ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2019, Volume 10, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

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KHARIN STANISLAV NIKOLAYEVICH

(to the 80th birthday)

Stanislav Nikolayevich Kharin was born on December 4, 1938 in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis "Heat phenomena in electrical contacts and associated singular integral equations", and in 1990 his doctoral thesis "Mathematical models of thermo-physical processes in electrical contacts" in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. Using models based on the new original methods for solving free boundary problems he described mathematically the phenomena of arcing, contact welding, contact floating, dynamics of contact blow-open phenomena, electrochemical mechanism of electron emission, arc-to-glow transition, thermal theory of the bridge erosion. For these achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research and the evidence of this in the new monograph "Mathematical models of phenomena in electrical contacts" published last year in Novosibirsk.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Stanislav Nikolayevich on the occasion of his 80th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 10, Number 1 (2019), 16 – 29

LAGRANGE FORMULA FOR DIFFERENTIAL OPERATORS AND SELF-ADJOINT RESTRICTIONS OF THE MAXIMAL OPERATOR ON A TREE

B. Kanguzhin, L. Zhapsarbayeva, Zh. Madibaiuly

Communicated by M. Otelbaev

Key words: directed graph, tree, Kirchhoff conditions, self-adjoint restrictions.

AMS Mathematics Subject Classification: 34B45,34L20.

Abstract. The paper is devoted to linear differential operators defined on a tree. We aim at obtaining complete descriptions of well-posed restrictions of a given maximal differential operator on a tree. In this paper all self-adjoint restrictions of the maximal operator and also all the invertible restrictions of the maximal operator are described. We also present the Lagrange formula for a differential operator on a tree with the Kirchhoff conditions at its interior vertices.

DOI: https://doi.org/10.32523/2077-9879-2019-10-1-16-29

1 Introduction

Differential equations that arise in many in applications can be interpreted as equations on graphs. Plenty of examples can be found in fields such as chemical kinetics, chemical technology, biology, and in Markov processes. Therefore, the study of differential equations is of interest beyond the field of mathematics.

Besides applications, mathematicians also intensively develop the theoretical foundation of differential equations on graphs [5, 9, 10]. One of the main questions is: what conditions at the vertices of the graph are the most "natural"? The standard answer is the Kirchhoff conditions. However, there is a possibility to impose conditions other than the Kirchhoff conditions at the vertices. The detailed answer to this question for second order differential equations can be found in [5, 12], and for higher order differential equations in [13]. In monograph [9], the vertices of graphs are divided into two types: boundary vertices and interior vertices. If we assume that the Kirchhoff conditions or conditions from [5, 12] hold at the interior vertices, there is still a problem of determining conditions at boundary vertices. The problem of determining the general boundary conditions at boundary vertices have not been studied in [5, 12, 13].

Five approaches (decomposition, scalarizing, vector, related and synthetic) of the study of differential equations on graphs are presented in monograph [9].

In this paper, we exploit a related approach, according to which differential equations at the edges of a graph and conditions at the interior vertices of a graph generate a differential operator in some function space on the graph. Then, the problem of describing the general boundary conditions at boundary vertices of a graph is reduced to determining the self-adjoint restrictions of the above-mentioned differential operator. Similar constructions for any graphs are given in [14]. In the present paper, in the case of a tree, it is possible to describe in a compact form (more clearly than in work [14]) not only all possible self-adjoint boundary value problems, but also well-posed problems for differential equations with the Kirchhoff conditions at interior vertices. In this paper all self-adjoint restrictions of the maximal operator (Theorem 5.1) and also all the invertible restrictions of the maximal operator (Theorem 6.2) are described.

We note that a related approach to finding the asymptotics of solutions to differential equations on a tree was used in [15].

2 Basic concepts

Let $\Im = \{\mathcal{V}, \mathcal{E}\}$ be a tree. Here $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges [1] of the graph $\Im$. A directed graph is a tree, if at each vertex, except for one vertex, there is one incoming edge. The vertex which does not have any incoming edges is said to be a root of the tree. We assign the number 0 to the root. One of the important properties of a tree is the existence of a unique path connecting the root and any vertex [11]. The length of the path determines the height of the vertex of the tree. Vertices which do not have the outgoing edges are called boundary vertices and denoted by Γ. Non-boundary vertices we call interior vertices and denote them by $\mathcal{I}$. First, we number the boundary vertices from 0 to p. Further, we assign the numbers from $p+1$ to r to the interior vertices by the rule: if the height of the vertex is greater, then its number is greater. We denote by m_j the number of edges outgoing from vertex j. Without loss of generality, we suppose that each edge has unit length. The edge which ended at vertex j, we denote by e_j. Function $y(x)$ defined on edge e_j we denote by $y_j(x_j), x_j \in e_j$. The path outgoing from the root and ending at vertex j, we denote by s_j, and its length by $|s_j| - 1$. Further, we assume that only one edge goes out from a root.

