that if $-r < \lambda < -r+1, r = 1, 2, \ldots$, then

$$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}\Big[e^{-t} - \sum_{i=0}^{r-1}\frac{(-t)^i}{i!}\Big]\,dt.$$

It was then proved in [5] that

$$\Gamma^{(s)}(\lambda) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{\lambda-1}e^{-t}\ln^s t\,dt \tag{1.1}$$

for $s = 0, 1, 2, \ldots$ and $\lambda \neq 0, -1, -2, \ldots$, where N is the neutrix, see van der Corput [11], having domain $N' = \{\epsilon :\ 0 < \epsilon < \infty\}$ with negligible functions finite linear sums of the functions

$$\epsilon^\lambda \ln^{s-1}\epsilon, \quad \ln^s\epsilon : \quad \lambda < 0, \quad s = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as ϵ tends to zero.

It was also proved in [5] that the neutrix limit in equality (1.1) also existed for $\lambda = 0, -1, -2, \ldots$, and this suggested that $\Gamma^{(s)}(-r)$ could be defined by

$$\Gamma^{(s)}(-r) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{-r-1}e^{-t}\ln^s t\,dt$$

for $r, s = 0, 1, 2, \ldots$.

It was proved in [4] that

$$\Gamma(-r) = \frac{(-1)^r}{r!}[\psi(r) - \gamma]$$

for $r = 0, 1, 2, \ldots$, where γ denotes the *Euler's constant* and

$$\psi(r) = \begin{cases} \sum_{i=1}^{r} \frac{1}{i}, & r > 0, \\ 0, & r = 0. \end{cases}$$

The *upper incomplete Gamma function* $\Gamma(\lambda, x)$ is defined by

$$\Gamma(\lambda, x) = \int_x^\infty t^{\lambda-1} e^{-t}\, dt, \qquad (\lambda > 0)$$

and more generally, the function $\Gamma_s(\lambda, x)$ is defined by

$$\Gamma_s(\lambda, x) = \int_x^\infty t^{\lambda-1} e^{-t} \ln^s t\, dt, \qquad (\lambda > 0)$$

for $s = 0, 1, 2, \ldots$.

The *lower incomplete Gamma function* or *incomplete Gamma function* $\gamma(\lambda, x)$ is defined by

$$\gamma(\lambda, x) = \int_0^x t^{\lambda-1} e^{-t}\, dt, \qquad (\lambda > 0, x \geq 0)$$

and more generally, the function $\gamma_s(\lambda, x)$ is defined by

$$\begin{aligned} \gamma_s(\lambda, x) &= \int_0^x t^{\lambda-1} e^{-t} \ln^s t\, dt, \qquad (\lambda > 0, x \geq 0), \\ \gamma_s(\lambda, x) &= \mathrm{N}\!-\!\lim_{\epsilon \to 0} \int_\epsilon^x t^{\lambda-1} e^{-t} \ln^s t\, dt, \qquad (\lambda \leq 0, x \geq 0) \end{aligned} \tag{1.2}$$

for $s = 0, 1, 2, \ldots$.

The existence of the neutrix limit in equality (1.2) follows from the existence of the neutrix limit in equality (1.1).

Note that if $-r < \lambda < -r + 1$, $r = 1, 2, \ldots$ and $x > 0$, we have

$$\gamma(\lambda, x) = \int_0^x t^{\lambda-1} \left[e^{-t} - \sum_{k=0}^{r-1} \frac{(-t)^k}{k!} \right] dt + \sum_{k=0}^{r-1} \frac{(-1)^k x^{\lambda+k}}{(\lambda+k)k!}.$$

It follows that

$$\lim_{x \to \infty} \gamma(\lambda, x) = \Gamma(\lambda)$$

for $\lambda \neq 0, -1, -2, \ldots$.

Alternatively, the incomplete Gamma function is defined by

$$\gamma(\lambda, x) = \int_0^x |t|^{\lambda-1} e^{-t}\, dt, \qquad (\lambda > 0, x < 0)$$

and more generally, the function $\gamma_s(\lambda, x)$ is defined by

$$\gamma_s(\lambda, x) = \int_0^x |t|^{\lambda-1} e^{-t} \ln^s |t| \, dt, \qquad (\lambda > 0, x < 0) \tag{1.3}$$

$$\gamma_s(\lambda, x) = \mathrm{N-}\lim_{\varepsilon\to 0} \int_\varepsilon^x |t|^{\lambda-1} e^{-t} \ln^s |t| \, dt, \qquad (\lambda \le 0, x < 0) \tag{1.4}$$

for $s = 0, 1, 2, \ldots$.

The corresponding locally summable functions $\gamma^+(\lambda, x)$ and $\gamma^-(\lambda, x)$ were defined in [6] by

$$\gamma^+(\lambda, x) = \begin{cases} \int_0^x t^{\lambda-1} e^{-t} dt, & x \ge 0, \\ 0, & x < 0, \end{cases}$$

$$\gamma^-(\lambda, x) = \begin{cases} \int_0^x |t|^{\lambda-1} e^{-t} dt, & x \le 0, \\ 0, & x > 0, \end{cases}$$

if $\lambda > 0$ and the distributions $\gamma^+(\lambda, x)$ and $\gamma^-(\lambda, x)$ were defined inductively by equalities

$$\gamma^+(\lambda, x) = \lambda^{-1}\gamma^+(\lambda+1, x) + \lambda^{-1}(x^\lambda)_+ e^{-x},$$
$$\gamma^-(\lambda, x) = -\lambda^{-1}\gamma^-(\lambda+1, x) - \lambda^{-1}(x^\lambda)_- e^{-x}$$

for $\lambda < 0$ and $\lambda \ne -1, -2, \ldots$. It follows that

$$\lim_{x\to-\infty} \gamma^-(\lambda, x) = \infty.$$

Note that the notations $\gamma(\lambda, x_+)$, $\gamma(\lambda, x_-), x_-^\lambda$ and x_+^λ in [6] are changed to $\gamma^+(\lambda, x)$, $\gamma^-(\lambda, x), (x^\lambda)_-$ and $(x^\lambda)_+$, respectively.

It was proved in [9] that, if $x \ge 0$, we have

$$\gamma(0, x) = e^{-x} \ln x + \gamma_1(1, x), \tag{1.5}$$

$$\gamma(-r, x) = -\frac{1}{r}\gamma(-r+1, x) - \frac{1}{r}x^{-r}e^{-x} + \frac{(-1)^r}{rr!}, \tag{1.6}$$

$$= \sum_{i=1}^{r} \left[\frac{(-1)^r}{ir!} - \frac{(-1)^{r-i}(i-1)!x^{-i}e^{-x}}{r!}\right] + \frac{(-1)^r}{r!}\gamma(0, x), \tag{1.7}$$

$r = 1, 2, \ldots$.

For the case $x < 0$, one can prove from equality (1.4) that

$$\gamma(0, x) = -e^{-x} \ln |x| - \gamma_1(1, x), \tag{1.8}$$

$$\gamma(-r, x) = \frac{1}{r}\gamma(-r+1, x) + \frac{1}{r}|x|^{-r}e^{-x} - \frac{1}{rr!}, \tag{1.9}$$

$$= \sum_{i=1}^{r} \frac{(i-1)!}{r!}|x|^{-i}e^{-x} - \frac{\psi(r)}{r!} + \frac{1}{r!}\gamma(0, x) \tag{1.10}$$

$r = 1, 2, \ldots$.

