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KHARIN STANISLAV NIKOLAYEVICH

(to the 80th birthday)

Stanislav Nikolayevich Kharin was born on December 4, 1938 in the village of Kaskelen, Alma-Ata region. In 1956 he graduated from high school in Voronezh with a gold medal. In the same year he entered the Faculty of Physics and Mathematics of the Kazakh State University and graduated in 1961, receiving a diploma with honors. After postgraduate studies he entered the Sector (since 1965 Institute) of Mathematics and Mechanics of the National Kazakhstan Academy of Sciences, where he worked until 1998 and progressed from a junior researcher to a deputy director of the Institute (1980). In 1968 he has defended the candidate thesis "Heat phenomena in electrical contacts and associated singular integral equations", and in 1990 his doctoral thesis "Mathematical models of thermo-physical processes in electrical contacts" in Novosibirsk. In 1994 S.N. Kharin was elected a corresponding member of the National Kazakhstan Academy of Sciences, the Head of the Department of Physics and Mathematics, and a member of the Presidium of the Kazakhstan Academy of Sciences.

In 1996 the Government of Kazakhstan appointed S.N. Kharin to be a co-chairman of the Committee for scientific and technological cooperation between the Republic of Kazakhstan and the Islamic Republic of Pakistan. He was invited as a visiting professor in Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, where he worked until 2001. For the results obtained in the field of mathematical modeling of thermal and electrical phenomena, he was elected a foreign member of the National Academy of Sciences of Pakistan. In 2001 S.N. Kharin was invited to the position of a professor at the University of the West of England (Bristol, England), where he worked until 2003. In 2005, he returned to Kazakhstan, to the Kazakh-British Technical University, as a professor of mathematics, where he is currently working.

Stanislav Nikolayevich paid much attention to the training of young researchers. Under his scientific supervision 10 candidate theses and 4 PhD theses were successfully defended.

Professor S.N. Kharin has over 300 publications including 4 monographs and 10 patents. He is recognized and appreciated by researchers as a prominent specialist in the field of mathematical modeling of phenomena in electrical contacts. Using models based on the new original methods for solving free boundary problems he described mathematically the phenomena of arcing, contact welding, contact floating, dynamics of contact blow-open phenomena, electrochemical mechanism of electron emission, arc-to-glow transition, thermal theory of the bridge erosion. For these achievements he got the International Holm Award, which was presented to him in 2015 in San Diego (USA).

Now he very successfully continues his research and the evidence of this in the new monograph "Mathematical models of phenomena in electrical contacts" published last year in Novosibirsk.

The mathematical community, many his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Stanislav Nikolayevich on the occasion of his 80th birthday and wish him good health, happiness and new achievements in mathematics and mathematical education.

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ON INDEX STABILITY OF NOETHERIAN DIFFERENTIAL OPERATORS IN ANISOTROPIC SOBOLEV SPACES

A. Darbinyan, A. Tumanyan

Communicated by V.I. Burenkov

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AMS Mathematics Subject Classification: 35H30, 47A53.

Abstract. In this paper index stability is studied for differential operators perturbed by lower order terms. Conditions are established under which lower order terms do not affect the index of Noetherian operators, acting in anisotropic Sobolev spaces on $\mathbb{R}^m$. Obtained result is applied to investigation of Noethericity and index of semi-elliptic operators.

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1 Introduction

This paper studies Noethericity and index stability of linear differential operators, acting in anisotropic Sobolev spaces on $\mathbb{R}^m$.

There are some main results obtained in this research area for elliptic operators acting in Sobolev spaces on compact manifolds. The equivalence of Noethericity and ellipticity is proved for operators, acting in certain Sobolev spaces on smooth compact manifolds (see [1]). Their index formula in topological terms is obtained in the work [2]. For elliptic operators in unbounded domains Noethericity is proved for the special class of operators acting in weighted Sobolev spaces in $\mathbb{R}^m$ (see [3]), and Noethericity is studied in terms of limiting operators in the work [9].

Noethericity of semi-elliptic operators with constant coefficients in $\mathbb{R}^m$ are studied in [5, 4, 6], Noethericity is proved for a class of semi-elliptic operators with variable coefficients in weighted Sobolev spaces (see [7]). A sufficient condition for index invariance on the scale of anisotropic spaces is established in [10].

2 Basic concepts and definitions

Definition 1. A bounded linear operator A, acting from a Banach space X to a Banach space Y is called Noetherian, if the following conditions are satisfied:

1. the image of the operator A is closed $\left(Im(A) = \overline{Im(A)}\right)$;
2. the kernel of the operator A is finite dimensional ($\dim Ker(A) < \infty$);
3. the cokernel of the operator A is finite dimensional ($\dim coker(A) = \dim Y/Im(A) < \infty$).