3 Definition of the maximal operator on a tree

We consider the space

$$L_2(\Im) \doteqdot \prod_{j=1}^{r} L_2(e_j)$$

with the elements

$$\vec{Y}(\vec{X}) \doteqdot [y_j(x_j), j = 1, \ldots, r]^T,$$

(where $\vec{X} = (x_j, j = 1, \ldots, r)$ and $\prod_{j=1}^{r}$ is the Cartesian product of the subspaces) and with a finite norm

$$||\vec{Y}||_{L_2(\Im)} = \sqrt{\sum_{j=1}^{r} \int_{e_j} |y_j(x_j)|^2 dx_j}.$$

In the standard way we introduce the space

$$W_2^2(\Im) \doteqdot \prod_{j=1}^{r} W_2^2(e_j).$$

We introduce the set of functions $D(\Lambda_{max}) \subset W_2^2(\Im)$ with the elements that satisfy the following Kirchhoff conditions [8]

$$y_k(1) = y_{s_1(k)}(0) = \ldots = y_{s_{m_k}(k)}(0), \tag{3.1}$$

$$y'_k(1) = y'_{s_1(k)}(0) + \ldots + y'_{s_{m_k}(k)}(0) \tag{3.2}$$

at each interior vertices $k = p+1, \ldots, r$. Here $s_1(k), \ldots, s_{m_k}(k)$ are the numbers of outgoing edges from vertex k (Fig.1).

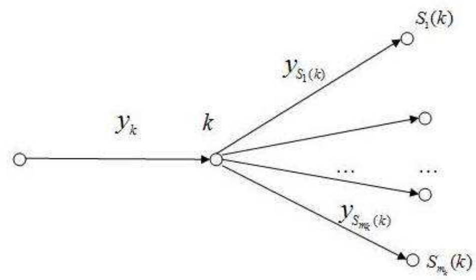

Fig. 1. The distribution of the solution at the interior k-th vertex.

An operator Λ_{max} with the domain of definition $D(\Lambda_{max})$ and given by the differential expressions

$$-y_j''(x_j) + q_j(x_j)y_j(x_j) = \rho^2 y_j(x_j),\ e_j \in \mathcal{E},\ 0 < x_j < 1, \quad j = 1, \dots, r \tag{3.3}$$

is called a maximal operator. Here $\{q_j(x_j), x_j \in e_j \in \mathcal{E},\ 0 < x_j < 1\}$ is the set of real-valued continuous functions, usually called potentials. We note that the total number of the Kirchhoff conditions at the interior vertices is equal to $2r - p - 1$.

4 Lagrange formula for differential operators on a tree

The Lagrange formula plays an important role in the study of differential operators on an interval. In this section we present the analogue of the Lagrange formula in the case of differential operators on a tree. The Lagrange formula for arbitrary connected graphs without loops is stated in [14]. When a graph becomes a tree, the Lagrange formula has a more illustrative form. First, we formulate some auxiliary lemmas [14].

Lemma 4.1. *The following identity*

$$\sum_{j=1}^{r} \int_{e_j} \Lambda_{max} y_j(x_j)\overline{v_j(x_j)}dx_j = \sum_{k=1}^{r}[-y_k'(1)\overline{v_k(1)} + y_k(1)\overline{v_k'(1)}] + \sum_{k=p+1}^{r}[y_k'(0)\overline{v_k(0)} - y_k(0)\overline{v_k'(0)}] + \sum_{j=1}^{r} \int_{e_j} y_j(x_j)\overline{\Lambda_{max} v_j(x_j)}dx_j \tag{4.1}$$

holds for all $\vec{Y}(x) = \{y_j(x_j), j = 1, \dots, r\}$, $\vec{V}(x) = \{v_j(x_j), j = 1, \dots, r\}$ from $W_2^2(\Im)$, where $\bar{z}$ is a complex conjugate of the number z.

Lemma 4.2. *The following identity*

$$\sum_{j=1}^{r} \int_{e_j} \Lambda_{max} y_j(x_j)\overline{v_j(x_j)}dx_j = \sum_{k=1}^{p}[-y_k'(1)\overline{v_k(1)} + y_k(1)\overline{v_k'(1)}] + [y_{p+1}'(0)\overline{v_{p+1}(0)} - y_{p+1}(0)\overline{v_{p+1}'(0)}] + \sum_{j=1}^{r} \int_{e_j} y_j(x_j)\overline{\Lambda_{max} v_j(x_j)}dx_j \tag{4.2}$$

holds for all $\vec{Y}(x) = \{y_j(x_j), j = 1, \dots, r\}$, $\vec{V}(x) = \{v_j(x_j), j = 1, \dots, r\}$ from the domain of definition of the maximal operator Λ_{max}.

Lemma 4.2 is proved in [14]. Here is a more illustrative proof of Lemma 4.2.

Proof. We assume that j takes one of the values $p+1,\dots,r$. Then the contribution of the vertex j to the terms

$$\sum_{k=1}^{r}[-y_k'(1)\overline{v_k(1)}+y_k(1)\overline{v_k'(1)}]+\sum_{k=p+1}^{r}[y_k'(0)\overline{v_k(0)}-y_k(0)\overline{v_k'(0)}], \tag{4.3}$$

that do not contain the integral, can be written in the following form

$$[-y_j'(1)\overline{v_j(1)}+y_j(1)\overline{v_j'(1)}]+\sum_{k=1}^{m_j}[y_{s_k(j)}'(0)\overline{v_{s_k(j)}(0)}-y_{s_k(j)}(0)\overline{v_{s_k(j)}'(0)}]. \tag{4.4}$$

By condition (3.1) we calculate the value of sum (4.4), and get

$$[-y_j'(1)\overline{v_j(1)}+y_j(1)\overline{v_j'(1)}]+\overline{v_j(1)}\sum_{k=1}^{m_j}y_{s_k(j)}'(0)-y_j(1)\sum_{k=1}^{m_j}\overline{v_{s_k(j)}'(0)}.$$