We now define distributions $\gamma^+(0, x)$, $\gamma^-(0, x)$, $\gamma^+(-r, x)$ and $\gamma^-(-r, x)$, $r = 1, 2, \ldots$, as follow:

$$\gamma^+(0, x) = e^{-x} \ln_+ x + \gamma_1^+(1, x), \tag{1.11}$$

$$\gamma^-(0, x) = -e^{-x} \ln_- x - \gamma_1^-(1, x), \tag{1.12}$$

$$\gamma^+(-r, x) = -\frac{1}{r}\gamma^+(-r+1, x) - \frac{1}{r}(x^{-r})_+ e^{-x} + \frac{(-1)^r}{rr!}H(x), \tag{1.13}$$

$$\gamma^-(-r, x) = \frac{1}{r}\gamma^-(-r+1, x) + \frac{1}{r}(x^{-r})_- e^{-x} - \frac{1}{rr!}H(-x), \tag{1.14}$$

where $H(x)$ denotes the *Heaviside function* and the functions $\gamma_s^+(\lambda, x)$ and $\gamma_s^-(\lambda, x)$ are defined by

$$\gamma_s^+(\lambda, x) = \begin{cases} \int_0^x t^{\lambda-1} e^{-t} \ln^s t dt, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{1.15}$$

$$\gamma_s^-(\lambda, x) = \begin{cases} \int_0^x |t|^{\lambda-1} e^{-t} \ln^s |t| dt, & x \leq 0, \\ 0, & x > 0, \end{cases} \tag{1.16}$$

for $\lambda > 0$ and $s = 0, 1, 2, \ldots$. It follows that

$$\begin{aligned} \lim_{x \to +\infty} \gamma^+(\lambda, x) &= \Gamma^{(s)}(\lambda), \\ \lim_{x \to -\infty} \gamma^-(\lambda, x) &= -\infty, \end{aligned}$$

for $\lambda > 0$ and $s = 0, 1, 2, \ldots$.

Note that

$$\Gamma^{(s)}(\lambda) = \gamma_s(\lambda, x) + \Gamma_s(\lambda, x)$$

for all λ and $s = 0, 1, 2, \ldots$. This suggests that the locally summable function $\Gamma_s^+(\lambda, x)$ could be defined by the relation

$$\Gamma^{(s)}(\lambda) = \gamma_s^+(\lambda, x) + \Gamma_s^+(\lambda, x) \tag{1.17}$$

for all λ and $s = 0, 1, 2, \ldots$.

The locally summable functions $\ln_+ x, \ln_- x, (x^r)_+, (x^r)_-, (x^{-r})_+$ and $(x^{-r})_-, r = 1, 2, \ldots$ are defined as follow:

- $\ln_+ x = \begin{cases} \ln x, & x > 0, \\ 0, & x < 0, \end{cases}$
- $\ln_- x = \begin{cases} \ln |x|, & x < 0, \\ 0, & x > 0, \end{cases}$
- $(x^r)_+ = \begin{cases} x^r, & x > 0, \\ 0, & x < 0, \end{cases}$
- $(x^r)_- = \begin{cases} |x|^r, & x < 0, \\ 0, & x > 0, \end{cases}$
- $(x^{-r})_+ = \frac{(-1)^{r-1}}{(r-1)!} (\ln_+ x)^{(r)},$
- $(x^{-r})_- = -\frac{1}{(r-1)!} (\ln_- x)^{(r)}.$

The distribution x^r is defined by

$$x^r = (x^r)_+ + (-1)^r (x^r)_-$$

for $r = 0, \pm 1, \pm 2, \ldots$.

Note that the distributions $(x^{-r})_+$ and $(x^{-r})_-$ are not the same as in Gel'fand and Shilov's definition and we denote those distributions by x_+^{-r} and x_-^{-r}, for $r = 1, 2, \ldots$, respectively. That is, for any $\varphi \in \mathcal{D}$, we have

$$\begin{aligned} \langle x_+^{-r}, \varphi(x) \rangle = \int_0^\infty x^{-r} \Bigg[&\varphi(x) - \sum_{i=0}^{r-2} \frac{x^i}{i!} \varphi^{(i)}(0) \\ &- \frac{x^{r-1}}{(r-1)!} \varphi^{(r-1)}(0) H(1-x) \Bigg] dx, \end{aligned} \tag{1.18}$$

$$\langle x_-^{-r}, \varphi(x)\rangle = \int_0^\infty x^{-r}\Bigg[\varphi(-x) - \sum_{i=0}^{r-2}\frac{(-x)^i}{i!}\varphi^{(i)}(0) - \frac{(-x)^{r-1}}{(r-1)!}\varphi^{(r-1)}(0)H(1-x)\Bigg]\,dx. \tag{1.19}$$

Note also that

$$\langle x_+^{-r}, \varphi(x)\rangle = \mathrm{N-}\lim_{\varepsilon\to 0}\int_\varepsilon^\infty x^{-r}\varphi(x)dx, \tag{1.20}$$

$$\langle x_-^{-r}, \varphi(x)\rangle = \mathrm{N-}\lim_{\varepsilon\to 0}\int_{-\infty}^\varepsilon |x|^{-r}\varphi(x)dx \tag{1.21}$$

for $r = 1, 2, \ldots$, where N is the neutrix defined in Section 1.

It was proved in [7] that

$$(x^{-r})_+ = x_+^{-r} + \frac{(-1)^r\psi(r-1)}{(r-1)!}\delta^{(r-1)}(x), \tag{1.22}$$

$$(x^{-r})_- = x_-^{-r} - \frac{(-1)^r\psi(r-1)}{(r-1)!}\delta^{(r-1)}(x) \tag{1.23}$$

for $r = 1, 2, \ldots$, where δ denotes the *Dirac delta function.*

It follows that

$$e^{-x}(x^{-r})_+ = e^{-x}x_+^{-r} + \frac{(-1)^r\psi(r-1)}{(r-1)!}e^{-x}\delta^{(r-1)}(x), \tag{1.24}$$

$$e^{-x}(x^{-r})_- = e^{-x}x_-^{-r} - \frac{(-1)^r\psi(r-1)}{(r-1)!}e^{-x}\delta^{(r-1)}(x) \tag{1.25}$$

for $r = 1, 2, \ldots$.

2 Convolution of distributions

If f and g are locally summable functions then the classical definition for the convolution $f * g$ of f and g is as follows:

Definition 1. Let f and g be functions. Then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^\infty f(t)g(x-t)dt$$

for all points x for which the integral exists.

It follows easily from the definition that if the classical convolution $f * g$ of f and g exists, then $g * f$ exists and

$$f * g = g * f. \tag{2.1}$$

Further, if $(f * g)'$ and $f * g'$ (or $f' * g$) exist, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{2.2}$$

The classical definition of the convolution can be extended to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ with the following definition, see [1].

Definition 2. Let f and g be distributions in $\mathcal{D}'$. Then the convolution $f * g$ is defined by the equality

$$\langle (f*g)(x), \varphi(x)\rangle = \langle f(y), \langle g(x), \varphi(x+y)\rangle\rangle$$

for arbitrary φ in $\mathcal{D}'$, provided that f and g satisfy either of the following conditions:

(a) either f or g has bounded support

(b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by this definition, then equalitys (2.1) and (2.2) are satisfied.

We first of all prove the following results that are needed to prove the next convolution products.

Lemma 2.1.

$$H(x) * (x^s)_+ = \frac{1}{s+1}(x^{s+1})_+, \tag{2.3}$$

$$H(-x) * (x^s)_- = \frac{1}{s+1}(x^{s+1})_-, \tag{2.4}$$

for $s = 0, 1, 2, \dots$.

Proof. It is obvious that $H(x) * (x^s)_+ = 0$, if $x < 0$. When $x > 0$, we have

$$\begin{aligned} H(x) * (x^s)_+ &= \int_{-\infty}^{\infty} H(t)(x-t)_+^s dt \\ &= \int_0^x (x-t)^s dt \\ &= \frac{1}{s+1} x^{s+1} \end{aligned}$$

proving equality (2.3).