The difference between the dimensions of the kernel and the cokernel is called the index of the operator:

$$ind(A) = \dim Ker(A) - \dim coker(A).$$

Definition 2. A bounded linear operator A, acting from a Banach space X to a Banach space Y is called a Fredholmian (F-operator) if it is a Noetherian operator with $ind(A) = 0$.

Definition 3. A bounded linear operator A, acting from a Banach space X to a Banach space Y, is called normally solvable if the image of operator A is closed $\left(Im(A) = \overline{Im(A)}\right)$.

Let $\mathbb{N}$ be the set of all natural numbers, $m \in \mathbb{N}$, $\mathbb{Z}_+^m$ – the set of all m-dimensional multi-indices, $\mathbb{N}^m$ – the set of m-dimensional multi-indices with natural components, $\mathbb{R}^m$ – the m-dimensional Euclidean space.

Denote

$$Q := \left\{ g \in C^\infty(\mathbb{R}^m) : g(x) > 0, \forall x \in \mathbb{R}^m; \ \sup_{x \in \mathbb{R}^m} \frac{|D^\beta g(x)|}{g(x)} < \infty, \forall \beta \in \mathbb{Z}_+^m \right\},$$

and

$$\widetilde{Q} := \{ \, g \in C^\infty(\mathbb{R}^m) : g(x) > 0, \forall x \in \mathbb{R}^m; \ \frac{|D^\beta g(x)|}{g(x)} \rightrightarrows 0 \text{ as } |x| \to \infty, \\ \forall \beta \in \mathbb{Z}_+^m, \beta \neq 0 \, \}.$$

For $k \in \mathbb{Z}_+$ and $\nu \in \mathbb{N}^m$ denote

$$C^{k,\nu}(\mathbb{R}^m) := \{ \, a : D^\beta a \in C(\mathbb{R}^m), \\ \sup_{x \in \mathbb{R}^m} |D^\beta a(x)| < \infty, \forall \beta \in \mathbb{Z}_+^m \text{ such that } (\beta : \nu) \equiv \frac{\beta_1}{\nu_1} + \cdots + \frac{\beta_n}{\nu_n} \leq k \, \}.$$

Definition 4. For $k \in \mathbb{Z}_+, \nu \in \mathbb{N}^m$ denote by $H^{k,\nu}(\mathbb{R}^m)$ the space of all measurable functions u for which the norm

$$\|u\|_{k,\nu} = \left(\sum_{(\alpha:\nu) \leq k} \int |D^\alpha u(x)|^2 \, dx \right)^{\frac{1}{2}} < \infty.$$

Hereinafter, for $\nu \in \mathbb{N}^m, \nu_1 = \cdots = \nu_m = 1$ the space $H^{k,\nu}(\mathbb{R}^m)$ will be denoted by $H^k(\mathbb{R}^m)$.

Definition 5. For $k \in \mathbb{Z}_+, \nu \in \mathbb{N}^m$ and a positive-valued function q denote by $H_q^{k,\nu}(\mathbb{R}^m)$ the space of all measurable functions u for which the norm

$$\|u\|_{k,\nu,q} = \left(\sum_{(\alpha:\nu) \leq k} \int \left| D^\alpha u(x) \, q(x)^{k-(\alpha:\nu)} \right|^2 dx \right)^{\frac{1}{2}} < \infty.$$

Definition 6. For $k \in \mathbb{Z}_+, \nu \in \mathbb{N}^m$ and $r \in Q$ denote by $\widetilde{H}_r^{k,\nu}(\mathbb{R}^m)$ the space of all functions u such that $ru \in H^{k,\nu}(\mathbb{R}^m)$ equipped with the norm

$$\|u\|_{k,\nu,r}' = \|ru\|_{k,\nu}.$$

Consider $k, s \in \mathbb{N}$ such that $k \geq s$.

Let

$$P(x, \mathbb{D}) = \sum_{(\alpha:\nu) \leq s} a_\alpha(x) D^\alpha, \tag{2.1}$$

where $\alpha \in \mathbb{Z}_+^m$, $\nu \in \mathbb{N}^m$, $D^\alpha = D_1^{\alpha_1} ... D_m^{\alpha_m}$, $D_j = i^{-1} \frac{\partial}{\partial x_j}, x = (x_1, ..., x_m) \in \mathbb{R}^m, a_\alpha \in C^{k-s,\nu}(\mathbb{R}^m)$.

Denote by

$$P_s(x,\mathbb{D}) = \sum_{(\alpha:\nu)=s} a_\alpha(x) D^\alpha, \tag{2.2}$$

the principal part of $P(x,\mathbb{D})$ and by

$$P_s(x,\xi) = \sum_{(\alpha:\nu)=s} a_\alpha(x) \xi^\alpha, \tag{2.3}$$

its symbol.

With the specified conditions on the coefficients of the differential form $P(x,\mathbb{D})$ it originates a bounded linear operator acting from $H^{k,\nu}(\mathbb{R}^m)$ to $H^{k-s,\nu}(\mathbb{R}^m)$. Notation $\left(P; H^{k,\nu}\right)$ will be used for it.