By Kirchhoff condition (3.2) we compute the last expression. We obtain

$$[-y_j'(1)\overline{v_j(1)}+y_j(1)\overline{v_j'(1)}]+\overline{v_j(1)}y_j'(1)-y_j(1)\overline{v_j'(1)}=0. \tag{4.5}$$

Relation (4.5) means that the contribution of interior vertices $p+1,\dots,r$ to the terms that do not contain the integral (4.3) is equal to zero. Therefore in (4.3) it is necessary to take into account only the contribution of boundary vertices $0,\dots,p$. Then we have the values of the functions $y_j(x)$, $y_j'(x)$, $\overline{v_j(x)}$, $\overline{v_j'(x)}, j=1,\dots,p$ at the point $x=1$. We recall that these functions are defined on the incoming edges $e_1,\dots,e_p$ to the boundary vertices $1,\dots,p$. There are also the values of the functions $y_{p+1}(x)$, $y_{p+1}'(x)$, $\overline{v_{p+1}(x)}$, $\overline{v_{p+1}'(x)}$ at the point $x=0$. These functions are defined on the outgoing edges e_{p+1} from the vertex 0 and directed to the vertex $p+1$ (Fig.2):

$$\sum_{k=1}^{p}[-y_k'(1)\overline{v_k(1)}+y_k(1)\overline{v_k'(1)}]-[-y_{p+1}'(0)\overline{v_{p+1}(0)}+y_{p+1}(0)\overline{v_{p+1}'(0)}].$$

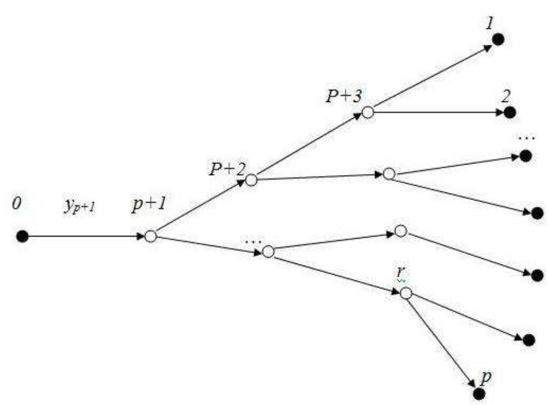

Fig. 2.Tree with r vertices (black vertices are boundary vertices).

$\square$

Lemma 4.2 implies that the contribution of interior vertices to the terms that do not contain integral (4.2) is equal to zero. In other words, the terms that are outside of integral (4.2) contain only the contribution of boundary vertices. By monograph [6] similar formulae are called Lagrange formula. Formula (4.2) can be generalized in the following direction.

For $k = 1, \dots, 2(p+1)$ we consider

$$U_k(\vec{Y}) = \sum_{j=1}^{p} [\alpha_{kj} y_j(1) + \beta_{kj} y_j'(1)] + [\alpha_{k,p+1} y_{p+1}(0) + \beta_{k,p+1} y_{p+1}'(0)], \tag{4.6}$$

where $\alpha_{kj}, \beta_{kj}, \alpha_{k,p+1}, \beta_{k,p+1}$ are some constants.

Theorem 4.1. *[Lagrange formula] Let $\{U_1, \dots, U_{2(p+1)}\}$ be a set of linear independent boundary forms. Then there exists a unique set of boundary forms $\{T_1, \dots, T_{2(p+1)}\}$ such that the following identity*

$$\sum_{j=1}^{r} \int_{e_j} \Lambda_{max} y_j(x_j) \overline{v_j(x_j)} dx_j = U_1(\vec{Y}) \overline{T_{2(p+1)}(\vec{V})} + U_2(\vec{Y}) \overline{T_{2(p+1)-1}(\vec{V})} + \dots$$
$$+ U_{2(p+1)}(\vec{Y}) \overline{T_1(\vec{V})} + \sum_{j=1}^{r} \int_{e_j} y_j(x_j) \overline{\Lambda_{max} v_j(x_j)} dx_j \tag{4.7}$$

holds for all functions $\vec{Y}(x) = \{y_j(x_j), j = 1, \dots, r\}$, $\vec{V}(x) = \{v_j(x_j), j = 1, \dots, r\}$ from the domain of definition of the maximal operator Λ_{max}.

Theorem 4.1 is proved in [14]. Here it is specified that the boundary forms $\{T_1, \dots, T_{2(p+1)}\}$ can be represented as

$$T_{2(p+1)-k+1}(\vec{V}) = \sum_{k=1}^{p} \left[-\overline{\epsilon_{jk}} v_j(1) + \overline{\gamma_{jk}} v_j'(1) \right] + \left[\overline{\epsilon_{j,p+1}} v_{p+1}(0) - \overline{\gamma_{j,p+1}} v_{p+1}'(0) \right],$$

where $\epsilon_{jk}, \gamma_{jk}$ is some set of numbers (may be complex).

Formula (4.7) is called the Lagrange formula.

Theorem 4.1 immediately implies the following statement.