Equality (2.4) follows from equality (2.3) by replacing x by $-x$. □

Lemma 2.2.

$$[e^{-x}\delta^{(r-1)}(x)] * x^s = \sum_{i=0}^{s} \binom{r-1}{i} \frac{s!}{(s-i)!} x^{s-i}, \tag{2.5}$$

for $s = 0, 1, \dots, r-1$ *and* $r = 1, 2, \dots$ *and*

$$[e^{-x}\delta^{(r-1)}(x)] * x^s = \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{s!}{(s-i)!} x^{s-i}, \tag{2.6}$$

for $s = r, r+1, \dots$ *and* $r = 1, 2, \dots$.

Proof. We have

$$e^{-x}\delta^{(r-1)}(x) = \sum_{i=0}^{r-1} \binom{r-1}{i} \delta^{(i)}(x), \tag{2.7}$$

for $r = 1, 2, \dots$. It follows that

$$\begin{aligned}
[e^{-x}\delta^{(r-1)}(x)] * x^s &= \sum_{i=0}^{r-1} \binom{r-1}{i} \delta^{(i)}(x) * x^s \\
&= \sum_{i=0}^{r-1} \binom{r-1}{i} (x^s)^{(i)}
\end{aligned} \tag{2.8}$$

where

$$(x^s)^{(i)} = \begin{cases} \dfrac{s!}{(s-i)!} x^{s-i}, & s \geq i, \\ 0, & s < i, \end{cases} \tag{2.9}$$

for $i, s = 0, 1, 2, \dots$.

Equalities (2.5) and (2.6) then follow easily from equalities (2.8) and (2.9). □

Lemma 2.3.

$$[e^{-x}\delta^{(r-1)}(x)] * (x^s)_+ = \sum_{i=0}^{s} \binom{r-1}{i} \frac{s!}{(s-i)!} (x^{s-i})_+ + s! \sum_{i=s+1}^{r-1} \binom{r-1}{i} \delta^{(i-s-1)}(x), \tag{2.10}$$

for $s = 0, 1, \dots, r-2$ *and* $r = 1, 2, \dots$ *and*

$$[e^{-x}\delta^{(r-1)}(x)] * (x^s)_+ = \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{s!}{(s-i)!} (x^{s-i})_+, \tag{2.11}$$

for $s = r-1, r, r+1, \dots$ *and* $r = 1, 2, \dots$.

$$[e^{-x}\delta^{(r-1)}(x)] * (x^s)_- = \sum_{i=0}^{s} \binom{r-1}{i} \frac{(-1)^i s!}{(s-i)!} (x^{s-i})_- - s! \sum_{i=s+1}^{r-1} \binom{r-1}{i} \delta^{(i-s-1)}(x), \tag{2.12}$$

for $s = 0, 1, \dots, r-2$ *and* $r = 1, 2, \dots$ *and*

$$[e^{-x}\delta^{(r-1)}(x)] * (x^s)_- = \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{(-1)^i s!}{(s-i)!} (x^{s-i})_-, \tag{2.13}$$

for $s = r-1, r, r+1, \dots$ *and* $r = 1, 2, \dots$.

Proof. The proof of (2.10) and (2.11) is similar to the proof of Lemma 2.2 on noting that

$$((x^s)_+)^{(i)} = \begin{cases} \dfrac{s!}{(s-i)!} (x^{s-i})_+, & s \geq i, \\ s!\delta^{(i-s-1)}(x), & s < i, \end{cases}$$

for $i, s = 0, 1, 2, \dots$.

The proof of (2.12) and (2.13) follow from (2.10), (2.11) and Lemma 2.2 on noting that

$$[e^{-x}\delta^{(r-1)}(x) * x^s = [e^{-x}\delta^{(r-1)}(x)] * (x^s)_+] + (-1)^s [e^{-x}\delta^{(r-1)}(x)] * (x^s)_-]$$

for $s = 0, 1, 2, \dots$. □

Theorem 2.1.

$$(e^{-x}\ln_+ x) * (x^s)_+ = \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\gamma_1^+(i+1,x), \tag{2.14}$$

$$(e^{-x}\ln_- x) * (x^s)_- = (-1)^{s-1}\sum_{i=0}^{s}\binom{s}{i}\gamma_1^-(i+1,x)x^{s-i}, \tag{2.15}$$

for $s = 0, 1, 2, \ldots$.

Proof. It is obvious that $(e^{-x}\ln_+ x) * (x^s)_+ = 0$ if $x < 0$. When $x > 0$, we have

$$\begin{aligned}
(e^{-x}\ln_+ x) * (x^s)_+ &= \int_{-\infty}^{\infty} e^{-t}\ln t_+ (x-t)_+^s\,dt \\
&= \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\int_0^x t^i e^{-t}\ln t\,dt \\
&= \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\gamma_1(i+1,x),
\end{aligned}$$

proving equality (2.14).

It is obvious that $(e^{-x}\ln_- x) * (x^s)_- = 0$ if $x > 0$. When $x < 0$, we have

$$\begin{aligned}
(e^{-x}\ln_- x) * (x^s)_- &= \int_x^0 |x-t|^s e^{-t}\ln|t|\,dt \\
&= (-1)^s\sum_{i=0}^{s}\binom{s}{i}x^{s-i}\int_x^0 |t|^i e^{-t}\ln|t|\,dt \\
&= (-1)^{s-1}\sum_{i=0}^{s}\binom{s}{i}\gamma_1^-(i+1,x)x^{s-i}
\end{aligned}$$

proving equality (2.15). □

Theorem 2.2.

$$\gamma_1^+(1,x) * (x^s)_+ = \frac{1}{s+1}x^{s+1}\gamma_1^+(1,x) + \sum_{i=0}^{s}\binom{s}{i}\frac{(-1)^{i+1}}{i+1}x^{s-i}\gamma_1^+(i+2,x), \tag{2.16}$$

$$\gamma_1^-(1,x) * (x^s)_- = \frac{(-1)^{s+1}}{s+1}x^{s+1}\gamma_1^-(1,x) + \sum_{i=0}^{s}\binom{s}{i}\frac{(-1)^{s+1}}{i+1}x^{s-i}\gamma_1^-(i+2,x), \tag{2.17}$$

for $s = 0, 1, 2, \ldots$.

Proof. It is obvious that $\gamma_1^+(1,x)*(x^s)_+ = 0$, if $x<0$. When $x>0$, we have

$$\begin{aligned}
\gamma_1^+(1,x)*(x^s)_+ &= \langle \gamma_1(1,t_+),(x-t)_+^s\rangle \\
&= \int_0^x \gamma_1^+(1,t)(x-t)^s dt \\
&= \int_0^x\int_0^t e^{-u}\ln u(x-t)^s dudt \\
&= \sum_{i=0}^s \binom{s}{i}(-1)^i x^{s-i}\int_0^x\int_0^t e^{-u}\ln u t^i dudt \\
&= \sum_{i=0}^s \binom{s}{i}(-1)^i x^{s-i}\int_0^x e^{-u}\ln u\int_u^x t^i dtdu \\
&= \sum_{i=0}^s\binom{s}{i}\frac{(-1)^i}{i+1}x^{s+1}\gamma_1(1,x)+\sum_{i=0}^s\binom{s}{i}\frac{(-1)^{i+1}}{i+1}x^{s-i}\gamma_1(i+2,x) \\
&= \frac{1}{s+1}x^{s+1}\gamma_1(1,x)+\sum_{i=0}^s\binom{s}{i}\frac{(-1)^{i+1}}{i+1}x^{s-i}\gamma_1(i+2,x)
\end{aligned}$$

proving equality (2.16).

The proof of equality (2.17) is similar to the proof of equality (2.16). □

Theorem 2.3.

$$\gamma^+(0,x)*(x^s)_+ = \frac{1}{s+1}x^{s+1}\gamma^+(0,x)+\sum_{i=0}^s\binom{s}{i}\frac{(-1)^{i+1}}{i+1}x^{s-i}\gamma^+(i+1,x), \tag{2.18}$$

$$\gamma^-(0,x)*(x^s)_- = \frac{(-1)^{s+1}}{s+1}x^{s+1}\gamma^-(0,x)+\sum_{i=0}^s\binom{s}{i}\frac{(-1)^{s+1}}{i+1}x^{s-i}\gamma^-(i+1,x), \tag{2.19}$$

for $s=0,1,2,\ldots$.