Corollary 4.1. *Let Λ be a restriction of the operator Λ_{max} in the domain of definition $D(\Lambda) = \{\vec{Y} \in D(\Lambda_{max}) : U_1(\vec{Y}) = 0, \dots, U_{p+1}(\vec{Y}) = 0\}$. Then the adjoint operator Λ^* is also a restriction of the operator Λ_{max} in the domain of definition $D(\Lambda^*) = \{\vec{V} \in D(\Lambda_{max}) : T_1(\vec{V}) = 0, \dots, T_{p+1}(\vec{V}) = 0\}$, and for all $\vec{Y} \in D(\Lambda)$ and $\vec{V} \in D(\Lambda^*)$ the following equality*

$$\sum_{j=1}^{r} \int_{e_j} \Lambda y_j(x_j) \overline{v_j(x_j)} dx_j = \sum_{j=1}^{r} \int_{e_j} y_j(x_j) \overline{\Lambda^* v_j(x_j)} dx_j$$

holds.

5 Self-adjoint restrictions of the maximal operator Λ_{max}

In this section we give the complete description of all self-adjoint restrictions of the operator Λ_{max}. First we introduce the minimal restriction Λ_0 of the operator Λ_{max}. We denote by $D(\Lambda_0)$ the set all functions $\vec{Y}(x) \in D(\Lambda_{max})$ which satisfy the conditions

$$y_j(1) = 0, \; y_j'(1) = 0 \quad \text{for} \quad j = 1, \dots, p, \tag{5.1}$$

$$y_{p+1}(0) = 0,\ y'_{p+1}(0) = 0.$$

Furthermore, we introduce the minimal restriction Λ_0 by formula

$$\Lambda_0\vec{Y} = \Lambda_{max}\vec{Y}, \quad \vec{Y} \in D(\Lambda_0).$$

The following assertions are valid

I) the equality

$$\langle \Lambda_0\vec{Y}, \vec{V}\rangle = \langle \vec{Y}, \Lambda_{max}\vec{V}\rangle \tag{5.2}$$

holds for all $\vec{Y} \in D(\Lambda_0)$, $\vec{V} \in D(\Lambda_{max})$,

II) the equality

$$\langle \Lambda_0\vec{Y}, \vec{V}\rangle = \langle \vec{Y}, \Lambda_0\vec{V}\rangle$$

holds for all $\vec{Y}, \vec{V} \in D(\Lambda_0)$. From equality (5.2) it follows that $\Lambda_{max} \subset \Lambda_0^*$.

To study the properties of the minimal operator it is convenient to introduce the operators Λ_1 and Λ_2 which are the restrictions of the maximal operator Λ_{max}. Let

$$D(\Lambda_1) = \{\vec{Y} \in D(\Lambda_{max}) : y_j(1) = 0,\ j = 1, \dots, p,\ y_{p+1}(0) = 0\}$$

and $\Lambda_1\vec{Y}(x) = \Lambda_{max}\vec{Y}(x)$ for $\vec{Y} \in D(\Lambda_1)$.

Let

$$D(\Lambda_2) = \{\vec{Y} \in D(\Lambda_{max}) : y'_j(1) = 0,\ j = 1, \dots, p,\ y'_{p+1}(0) = 0\}$$

and $\Lambda_2\vec{Y}(x) = \Lambda_{max}\vec{Y}(x)$ for $\vec{Y} \in D(\Lambda_2)$.

Assumption 1. *The equation*

$$\Lambda_i\vec{Y}(x) = \vec{F}(x), \quad i = 1, 2 \tag{5.3}$$

has a unique solution in $D(\Lambda_i)$, $i = 1, 2$ for all functions $\vec{F}(x)$ in $L_2(\Im)$.

Remark 1. *Operator equation (5.3) in the set $D(\Lambda_i)$, $i = 1, 2$ is equivalent to the system of second order linear differential equations on the set of edges $\mathcal{E}$ with $2|\mathcal{E}| - p - 1$ the Kirchhoff conditions at the interior vertices $\mathcal{I}$ and $p + 1$ conditions at the boundary vertices Γ. Thus, we have the system of second order non-homogeneous linear differential equations on the set of edges $\mathcal{E}$, whose general solution contains $2|\mathcal{E}|$ constants. There are $(2|\mathcal{E}| - p - 1) + p + 1 = 2|\mathcal{E}|$ linear conditions to determine them. Consequently, one can write some determinant D_i, $i = 1, 2$ of size $2|\mathcal{E}|$. Then the unique solvability of equation (5.3) is equivalent to the fact that the determinant D_i, $i = 1, 2$ is not equal to zero.*

By following monograph [6], we formulate two lemmas.

Lemma 5.1. *Let Λ_{max} be a maximal operator on the tree $\Im$ that was introduced in Section 3, and let $\vec{F}(x)$ be a function in $L_2(\Im)$. If assumption 1 hold, then the equation*

$$\Lambda_{max}\vec{Y}(x) = \vec{F}(x)$$

has a solution $\vec{Y}(x)$ that satisfies condition (5.1) if and only if $\vec{F}(x)$ is orthogonal to all elements from $Ker\Lambda_{max}$.

Proof. By assumption 1 we denote by $\vec{Y}(x)$ the unique solution of operator equation (5.3) for $i = 1$. We denote by $\vec{V}_1, \dots, \vec{V}_{p+1}$ the fundamental set of solutions of homogeneous operator equation $\Lambda_{max}\vec{V} = 0$ that satisfies the following conditions: all boundary forms $v_{jl}(1)$ for $j = 1, \dots, p$ и $v_{p+1,l}(0)$, except for one of them, are equal to zero, and one of the forms is equal to 1. There exists such a fundamental system. This follows from Remark 1, because the condition of solvability is equivalent to the fact that the determinant D_1 is not equal to zero.

By applying the Lagrange formula to the functions $\vec{Y}(x)$ and $\vec{V}_k(x)$, we have

$$\langle \vec{F}, \vec{V}_k \rangle = \langle \Lambda_{max}\vec{Y}, \vec{V}_k \rangle - \langle \vec{Y}, \Lambda_{max}\vec{V}_k \rangle. \tag{5.4}$$

However $\Lambda_{max}\vec{V}_k = 0$. Moreover, the inclusion $\vec{Y} \in D(\Lambda_1)$ implies that

$$\sum_{l=1}^{p} y_l(1)v'_{kl}(1) - y_{p+1}(0)v'_{k,p+1}(0) = 0.$$

Hence, formula (5.4) has the following form

$$\langle \vec{F}, \vec{V}_k \rangle = -\sum_{l=1}^{p} y'_l(1)v_{kl}(1) + y'_{p+1}(0)v_{k,p+1}(0)$$

$$= \begin{cases} -y'_l(1) & \text{if} \quad v_{kl}(1) = 1, \\ y'_{p+1}(0) & \text{if} \quad v_{k,p+1}(0) = 1. \end{cases} \tag{5.5}$$

Relation (5.5) implies the statement of Lemma 5.1: equalities (5.1) are valid if and only if $\langle \vec{F}, \vec{V}_k \rangle = 0$, $k = 1, \dots, p+1$. Namely, $\vec{F}(x)$ is orthogonal to all solutions of the equation $\Lambda_{max}\vec{V} = 0$. □

Lemma 5.2. *Let assumption 1 hold. Then there exists a function such that $\vec{Y}(x) \in D(\Lambda_{max})$ which satisfies the following conditions*

$$y'_k(1) = \beta_k,\ y_k(1) = \alpha_k,\ k = 1, \dots, p,$$

$$y'_{p+1}(0) = \beta_{p+1},\ y_{p+1}(0) = \alpha_{p+1}$$

for any numbers α_k, β_k for $k = 1, \dots, p$ and $\alpha_{p+1}, \beta_{p+1}$.

Proof. First, we prove Lemma 5.2 for the case $\alpha_k = 0$. We choose $\vec{F}(x) \in L_2(\Im)$ such that

$$\langle \vec{F}, \vec{V}_k \rangle = \begin{cases} -\beta_l & \text{if} \quad v_{kl}(1) = 1, \\ \beta_{p+1} & \text{if} \quad v_{k,p+1}(0) = 1, \end{cases} \tag{5.6}$$

where $\vec{V}_k$, $k = 1, \dots, p+1$ is the same system of fundamental solution, that in the proof of Lemma 5.1. There exists such an element and moreover it is in $Ker\Lambda_{max}$. Indeed, if we take

$$\vec{F} = \sum_{k=1}^{p+1} \mu_k \vec{V}_k,$$

then condition (5.6) is the system of equations with respect to the constants $\mu_1, \dots, \mu_{p+1}$, whose determinant is the Gram determinant of linear independent functions $\vec{V}_1, \dots, \vec{V}_{p+1}$. Consequently, it is not equal to zero. We denote by $\vec{V}$ the solution to the equation $\Lambda_1\vec{V} = \vec{F}$. Then the Lagrange formula implies

$$\vec{V}'(1) = \beta_j \quad \text{for} \quad j = 1, \dots, p,$$

$$\vec{V}'(0) = \beta_{p+1}.$$

So, the constructed function $\vec{V}(x) \in D(\Lambda_{max})$ and

$$\vec{V}'(1) = \beta_j,\ \vec{V}(1) = 0 \quad \text{for} \quad j = 1, \dots, p,$$

$$\vec{V}'(0) = \beta_{p+1},\ \vec{V}(0) = 0.$$

Replacing the set α_j, α_{p+1} by β_j, β_{p+1}, and replacing the operator Λ_1 by Λ_2, we obtain the proof of Lemma 5.2. □

Now we formulate the statement about the minimal operator Λ_0.

Lemma 5.3. $\Lambda_0 \subset \Lambda_0^* = \Lambda_{max}$, $\Lambda_{max}^* = \Lambda_0$.

Lemma 5.3 can be proved by arguing as in the proof of a similar statement in monograph [6]. We state the main theorem of this section.

Theorem 5.1. *If assumption 1 hold, then any self-adjoint restriction Λ of operator Λ_{max} can be determined by $k = 1, \dots, p+1$ linear independent boundary conditions*

$$U_k(\vec{Y}) = \sum_{j=1}^{p} [\alpha_{jk} y_j(1) + \beta_{jk} y_j'(1)] + [\alpha_{p+1,k} y_{p+1}(0) + \beta_{p+1,k} y_{p+1}'(0)] = 0, \tag{5.7}$$

where $\alpha_{jk}, \beta_{jk}, \alpha_{p+1,k}, \beta_{p+1,k}$ are some constants. Moreover

$$\sum_{j=1}^{p} [\alpha_{jk} \bar{\beta}_{jk} - \bar{\alpha}_{jk} \beta_{jk}] = \alpha_{p+1,k} \bar{\beta}_{p+1,k} - \bar{\alpha}_{p+1,k} \beta_{p+1,k} \tag{5.8}$$

for $k = 1, \dots, p+1$.

Conversely, if assumption 1 holds, any linear independent boundary conditions of form (5.7) which satisfy relations (5.8), specify the domain of definition of some self-adjoint restriction Λ of the operator Λ_{max}.