Proof. We have from equality (1.11) that

$$\gamma^+(0,x)*(x^s)_+ = [(e^{-x}\ln_+ x)*(x^s)_+]+[\gamma_1^+(1,x)*(x^s)_+], \tag{2.20}$$

for $s=0,1,2,\ldots$. It follows from equalities (2.14), (2.16) and (2.20) that

$$\begin{aligned}
\gamma^+(0,x)*(x^s)_+ &= \sum_{i=0}^s\binom{s}{i}(-1)^i x^{s-i}\gamma_1^+(i+1,x) \\
&\quad+\sum_{i=0}^s\binom{s}{i}\frac{(-1)^{i+1}}{i+1}x^{s-i}\gamma_1^+(i+2,x)+\frac{1}{s+1}x^{s+1}\gamma_1^+(1,x) \\
&= \sum_{i=0}^s\binom{s}{i}(-1)^i x^{s-i}\left[\gamma_1^+(i+1,x)-\frac{1}{i+1}\gamma_1^+(i+2,x)\right] \\
&\quad+\frac{1}{s+1}x^{s+1}\gamma_1^+(1,x) \\
&= \sum_{i=0}^s\binom{s}{i}\frac{(-1)^{i+1}}{i+1}x^{s-i}\gamma^+(i+1,x)+\frac{1}{s+1}x^{s+1}\gamma^+(0,x),
\end{aligned}$$

on noting that

$$\gamma_1^+(i+1,x)-\frac{1}{i+1}\gamma_1^+(i+2,x) = \frac{1}{i+1}x^{i+1}e^{-x}\ln_+ x-\frac{1}{i+1}\gamma^+(i+1,x),$$

for $i = 0, 1, 2, \ldots$. This proves equality (2.18).

Equality (2.19) follows from equalities (1.12), (2.15) and (2.17) by noting that

$$\gamma_1^-(i+1,x) + \frac{1}{i+1}\gamma_1^-(i+2,x) = \frac{(-1)^i}{i+1}x^{i+1}e^{-x}\ln|x| - \frac{1}{i+1}\gamma^-(i+1,x),$$

for $i = 0, 1, 2, \ldots$. □

Theorem 2.4.

$$[e^{-x}x_+^{-r}] * (x^s)_+ = \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\gamma^+(-r+i+1,x), \tag{2.21}$$

$$[e^{-x}x_-^{-r}] * (x^s)_- = (-1)^{s-1}\sum_{i=0}^{s}\binom{s}{i}x^{s-i}\gamma^-(-r+i+1,x) \tag{2.22}$$

for $s = 0, 1, 2, \ldots$ *and* $r = 1, 2, \ldots$.

Proof. It is obvious that $[e^{-x}x_+^{-r}] * (x^s)_+ = 0$, if $x < 0$. When $x > 0$, we have

$$\begin{aligned}[e^{-x}x_+^{-r}] * (x^s)_+ &= \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_\varepsilon^x e^{-t}t^{-r}(x-t)^s dt \\ &= \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\,\mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_\varepsilon^x e^{-t}t^{-r+i}dt \\ &= \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\gamma(-r+i+1,x),\end{aligned}$$

proving equality (2.21).

It is obvious that $[e^{-x}x_-^{-r}] * (x^s)_+ = 0$, if $x > 0$. When $x < 0$, we have

$$\begin{aligned}[e^{-x}x_-^{-r}] * (x^s)_- &= \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_x^\varepsilon e^{-t}|t|^{-r}|x-t|^s dt \\ &= (-1)^s\sum_{i=0}^{s}\binom{s}{i}x^{s-i}\,\mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_x^\varepsilon e^{-t}|t|^{-r+i}dt \\ &= (-1)^{s-1}\sum_{i=0}^{s}\binom{s}{i}x^{s-i}\gamma^-(-r+i+1,x)\end{aligned}$$

proving equality (2.22). □

Theorem 2.5.

$$\begin{aligned}[e^{-x}x_+^{-r}] * (x^s)_+ &= \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\gamma^+(-r+i+1,x) + \sum_{i=0}^{s}\binom{s}{i}\frac{(-1)^r\psi(r-1)}{(r-1-i)!}(x^{s-i})_+ \\ &\quad + \frac{(-1)^r\psi(r-1)s!}{(r-1)!}\sum_{i=s+1}^{r-1}\binom{r-1}{i}\delta^{(i-s-1)}(x),\end{aligned} \tag{2.23}$$

for $s = 0, 1, \ldots, r-2$ *and* $r = 1, 2, \ldots$ *and*

$$\begin{aligned}[e^{-x}x_+^{-r}] * (x^s)_+ &= \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\gamma^+(-r+i+1,x) \\ &\quad + \sum_{i=0}^{r-1}\binom{s}{i}\frac{(-1)^r\psi(r-1)}{(r-1-i)!}(x^{s-i})_+,\end{aligned} \tag{2.24}$$

for $s = r-1, r, r+1, \ldots$ *and* $r = 1, 2, \ldots$.

$$\begin{aligned}[e^{-x}x_-^{-r}] * (x^s)_- &= (-1)^{s-1}\sum_{i=0}^{s}\binom{s}{i}x^{s-i}\gamma^-(-r+i+1,x) \\ &\quad -\sum_{i=0}^{s}\binom{s}{i}\frac{(-1)^{r+i}\psi(r-1)}{(r-1-i)!}(x^{s-i})_- \\ &\quad -\frac{(-1)^r\psi(r-1)s!}{(r-1)!}\sum_{i=s+1}^{r-1}\binom{r-1}{i}\delta^{(i-s-1)}(x), \end{aligned} \tag{2.25}$$

for $s = 0, 1, \ldots, r-2$ *and* $r = 1, 2, \ldots$ *and*

$$\begin{aligned}[e^{-x}x_-^{-r}] * (x^s)_- &= (-1)^{s-1}\sum_{i=0}^{s}\binom{s}{i}x^{s-i}\gamma^-(-r+i+1,x) \\ &\quad -\sum_{i=0}^{r-1}\binom{s}{i}\frac{(-1)^{r+i}\psi(r-1)}{(r-1-i)!}(x^{s-i})_-, \end{aligned} \tag{2.26}$$

for $s = r-1, r, r+1, \ldots$ *and* $r = 1, 2, \ldots$.

Proof. The proof of equalities (2.23) and (2.24) are quite straitforward on using equalities (1.24), (2.10) and (2.21).

The proof of equalities (2.25) and (2.26) are quite straightforward on using equalities (1.25), (2.12) and (2.22). □

Theorem 2.6. *The convolution* $\gamma^+(-r,x) * (x^s)_+$ *and* $\gamma^-(-1,x) * (x^s)_-$ *exist for* $s = 0, 1, 2, \ldots$ *and* $r = 1, 2, \ldots$. *In particular,*

$$\begin{aligned}\gamma^+(-1,x) * (x^s)_+ &= \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\left[\frac{1}{i+1}\gamma^+(i+1,x) - \gamma^+(i,x)\right] \\ &\quad -\frac{1}{s+1}x^{s+1}\gamma^+(0,x) - \frac{1}{s+1}x_+^{s+1}, \end{aligned} \tag{2.27}$$

$$\begin{aligned}\gamma^-(-1,x) * (x^s)_- &= (-1)^{s+1}\sum_{i=0}^{s}\binom{s}{i}x^{s-i}\left[\frac{1}{i+1}\gamma^-(i+1,x) + \gamma^-(i,x)\right] \\ &\quad +\frac{1}{s+1}x_-^{s+1}\gamma^-(0,x) - \frac{1}{s+1}x_-^{s+1}, \end{aligned} \tag{2.28}$$

for $s = 0, 1, 2, \ldots$.