An analogue of Theorem 5.1 can be found in [14].

Proof. In monograph [6] there is a similar theorem. Following monograph [6] we introduce the functions $\vec{V}_1, \dots, \vec{V}_{p+1}$. More precisely, $\vec{V}_k \in D(\Lambda_{max})$ with conditions

$$v_{kj}'(1) = \alpha_{jk},\ v_{kj}(1) = -\beta_{jk} \quad \text{for} \quad j = 1, \dots, p, \tag{5.9}$$

$$v_{k,p+1}'(0) = -\alpha_{p+1,k},\ v_{k,p+1}(0) = \beta_{p+1,k}.$$

By Lemma 5.2 there exist such solutions. Then condition (5.7) for $k = 1, \dots, p+1$ has the form

$$U_k(\vec{Y}) = \sum_{j=1}^{p} [y_j(1) v_{kj}'(1) - y_j'(1) v_{kj}(1)] - [y_{p+1}(0) v_{k,p+1}'(0) - y_{p+1}'(0) v_{k,p+1}(0)] = 0.$$

By results of monograph [6] boundary conditions (5.7) specify the domain of definition of a self-adjoint restriction Λ of the operator Λ_{max}. Suppose that the domain of definition of the restriction Λ is defined by boundary conditions (5.7). Then the following equalities

$$U_k(\vec{V}_j) = 0, \quad k, j = 1, \dots, p+1$$

hold. Hence, Λ is the self-adjoint restriction. The inverse statement is true. The proof of Theorem 5.1 is complete. □

6 Well-posed restrictions of the maximal operator Λ_{max}

In the preceding section, we gave a complete description for all self-adjoint restrictions of the maximal operator Λ_{max}. Now we will describe well-posed restrictions of the maximal operator Λ_{max}.

An operator Λ is called a well-posed restriction of the maximal operator Λ_{max}, if the following conditions

(i) $\Lambda \subset \Lambda_{max}$,

(ii) $\exists\, \Lambda^{-1}$ is a bounded operator in $L_2(\Im)$

hold. Well-posed restrictions of various classes of differential operators have been studied in [4, 7]. Motivated by works [4, 7], first, we need to choose some fixed well-posed restriction Λ_1 of the maximal operator Λ_{max}. Afterwards, knowing the inverse operator Λ_1^{-1}, we need to describe all well-posed restrictions.

6.1 Green function for a fixed well-posed restriction

Further, we suppose that assumption 1 holds for $i = 1$. By assumption 1 there exists a bounded operator Λ_1^{-1} in the space $L_2(\Im)$ for $i = 1$. In this subsection we find out the structure of the inverse operator Λ_1^{-1}. So, we consider operator equation $\Lambda_1 \vec{Y} = \vec{F}$ for an arbitrary $\vec{F}$ from $L_2(\Im)$. We need to express its solution $\vec{Y} = \{y_j(x_j), x_j \in e_j, j = 1, \ldots, r\} \in D(\Lambda_1)$ by $\vec{F} = \{f_j(x_j), x_j \in e_j, j = 1, \ldots, r\} \in D(\Lambda_1)$. It is well-known [6] that on the edge e_j the function $y_j(x_j)$ that satisfies the equation

$$-y_j''(x_j) + q_j(x_j)y_j(x_j) = f_j(x_j),\ \ x_j \in e_j$$

has the following representation

$$y_j(x_j) = y_j(0)c_j(x_j) + y_j'(0)s_j(x_j) + \int_0^{x_j} \tilde{g}_j(x_j, t)f_j(t)dt, \tag{6.1}$$

where $\{c_j(x_j), s_j(x_j)\}$ is the fundamental set of solutions of homogeneous equation $-y_j''(x_j) + q_j(x_j)y_j(x_j) = 0$ subordinated to Cauchy data $s_j'(0) = c_j(0) = 1$, $s_j(0) = c_j'(0) = 0$. In formula (6.1) the Cauchy function $\tilde{g}_j(x_j, t)$ also appears. It is defined by the formula

$$\tilde{g}_j(x_j, t) = \begin{vmatrix} c_j(t) & s_j(t) \\ c_j(x) & s_j(x) \end{vmatrix},\ t < x_j.$$

Further, it is convenient to introduce the notation

$$g_j(x_j, t) = \begin{cases} \quad 0 \quad \text{for } x_j \le t < 1, \\ \tilde{g}_j(x_j, t) \text{ for } 0 \le t < x_j. \end{cases}$$

Then formula (6.1) for $x_j \in e_j$ has the following form

$$y_j(x_j) = y_j(0)c_j(x_j) + y_j'(0)s_j(x_j) + \int_{e_j} g_j(x_j, t)f_j(t)dt. \tag{6.2}$$

Since Λ_1^{-1} exists, the values $y_j(0)$ and $y_j'(0)$ can be uniquely determined by $\vec{F}$. The inverse operator Λ_1^{-1} is linear because Λ_1 is linear. Consequently, the functionals $y_j(0)$ and $y_j'(0)$ depend on $\vec{F}$ linearly. From the boundedness of Λ_1^{-1} it follows the boundedness of the linear functionals $y_j(0)$

and $y_j'(0)$ in $L_2(\Im)$. Hence, the Riesz representation theorem on continuous linear functionals in $L_2(\Im)$ implies the following statement: there exist functions $\vec{A}_j$ and $\vec{B}_j$ in $L_2(\Im)$ such that

$$y_j(0) = \sum_{k=1}^{r} \int_{e_k} a_{jk}(x) f_k(x) dx, \tag{6.3}$$

$$y_j'(0) = \sum_{k=1}^{r} \int_{e_k} b_{jk}(x) f_k(x) dx.$$

From relations (6.2) by (6.3), it follows that

$$y_j(x_j) = \sum_{k=1}^{r} \int_{e_k} \{a_{jk}(t) c_j(x_j) + b_{jk}(t) s_j(x_j)\} f_k(t) dt + \int_{e_j} g_j(x_j, t) f_j(t) dt. \tag{6.4}$$

Thus, for fixed $x_j \in e_j$ the right-hand side of relation (6.4) has the form of inner product of the space $L_2(\Im)$. Therefore, formula (6.4) for $j = 1, \dots, r$ can be rewritten in the form

$$y_j(x_j) = \sum_{k=1}^{r} \int_{e_k} d_{jk}(x_j, t_k) f_k(t_k) dt_k, \; x_j \in e_j, \tag{6.5}$$

where $d_{jk}(x_j, t_k)$ is some set of functions.

So, the inverse operator Λ_1^{-1} is defined by formulae (6.5). The matrix $D = \|d_{jk}(x_j, t_k)\|$ is usually called the Green function of the operator Λ_1.

Remark 2. *Instead of restriction Λ_1 one can choose other invertible restrictions of maximal operator Λ_{max}. Finally, we arrive at the following question. Which restriction has the inverse operator to be of the simplest form? For example, the inverse operator of the restriction $\Lambda_3 \subset \Lambda_{max}$ in the domain of definition*

$$D(\Lambda_3) = \{\vec{Y} \in D(\Lambda_{max}) : y_j(1) = 0, \; j = 1, \dots, p-1, \; y_{p+1}(0) = y_{p+1}'(0) = 0\}$$

has the following form

$$y_j(x_j) = \sum_{k=2}^{|s_j|-1} \int_{e_{n_{k,j}}} d_{n_{k,j}}(x_j, t) f_{n_{k,j}}(t) dt, \; x_j \in e_j. \tag{6.6}$$

Here $s_j = \{n_{1j}, n_{2j}, \dots, n_{|s_j|,j}\}$ is the path connecting the vertices 0 and j. It is clear that $n_{1j} = 0, n_{2j} = p+1, \dots, n_{|s_j|,j} = j$. In contrast to formula (6.5), only edges that form the path s_j appear in formula (6.6). At the same time in the right-hand side of formula (6.5) it is essential that the graph is a tree, i.e. all the edges will participate in formula (6.6).

6.2 The description of the well-posed restrictions

In this subsection, the full description of well-posed restrictions of the operator Λ_{max} is given by following the scheme proposed by M. Otelbayev [4, 7].

Let $\vec{H} = \{h_j(x_j), x_j \in e_j, j = 1, \dots, r\}$ be an arbitrary element of the set $D(\Lambda_{max})$. We introduce a new function $\vec{Z} = \{z_j(x_j), x_j \in e_j, j = 1, \dots, r\}$ by formula (6.5)

$$z_j(x_j) = \sum_{k=1}^{r} \int_{e_k} d_{jk}(x_j, t_k) \left(-h_k''(t_k) + q_k(t_k) h_k(t_k)\right) dt_k, \; x_j \in e_j. \tag{6.7}$$

It is clear that the function $\vec{Z}$ has the following properties:

$$-z_j''(x_j)+q_j(x_j)z_j(x_j)=-h_j''(x_j)+q_j(x_j)h_j(x_j),\ x_j\in e_j, \tag{6.8}$$

$$z_{p+1}(0)=0, z_1(1)=z_2(1)=\ldots=z_p(1)=0, \tag{6.9}$$

$$\vec{Z}\in D(\Lambda_1)\subset D(\Lambda_{max}). \tag{6.10}$$

On the other hand, using Lagrange formula (4.2) to the right-hand side of relation (6.7) we can rewrite in the following way

$$z_j(x_j)=\sum_{k=1}^{r}\int_{e_k}\left(-d_{jk}''(x_j,t_k)+q_k(t_k)d_{jk}(x_j,t_k)\right)h_k(t_k)dt_k$$

$$+\sum_{k=1}^{p}[-h_k'(1)d_{jk}(x_j,1)+h_k(1)\frac{\partial}{\partial t_k}d_{jk}(x_j,1)]$$

$$+[h_{p+1}'(0)d_{j,p+1}(x_j,0)-h_{p+1}(0)\frac{\partial}{\partial t_{p+1}}d_{j,p+1}(x_j,0)].$$

Hence, by (6.8) and (6.9), we have

$$z_j(x_j)=h_j(x_j)-h_{p+1}(0)\frac{\partial}{\partial t_{p+1}}d_{j,p+1}(x_j,0)+\sum_{k=1}^{p}h_k(1)\frac{\partial}{\partial t_k}d_{jk}(x_j,1). \tag{6.11}$$

Thus, the following statement is valid.

Lemma 6.1. *Identity (6.11) holds for all functions $\vec{H}=\{h_j(x_j),x_j\in e_j,j=1,\ldots,r\}$, where $\vec{Z}=\Lambda_1^{-1}\left(\Lambda_{max}\vec{H}\right)$.*

Lemma 6.1 immediately implies the following corollary.