Proof. We have proved that the convolution $\gamma^+(0,x) * (x^s)_+$ exists. Now assuming that the convolution $\gamma^+(-r,x) * (x^s)_+$ exists for some r. We have from equality (1.13) that

$$\begin{aligned}\gamma^+(-r-1,x) * (x^s)_+ &= -\frac{1}{r+1}\left[\gamma^+(-r,x) * (x^s)_+\right] - \frac{1}{r+1}[e^{-x}x_+^{-r-1} * (x^s)_+] \\ &\quad -\frac{(-1)^r}{(r+1)(r+1)!}H(x) * (x^s)_+. \end{aligned}$$

Since each term of the right-hand side exists, therefore the convolution $\gamma^+(-r-1,x) * (x^s)_+$ exists.

In particular, we have

$$\gamma^+(-1,x) * (x^s)_+ = \quad -\left[\gamma^+(0,x) * (x^s)_+\right] - [e^{-x}x_+^{-1} * (x^s)_+] - [H(x) * (x^s)_+], \tag{2.29}$$

for $s = 0, 1, 2, \ldots$.

Equality (2.27) then follows from equalities (2.3), (2.18), (2.24) and (2.29).

We have proved that the convolution $\gamma^{-}(0, x) * (x^s)_-$ exists. Now assuming that the convolution $\gamma^{(} - r, x) * (x^s)_-$ exists for some r. We have from equality (1.14) that

$$\begin{aligned}\gamma^{-}(-r-1, x) * (x^s)_- &= \frac{1}{r+1}\left[\gamma^{-}(-r, x) * (x^s)_-\right] + \frac{1}{r+1}[e^{-x}x_-^{-r-1} * (x^s)_-] \\ &\quad -\frac{1}{(r+1)(r+1)!}H(-x) * (x^s)_-.\end{aligned}$$

Since each term of the right-hand side exists, therefore the convolution $\gamma^{-}(-r-1, x) * (x^s)_-$ exists.

In particular, we have

$$\gamma^{-}(-1, x) * (x^s)_- = \left[\gamma^{-}(0, x) * (x^s)_-\right] + [e^{-x}x_-^{-1} * (x^s)_-] - [H(-x) * (x^s)_-], \tag{2.30}$$

for $s = 0, 1, 2, \ldots$.

Equality (2.28) then follows from equalities (2.4), (2.19), (2.26) and (2.30). □

3 Neutrix convolution of distributions

The definition of the convolution is rather restrictive and so the non-commutative neutrix convolution was introduced in [2]. In order to define the neutrix convolution product we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \le \tau(x) \le 1$,

(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,

(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function τ_n is then defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n \end{cases}$$

for $n = 1, 2, \ldots$.

The following definitions were given by van der Corput [11].

Definition 3. A neutrix N is defined as a commutative additive group of functions $\nu(\xi)$ defined on a domain N' with values in an additive group N'', where further, if for some $\nu \in N$, $\nu(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in N are called negligible functions.

Definition 4. Let N' be a set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function in N' with values in N'' and it is possible to find a constant c such that $f(\xi) - c \in N$, then c is called the neutrix limit of f as ξ tends to b and we write $\mathrm{N}-\lim_{\xi\to b} f(\xi) = c$.

Note that f tends to c in the normal sense as ξ tends to b, then it converges to c in the neutrix sense.

The following definition was given in [2].

Definition 5. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the *neutrix convolution* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit h exists in the sense

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [11], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'', the real numbers, with negligible functions being finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0, r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.
In particular, if

$$\lim_{n\to\infty}\langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, we say that the *convolution* $f * g$ exists and equals h.

Note that in this definition the convolution $f_n * g$ is as defined in Gel'fand and Shilov's sense, the distribution f_n having compact support. Note also that because of the lack of symmetry in the definition of $f \circledast g$, the neutrix convolution is in general non-commutative.

The following theorem was proved in [2], showing that the neutrix convolution is a generalization of the convolution.

Theorem 3.1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

For our next results, we need to extends of our set of negligible functions to include finite linear sums of

$$n^{r-1}e^n, \quad e^n \ln n, \quad n^{r-1}\gamma_1^-(i, -n), \quad \gamma^-(i, -n): \quad r, i = 1, 2, \ldots.$$

The following results were proved in [8] and [6], respectively:

$$\begin{aligned}
\gamma^+(\lambda, x) \circledast x^s &= \frac{1}{s+1}\sum_{i=1}^{s+1}\binom{s+1}{i}(-1)^i\Gamma(\lambda+i)x^{s-i+1}, \\
\gamma^-(\lambda, x) \circledast x^s &= 0,
\end{aligned}$$

for $s = 0, 1, 2, \ldots$ and $\lambda \neq 0, -1, -2, \ldots$.

We now prove

Theorem 3.2. *The neutrix convolution $H(x) \circledast x^s$ exists and*

$$H(x) \circledast x^s = 0, \tag{3.1}$$
$$H(-x) \circledast x^s = 0, \tag{3.2}$$

for $s = 0, 1, 2, \ldots$.

Proof. We have

$$[H(x)\tau_n(x)] * x^s = \int_0^n (x-t)^s dt + \int_n^{n+n^{-n}} \tau_n(t)(x-t)^s dt$$

for $s = 0, 1, 2, \ldots$. It is easily seen that

$$\operatorname*{N-lim}_{n\to\infty} [H(x)\tau_n(x)] * x^s = 0,$$

proving equality (3.1).

The proof of equality (3.2) is similar to the proof of equality (3.1). □

Theorem 3.3. *The neutrix convolution* $(e^{-x}\ln_+ x) \circledast x^s$ *and* $(e^{-x}\ln_- x) \circledast x^s$ *exist and*

$$\begin{aligned}(e^{-x}\ln_+ x) \circledast x^s &= \sum_{i=0}^{s} \binom{s}{i} (-1)^i \Gamma'(i+1) x^{s-i}, &(3.3)\\ (e^{-x}\ln_- x) \circledast x^s &= 0, &(3.4)\end{aligned}$$

for $s = 0, 1, 2, \ldots$.

Proof. Put $(e^{-x}\ln_+ x)_n = (e^{-x}\ln_+ x)\tau_n(x)$. Then

$$\begin{aligned}(e^{-x}\ln_+ x)_n * x^s &= \int_0^n e^{-t}\ln t (x-t)^s dt + \int_n^{n+n^{-n}} e^{-t}\ln t \tau_n(t)(x-t)^s dt\\ &= J_1 + J_2. &(3.5)\end{aligned}$$

Now

$$\begin{aligned}J_1 &= \sum_{i=0}^{s} \binom{s}{i} (-1)^i x^{s-i} \int_0^n e^{-t} t^i \ln t dt\\ &= \sum_{i=0}^{s} \binom{s}{i} (-1)^i x^{s-i} \gamma_1(i+1, n)\end{aligned}$$

and so

$$\operatorname*{N-lim}_{n\to\infty} J_1 = \sum_{i=0}^{s} \binom{s}{i} (-1)^i x^{s-i} \Gamma'(i+1). \tag{3.6}$$

Further, it is easily seen that

$$\operatorname*{N-lim}_{n\to\infty} J_2 = 0. \tag{3.7}$$

It follows from equalities (3.5), (3.6) and (3.7) that

$$\operatorname*{N-lim}_{n\to\infty} [(e^{-x}\ln_+ x)_n * x^s] = \sum_{i=0}^{s} \binom{s}{i} (-1)^i x^{s-i} \Gamma'(i+1)$$

proving equality (3.3).

Put $(e^{-x}\ln_- x)_n = (e^{-x}\ln_- x)\tau_n(x)$. Then

$$\begin{aligned}(e^{-x}\ln_- x)_n * x^s &= \int_{-n}^{0} e^{-t}\ln|t|(x-t)^s dt + \int_{-n-n^{-n}}^{-n} e^{-t}\ln|t|\tau_n(t)(x-t)^s dt\\ &= I_1 + I_2. &(3.8)\end{aligned}$$

Now

$$
\begin{aligned}
I_1 = \int_{-n}^{0} e^{-t}\ln|t|(x-t)^s dt &= \sum_{i=0}^{s}\binom{s}{i}x^{s-i}\int_{-n}^{0} e^{-t}|t|^i \ln|t|dt \\
&= -\sum_{i=0}^{s}\binom{s}{i}x^{s-i}\gamma_1(i+1,-n).
\end{aligned}
$$

It follows that

$$
\mathrm{N}\!-\!\lim_{n\to\infty} I_1 = 0. \tag{3.9}
$$

Further, it is easily seen that

$$
\mathrm{N}\!-\!\lim_{n\to\infty} I_2 = 0,
$$

Combining equalities (3.8), (3.9) and (3.10), we get

$$
\mathrm{N}\!-\!\lim_{n\to\infty}[(e^{-x}\ln_- x)_n * x^s] = 0, \tag{3.10}
$$

for $s = 0, 1, 2, \ldots$. This completes the proof of the theorem. □

Theorem 3.4. *The neutrix convolution* $\gamma_1^+(1,x) \circledast x^s$ *and* $\gamma_1^-(1,x) \circledast x^s$ *exist and*

$$
\gamma_1^+(1,x) \circledast x^s = \sum_{i=0}^{s}\binom{s}{i}\frac{(-1)^{i+1}}{i+1}\Gamma'(i+2)x^{s-i}, \tag{3.11}
$$

$$
\gamma_1^-(1,x) \circledast x^s = 0, \tag{3.12}
$$

for $s = 0, 1, 2, \ldots$.

Proof. Put $\gamma_1^+(1,x)_n = \gamma_1^+(1,x)\tau_n(x)$. Then

$$
\begin{aligned}
\gamma_1^+(1,x)_n * x^s &= \int_0^n \gamma_1^+(1,t)(x-t)^s dt + \int_n^{n+n^{-n}} \gamma_1^+(1,t)\tau_n(t)(x-t)^s dt \\
&= J_3 + J_4.
\end{aligned} \tag{3.13}
$$

We have

$$
\begin{aligned}
J_3 &= \int_0^n \gamma_1^+(1,t)(x-t)^s dt \\
&= \int_0^n \int_0^t e^{-u}\ln u(x-t)^s du dt \\
&= \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\int_0^n\int_0^t e^{-u}\ln u t^i du dt \\
&= \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\int_0^n e^{-u}\ln u\int_u^n t^i dt du \\
&= \sum_{i=0}^{s}\binom{s}{i}\frac{(-1)^i}{i+1}x^{s-i}\left[n^{i+1}\gamma_1(1,n) - \gamma_1(i+2,n)\right].
\end{aligned}
$$

It follows that

$$\underset{n\to\infty}{\text{N}-\lim}\ J_3 = \sum_{i=0}^{s}\binom{s}{i}\frac{(-1)^{i+1}}{i+1}\Gamma'(i+2)x^{s-i} \tag{3.14}$$

Further, it is easily seen that

$$\underset{n\to\infty}{\text{N}-\lim}\ J_4 = 0. \tag{3.15}$$

It now follows from equalities (3.13), (3.14) and (3.15) that

$$\underset{n\to\infty}{\text{N}-\lim}[\gamma_1^+(1,x)_n * x^s] = \sum_{i=0}^{s}\binom{s}{i}\frac{(-1)^{i+1}}{i+1}\Gamma'(i+2)x^{s-i}$$

for $s = 0, 1, 2, \ldots$.

This completes the proof of (3.11).

Put $\gamma_1^-(1,x)_n = \gamma_1^-(1,x)\tau_n(x)$. Then

$$\begin{aligned}\gamma_1^-(1,x)_n * x^s &= \int_{-n}^{0}\gamma_1^-(1,t)(x-t)^s dt + \int_{-n-n^{-n}}^{-n}\gamma_1^-(1,t)\tau_n(t)(x-t)^s dt\\ &= I_3 + I_4.\end{aligned} \tag{3.16}$$

We have

$$\begin{aligned} I_3 &= \int_{-n}^{0}\gamma_1^-(1,t)(x-t)^s dt\\ &= \int_{-n}^{0}\int_{0}^{t} e^{-u}\ln|u|(x-t)^s du dt\\ &= \sum_{i=0}^{s}\binom{s}{i}x^{s-i}\int_{-n}^{0}\int_{0}^{t} e^{-u}\ln|u||t|^i du dt\\ &= -\sum_{i=0}^{s}\binom{s}{i}x^{s-i}\int_{0}^{-n} e^{-u}\ln|u|\int_{u}^{-n}|t|^i dt du\\ &= \sum_{i=0}^{s}\binom{s}{i}\frac{1}{i+1}x^{s-i}n^{i+1}\gamma_1(1,-n) - \sum_{i=0}^{s}\binom{s}{i}\frac{1}{i+1}x^{s-i}\gamma_1(i+2,-n).\end{aligned}$$

It follows that

$$\underset{n\to\infty}{\text{N}-\lim}\ I_3 = 0. \tag{3.17}$$

Further, it is easily seen that

$$\underset{n\to\infty}{\text{N}-\lim}\ I_4 = 0. \tag{3.18}$$

Combining equalities (3.16), (3.17) and (3.18), we get

$$\underset{n\to\infty}{\text{N}-\lim}[\gamma_1^-(1,x)_n * x^s] = 0,$$

for $s = 0, 1, 2, \ldots$.

This completes the proof of the theorem. □

Theorem 3.5. *The neutrix convolutions* $\gamma^+(0,x) \circledast x^s$ *and* $\gamma^-(0,x) \circledast x^s$ *exist and*

$$\gamma^+(0,x) \circledast x^s = \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^{i+1}}{i+1} \Gamma(i+1) x^{s-i}, \tag{3.19}$$
$$\gamma^-(0,x) \circledast x^s = 0, \tag{3.20}$$

for $s = 0, 1, 2, \ldots$.

Proof. Equality (3.19) follows from equalities (3.3) and (3.11) by noting that

$$\Gamma(i+1) = \Gamma'(i+2) - (i+1)\Gamma'(i+1),$$

for $i = 0, 1, 2, \ldots$.

Equality (3.20) follows from equalities (1.12), (3.4) and (3.12). □

Corollary 3.1. *The neutrix convolutions* $\gamma^+(0,x) \circledast (x^s)_-$ *and* $\gamma^-(0,x) \circledast (x^s)_+$ *exist and*

$$\gamma^+(0,x) \circledast (x^s)_- = \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^{s+i+1}}{i+1} \gamma^+(i+1,x) x^{s-i} + \frac{(-1)^{s+1}}{s+1} x^{s+1} \gamma^+(0,x), \tag{3.21}$$
$$\gamma^-(0,x) \circledast (x^s)_+ = \frac{1}{s+1} x^{s+1} \gamma^-(0,x) + \sum_{i=0}^{s} \binom{s}{i} \frac{1}{i+1} x^{s-i} \gamma^-(i+1,x), \tag{3.22}$$

for $s = 0, 1, 2, \ldots$.

Proof. We have

$$\gamma^+(0,x) \circledast x^s = [\gamma^+(0,x) \circledast (x^s)_+] + (-1)^s [\gamma^+(0,x) \circledast (x^s)_-], \tag{3.23}$$
$$\gamma^-(0,x) \circledast x^s = [\gamma^-(0,x) \circledast (x^s)_+] + (-1)^s [\gamma^-(0,x) \circledast (x^s)_-], \tag{3.24}$$

for $s = 0, 1, 2, \ldots$.

Equality (3.21) then follows from equalities (1.17), (2.18), (3.19) and (3.23).