Corollary 6.1. *The following equalities are valid*

$$\frac{\partial}{\partial t_{p+1}}d_{p+1,p+1}(x_{p+1},0)|_{x_{p+1}=0}=1,$$

$$\frac{\partial}{\partial t_k}d_{p+1,k}(x_{p+1},1)|_{x_{p+1}=0}=0,\ k=1,\ldots,p,$$

$$\frac{\partial}{\partial t_{p+1}}d_{j,p+1}(x_j,0)|_{x_j=1}=0,$$

$$\frac{\partial}{\partial t_k}d_{j,k}(x_j,1)|_{x_j=1}=\delta_{jk},\ k=1,\ldots,p$$

for $j=1,\ldots,p$.

It follows that firstly, identity (6.11) holds for all $\vec{H}\in D(\Lambda_{max})$, and, the function $\vec{Z}$ satisfies relations (6.9).

Now, we form new functions

$$w_j(x_j)=y_j(x_j)-h_{p+1}(0)\frac{\partial}{\partial t_{p+1}}d_{j,p+1}(x_j,0)+\sum_{k=1}^{p}h_k(1)\frac{\partial}{\partial t_k}d_{jk}(x_j,1),\ x_j\in e_j \tag{6.12}$$

for $j=1,\ldots,r$, where $\vec{H}$ is an arbitrary function from $D(\Lambda_{max})$, and $y_j(x_j)$ are functions defined by (6.5).

Theorem 6.1. *Function $\vec{W} = \{w_j(x_j), x_j \in e_j, j = 1, \dots, r\}$, introduced by formula (6.12), is the solution of the following problem*

$$\Lambda_{max}\vec{W} = \vec{F},\ \vec{W} \in D(\Lambda_{max}), \tag{6.13}$$

$$w_{p+1}(0) = h_{p+1}(0),\ w_j(1) = h_j(1),\ j = 1, \dots, r. \tag{6.14}$$

Moreover, a solution to problem (6.13)-(6.14) is unique, i.e. a solution to problem (6.13)-(6.14) depends only on the set $\{h_{p+1}(0), h_1(1), \dots, h_p(1)\}$, but does not depend on the functions $h_j(x_j), x_j \in e_j, j = 1, \dots, r$.

Proof of Theorem 6.1. Corollary 1 implies the validity of equality (6.14). To verify equality (6.13) it is enough to recall that the Green function $d_{jk}(x_j, t_k)$ is the solution to corresponding homogeneous equation for $\vec{X} = (x_1, \dots, x_r) \neq \vec{T} = (t_1, \dots, t_r)$. Assumption 1 implies the uniqueness of a solution of problem (6.13)-(6.14) for $j = 1$. Thereby, the proof of Theorem 6.1 is complete. □

Now we show how to construct well-posed boundary problems for the equation $\Lambda_{max}\vec{Y} = \vec{F}$ by applying Theorem 6.1. It suffices to prove that $\vec{H}$ depends continuously on $\vec{F}$ in Theorem 6.1, i.e. there exists a continuous operator K mapping $\vec{F}$ belonging to $L_2(\Im)$ to $\vec{H}$ belonging to $D(\Lambda_{max})$.

So, let $\vec{H} = K\vec{F}$. Then problem (6.13)-(6.14) have the form

$$\Lambda_{max}\vec{W} = \vec{F},\ \vec{W} \in D(\Lambda_{max}), \tag{6.15}$$

$$w_j(1) = \left(K\Lambda_{max}\vec{W}\right)_j(1),\ j = 1, \dots, r,\ w_{p+1}(0) = \left(K\Lambda_{max}\vec{W}\right)_{p+1}(0). \tag{6.16}$$

Conditions (6.16) imposed on the functions $\vec{W}$ can be interpreted as additional conditions in order for equation (6.15) to have a unique solution for any right hand side $\vec{F}$. Thus, each problem (6.15)-(6.16) present a well-posed problem with new "boundary" condition (6.16). Thus, the next statement is true.

Theorem 6.2. *For all continuous operators K mapping the space $L_2(\Im)$ in $D(\Lambda_{max})$, problems (6.15)-(6.16) have unique stable solutions for all $\vec{F}$ in $L_2(\Im)$.*

The inverse statement is also true.

The restriction corresponding to the operator K from Theorem 6.2 we denote by Λ_K. Theorem 6.1 implies the proof of the direct statement of the theorem. The proof of the inverse statement is similar to that of Theorem 5 in [2].

Example. Let K be the operator in Theorem 6.2 defined by formula

$$\left(K\vec{F}\right)_j(x_j) = \sum_{s=1}^{r} \int_{e_s} d_{js}(x_j, t_s) f_s(t_s) dt_s.$$

Then the well-posed restriction $\Lambda_K \subset \Lambda_{max}$ corresponds to the boundary value problem, i.e. its domain of definition is given by the following boundary conditions

$$D(\Lambda_K) = \{\vec{Y} \in D(\Lambda_{max}) : U_j(\vec{Y}) = 0, j = 1, \dots, p+1\},$$

where $U_1(\cdot), \dots, U_{p+1}(\cdot)$ are boundary forms defined by formulas (4.6) with some scalar coefficients.

The presented scheme of M. Otelbaev for the description of well-posed restrictions is used for partial differential equations in works [2, 3].

Acknowledgments

We would like to thank the anonymous referees for their valuable comments and suggestions. This research is financially supported by the Ministry of Science and Education of the Republic of Kazakhstan (Grants no. AP05131292 and no. AP05131845).

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Received: 08.08.2017