Equality (3.22) then follows from equalities (2.19), (3.20) and (3.24). □

Theorem 3.6. *The neutrix convolution* $[e^{-x} x_+^{-r}] \circledast x^s$ *exists and*

$$[e^{-x} x_+^{-r}] \circledast x^s = \sum_{i=0}^{s} \binom{s}{i} (-1)^i x^{s-i} \Gamma(-r+i+1), \tag{3.25}$$

for $s = 0, 1, 2, \ldots$ *and* $r = 1, 2, \ldots$.

Proof. We have

$$\begin{aligned}[e^{-x} x_+^{-r} \tau_n(x)] * x^s &= \operatorname*{N-lim}_{\varepsilon \to 0} \int_{\varepsilon}^{n} e^{-t} t^{-r} (x-t)^s \, dt + \int_{n}^{n+n^{-n}} e^{-t} t^{-r} \tau_n(t) (x-t)^s \, dt \\ &= J_5 + J_6, \end{aligned} \tag{3.26}$$

where

$$J_5 = \sum_{i=0}^{s} \binom{s}{i} (-1)^i x^{s-i} \gamma(-r+i+1, n).$$

It follows that

$$\underset{n\to\infty}{\text{N}-\lim} J_5 = \sum_{i=0}^{s} \binom{s}{i} (-1)^i x^{s-i} \Gamma(-r+i+1), \tag{3.27}$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$.

Note that from equalities (1.5) and (1.7), we have

$$\begin{aligned} \gamma(-r, n) &= \sum_{i=1}^{r} \left[\frac{(-1)^r}{ir!} - \frac{(-1)^{r-i}(i-1)! n^{-i} e^{-n}}{r!} \right] \\ &\quad + \frac{(-1)^r}{r!} e^{-n} \ln n + \frac{(-1)^r}{r!} \gamma_1(1, n), \end{aligned}$$

for $r = 0, 1, 2, \ldots$, where the sum is being empty when $r = 0$. Applying the neutrix limit, we get

$$\underset{n\to\infty}{\text{N}-\lim} \gamma(-r, n) = \frac{(-1)^r}{r!} [\psi(r) - \gamma] = \Gamma(-r),$$

for $r = 0, 1, 2, \ldots$.

Further, it is easily seen that

$$\underset{n\to\infty}{\text{N}-\lim} J_6 = 0. \tag{3.28}$$

It now follows from equalities (3.26), (3.27) and (3.28) that

$$\underset{n\to\infty}{\text{N}-\lim} [e^{-x} x_+^{-r} \tau_n(x)] * x^s = \sum_{i=0}^{s} \binom{s}{i} (-1)^i x^{s-i} \Gamma(-r+i+1),$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$.

This completes the proof of the theorem. □

Theorem 3.7. *The neutrix convolution* $[e^{-x} x_-^{-r}] \circledast x^s$ *exists and*

$$[e^{-x} x_-^{-r}] \circledast x^s = \sum_{i=0}^{s} \binom{s}{i} \frac{\psi(r-i-1)}{(r-i-1)!} x^{s-i}, \tag{3.29}$$

for $s = 0, 1, 2, \ldots, r-2$ *and* $r = 1, 2, \ldots$ *and*

$$[e^{-x} x_-^{-r}] \circledast x^s = \sum_{i=0}^{r-2} \binom{s}{i} \frac{\psi(r-i-1)}{(r-i-1)!} x^{s-i}, \tag{3.30}$$

for $s = r-1, r, r+1, \ldots$ *and* $r = 1, 2, \ldots$, *where the sum is empty when* $r = 1$.

Proof. We have

$$\begin{aligned} [e^{-x} x_-^{-r} \tau_n(x)] * x^s &= \underset{\varepsilon\to 0}{\text{N}-\lim} \int_{-n}^{\varepsilon} e^{-t} |t|^{-r} (x-t)^s dt + \int_{-n-n^{-n}}^{-n} e^{-t} |t|^{-r} \tau_n(t) (x-t)^s dt \\ &= I_5 + I_6, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} I_5 &= \sum_{i=0}^{s} \binom{s}{i} x^{s-i} \underset{\varepsilon\to 0}{\text{N}-\lim} \int_{-n}^{\varepsilon} e^{-t} |t|^{-r+i} dt \\ &= -\sum_{i=0}^{s} \binom{s}{i} x^{s-i} \gamma^-(-r+i+1, -n). \end{aligned} \tag{3.32}$$

We now have from equalities (1.8) and (1.10) that

$$\gamma^-(-r,-n) = \sum_{i=1}^{r} \frac{(i-1)!}{r!} n^{-i} e^n - \frac{\psi(r)}{r!} + \frac{1}{r!}\gamma^-(0,-n).$$

It follows that

$$\operatorname*{N-lim}_{n\to\infty} \gamma^-(-r,-n) = -\frac{\psi(r)}{r!}, \tag{3.33}$$

for $r = 1, 2, \ldots$.

It now follows from equality (3.32) that

$$I_5 = -\sum_{i=0}^{s} \binom{s}{i} x^{s-i}\gamma^-(-r+i+1,-n), \tag{3.34}$$

for $s = 0, 1, 2, \ldots, r-2$ and $r = 1, 2, \ldots$ and

$$\begin{aligned} I_5 \;=\; & -\sum_{i=0}^{r-2} \binom{s}{i} x^{s-i}\gamma^-(-r+i+1,-n) \\ & -\sum_{i=r-1}^{s} \binom{s}{i} x^{s-i}\gamma^-(-r+i+1,-n), \end{aligned} \tag{3.35}$$

for $s = r-1, r, r+1, \ldots$ and $r = 1, 2, \ldots$.

It now follows from equalities (3.33), (3.34) and (3.35) that

$$\operatorname*{N-lim}_{n\to\infty} I_5 = \sum_{i=0}^{s} \binom{s}{i} \frac{\psi(r-i-1)}{(r-i-1)!} x^{s-i}, \tag{3.36}$$

for $s = 0, 1, 2, \ldots, r-2$ and $r = 1, 2, \ldots$ and

$$\operatorname*{N-lim}_{n\to\infty} I_5 = \sum_{i=0}^{r-2} \binom{s}{i} \frac{\psi(r-i-1)}{(r-i-1)!} x^{s-i}, \tag{3.37}$$

for $s = r-1, r, r+1, \ldots$ and $r = 1, 2, \ldots$, where the sum is empty when $r = 1$.

Further, it is easily seen that

$$\operatorname*{N-lim}_{n\to\infty} I_6 = 0. \tag{3.38}$$

Equality (3.29) follows from equalities (3.31), (3.36) and (3.38).

Equality (3.30) follows from equalities (3.31), (3.37) and (3.38). □

Corollary 3.2. *The neutrix convolution $[e^{-x}(x^{-r})_+] \circledast x^s$ exists and*

$$[e^{-x}(x^{-r})_+] \circledast x^s = \sum_{i=0}^{s} \binom{s}{i} x^{s-i}\left[(-1)^i\Gamma(-r+i+1) + \frac{(-1)^r\psi(r-1)}{(r-i-1)!}\right], \tag{3.39}$$

for $s = 0, 1, \ldots, r-1$ and $r = 1, 2, \ldots$ and

$$[e^{-x}(x^{-r})_+] \circledast x^s = \sum_{i=0}^{s} \binom{s}{i} (-1)^i x^{s-i}\Gamma(-r+i+1) + \sum_{i=0}^{r-1} \binom{s}{i} \frac{(-1)^r\psi(r-1)}{(r-i-1)!} x^{s-i}, \tag{3.40}$$

for $s = r, r+1, \ldots$ and $r = 1, 2, \ldots$.

Proof. It follows from equality (1.24) that

$$[e^{-x}(x^{-r})_+] \circledast x^s = [e^{-x}x_+^{-r} \circledast x^s] + \frac{(-1)^r\psi(r-1)}{(r-1)!}[e^{-x}\delta^{(r-1)}(x) \circledast x^s], \tag{3.41}$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$.

Equality (3.39) follows from equalities (2.5), (3.25) and (3.41).

Equality (3.40) follows from equalities (2.6), (3.25) and (3.41). □

Corollary 3.3. *The neutrix convolution $[e^{-x}(x^{-r})_-] \circledast x^s$ exists and*

$$[e^{-x}(x^{-r})_-] \circledast x^s = \sum_{i=0}^{s}\binom{s}{i}\frac{1}{(r-i-1)!}\left[\psi(r-i-1) - (-1)^r\psi(r-1)\right]x^{s-i}, \tag{3.42}$$

for $s = 0, 1, \ldots, r-2$ and $r = 1, 2, \ldots$ and

$$[e^{-x}(x^{-r})_-] \circledast x^s = \sum_{i=0}^{r-2}\binom{s}{i}\frac{\psi(r-i-1)}{(r-i-1)!}x^{s-i} - \sum_{i=0}^{r-1}\binom{s}{i}\frac{(-1)^r\psi(r-1)}{(r-i-1)!}x^{s-i}, \tag{3.43}$$

for $s = r-1, r, r+1, \ldots$ and $r = 1, 2, \ldots$, where the first sum is being empty when $r = 1$.

Proof. It follows from equality (1.25) that

$$[e^{-x}(x^{-r})_-] \circledast x^s = [e^{-x}x_-^{-r} \circledast x^s] - \frac{(-1)^r\psi(r-1)}{(r-1)!}[e^{-x}\delta^{(r-1)}(x) \circledast x^s] \tag{3.44}$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$.

Equality (3.42) follows from equalities (2.5), (3.29) and (3.44).

Equality (3.43) follows from equalities (2.6), (3.30) and (3.44). □

Theorem 3.8. *The neutrix convolution $\gamma^+(-r, x) \circledast x^s$ exists for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$. In particular,*

$$\gamma^+(-1, x) \circledast x^s = \sum_{i=0}^{s}\binom{s}{i}(-1)^i x^{s-i}\left[\frac{1}{i+1}\Gamma(i+1) - \Gamma(i)\right], \tag{3.45}$$

for $s = 0, 1, 2, \ldots$.

Proof. We have proved that the convolution $\gamma^+(0, x) \circledast x^s$ exists. Now assuming that the convolution $\gamma^+(-r, x) \circledast x^s$ exists for some r. We then have from equality (1.13) that

$$\begin{aligned}\gamma^+(-r-1, x) \circledast x^s &= -\frac{1}{r+1}\left[\gamma^+(-r, x) \circledast x^s\right] - \frac{1}{r+1}[e^{-x}(x^{-r-1})_+ \circledast x^s] \\ &\quad - \frac{(-1)^r}{(r+1)(r+1)!}[H(x) \circledast x^s].\end{aligned}$$

Since each term of the right-hand side exists, the convolution $\gamma^+(-r-1, x) \circledast x^s$ exists.

In particular, we have

$$\gamma^+(-1, x) \circledast x^s = -\left[\gamma^+(0, x) \circledast x^s\right] - [e^{-x}(x^{-1})_+ \circledast x^s] - [H(x) \circledast x^s], \tag{3.46}$$

for $s = 0, 1, 2, \ldots$.

Equality (3.45) follows from equalities (3.1), (3.19), (3.39), (3.40) and (3.46). □

Theorem 3.9. *The neutrix convolution $\gamma^-(-r,x) \circledast x^s$ exists for $s = 0,1,2,\dots$ and $r = 1,2,\dots$. In particular,*

$$\gamma^-(-1,x) \circledast x^s \;=\; 0, \tag{3.47}$$

for $s = 0,1,2,\dots$.

Proof. The existence of the convolution $\gamma^-(-r,x) \circledast x^s$ is easily proved by induction.

We have in particular,

$$\gamma^-(-1,x) \circledast x^s = \left[\gamma^-(0,x) \circledast x^s\right] + [e^{-x}(x^{-1})_- \circledast x^s] - [H(-x) \circledast x^s] \tag{3.48}$$

for $s = 0,1,2,\dots$.

Equality (3.47) follows from equalities (3.2), (3.20), (3.43) and (3.48). □

Corollary 3.4. *The neutrix convolutions $\gamma^+(-r,x) \circledast (x^s)_-$ and $\gamma^-(-r,x) \circledast (x^s)_+$ exist for $s = 0,1,2,\dots$ and $r = 1,2,\dots$. In particular,*

$$\begin{aligned}\gamma^+(-1,x) \circledast (x^s)_- \;&=\; \sum_{i=0}^{s} \binom{s}{i} (-1)^{s+i} x^{s-i} \left[\frac{1}{i+1}\gamma^+(i+1,x) - \Gamma^+(i,x)\right] \\ &\quad \frac{(-1)^s}{s+1} x^{s+1}\gamma^+(0,x) + \frac{(-1)^s}{s+1}(x^{s+1})_+, \end{aligned} \tag{3.49}$$

$$\begin{aligned}\gamma^-(-1,x) \circledast (x^s)_+ \;&=\; \sum_{i=0}^{s} \binom{s}{i} x^{s-i} \left[\frac{1}{i+1}\gamma^-(i+1,x) + \gamma^-(i,x)\right] \\ &\quad + \frac{(-1)^{s+1}}{s+1}(x^{s+1})_-\gamma^-(0,x) - \frac{(-1)^{s+1}}{s+1}(x^{s+1})_-, \end{aligned} \tag{3.50}$$

for $s = 0,1,2,\dots$.

Proof. The existence of the convolution $\gamma^+(-r,x) \circledast (x^s)_-$ follows from the existence of the convolutions $\gamma^+(-r,x) \circledast x^s$ and $\gamma^+(-r,x) * (x^s)_+$.

The existence of the convolution $\gamma^-(-r,x) \circledast (x^s)_+$ follows from the existence of the convolutions $\gamma^-(-r,x) \circledast x^s$ and $\gamma^-(-r,x) * (x^s)_-$.

Equality (3.49) follows from equalities (2.27) and (3.45) by noting that

$$\Gamma(i+1) = \gamma^+(i+1,x) + \gamma^+(i+1,x),$$

for $i = 0,1,2,\dots$.

Equality (3.50) follows from equalities (2.28) and (3.47). □

References

[1] I.M. Gel'fand, G.E. Shilov, *Generalized functions*, Vol. I, Academic Press, 1964.

[2] B. Fisher, *Neutrices and the convolution of distributions*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 17 (1987), 119–135.

[3] B. Fisher, B. Jolevska-Tuneska, A. Kılıçman, On defining the incomplete gamma function, Integral Transform and Special Functions. 14 (2003), no. 4, 293–299.

[4] B. Fisher, A. Kılıçman, *Some results on the Gamma function for negative integers*, Appl. Math. Inf. Sci. 6 (2012), no. 2, 173–176.

[5] B. Fisher, Y. Kuribayashi, *Neutrices and the Gamma function,* J. Fac. Ed. Tottori Univ. 36 (1987), no. 1-2, 1–7.

[6] B. Fisher, B. Jolevska-Tuneska, A. Takaci, *On the incomplete Gamma function and the neutrix convolution,* Math. Bohemica. 4 (2003), no. 128, 367–377.

[7] B. Fisher, *Some notes on distributions,* Math. Student. 48 (1980), 269–281.

[8] B. Fisher, S. Yakubovich, M. Telci, *On the incomplete Gamma function*, Rad. Mat. 11 (2002), no. 1, 37–47.

[9] M. Lin, B. Fisher, S. Orankitjaroen, *On the incomplete Gamma function*, Int. J. App. Math. 27 (2014), no. 1, 31–43.

[10] I.N. Sneddon, *Special functions of mathematical physics and chemistry,* Oliver & Boyd, (1956).

[11] J.G. van der Corput, *Introduction to the neutrix calculus, J. Analyse Math.* 7 (1959-60), 291–398.

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