As $r \in Q$ the differential form $P(x,\mathbb{D})$ originates a bounded linear operator, acting from $\widetilde{H}_r^{k,\nu}(\mathbb{R}^m)$ to $\widetilde{H}_r^{k-s,\nu}(\mathbb{R}^m)$. Denote it by $\left(P; \widetilde{H}_r^{k,\nu}\right)$.

For a positive-valued function q, such that $\frac{1}{q(x)} \rightrightarrows 0$ as $|x| \to \infty$, the differential form $P(x,\mathbb{D})$ originates a bounded linear operator, acting from $H_q^{k,\nu}(\mathbb{R}^m)$ to $H_q^{k-s,\nu}(\mathbb{R}^m)$. Denote it by $\left(P; H_q^{k,\nu}\right)$.

Definition 7. A differential expression $P(x,\mathbb{D})$ of the form (2.1) is called semi-elliptic at a point $x = x_0$, if

$$P_s(x_0,\xi) \neq 0, \forall \xi \in \mathbb{R}^m, |\xi| \neq 0.$$

Definition 8. A differential expression $P(x,\mathbb{D})$ of the form (2.1) is called semi-elliptic in $\mathbb{R}^m$ or just semi-elliptic, if it is semi-elliptic at each point $x \in \mathbb{R}^m$.

3 Main results

Let $r \in Q$ and M_r be the operator of multiplication by r :

$$M_r : \widetilde{H}_r^{k,\nu}(\mathbb{R}^m) \to H^{k,\nu}(\mathbb{R}^m),\ (M_r u)(x) = r(x)u(x),\ \forall u \in \widetilde{H}_r^{k,\nu}(\mathbb{R}^m), \forall x \in \mathbb{R}^m$$

and

$$M_r^{-1} : H^{k,\nu}(\mathbb{R}^m) \to \widetilde{H}_r^{k,\nu}(\mathbb{R}^m),\ (M_r^{-1} v)(x) = \frac{v(x)}{r(x)},\ \forall v \in H^{k,\nu}(\mathbb{R}^m), \forall x \in \mathbb{R}^m.$$

Consider

$$P_r := M_r P(x,\mathbb{D}) M_r^{-1}.$$

As $\sup_{x\in\mathbb{R}^m} \frac{|D^\beta r(x)|}{r(x)} < \infty$ for all $\beta \in \mathbb{Z}_+^m$, we get that P_r originates a bounded linear operator acting from $H^{k,\nu}(\mathbb{R}^m)$ to $H^{k-s,\nu}(\mathbb{R}^m)$. Denote it by $\left(P_r; H^{k,\nu}\right)$.

By considering the properties of weighted spaces and the Noethericity conditions, the following statements can be proven.

Lemma 3.1. *An operator $\left(P_r; H^{k,\nu}\right)$ is a Noetherian operator if and only if $\left(P; \widetilde{H}_r^{k,\nu}\right)$ is Noetherian, and the following equalities hold:*

$$\dim Ker\left(P_r; H^{k,\nu}\right) = \dim Ker\left(P; \widetilde{H}_r^{k,\nu}\right),$$
$$\dim coker\left(P_r; H^{k,\nu}\right) = \dim coker\left(P; \widetilde{H}_r^{k,\nu}\right),$$
$$ind\left(P_r; H^{k,\nu}\right) = ind\left(P; \widetilde{H}_r^{k,\nu}\right).$$

Corollary 3.1. *Let $r \in \widetilde{Q}$. Then an operator $\left(P; H^{k,\nu}\right)$ is a Noetherian operator if and only if $\left(P; \widetilde{H}_r^{k,\nu}\right)$ is Noetherian, and the following equalities hold:*

$$\dim Ker\left(P; H^{k,\nu}\right) = \dim Ker\left(P; \widetilde{H}_r^{k,\nu}\right),$$
$$\dim coker\left(P; H^{k,\nu}\right) = \dim coker\left(P; \widetilde{H}_r^{k,\nu}\right),$$
$$ind\left(P; H^{k,\nu}\right) = ind\left(P; \widetilde{H}_r^{k,\nu}\right).$$

Lemma 3.2. *Let $q \in \widetilde{Q}$ be such that $\frac{1}{q(x)} \rightrightarrows 0$ as $|x| \to \infty$. Let $\left(P; H^{k,\nu}\right)$ be a Noetherian operator and $\left(P; H_q^{k,\nu}\right)$ be normally solvable. Then $\left(P; H_q^{k,\nu}\right)$ is also Noetherian with*

$$\dim Ker\left(P; H_q^{k,\nu}\right) = \dim Ker\left(P; H^{k,\nu}\right),$$
$$\dim coker\left(P; H_q^{k,\nu}\right) = \dim coker\left(P; H^{k,\nu}\right),$$
$$ind\left(P; H_q^{k,\nu}\right) = ind\left(P; H^{k,\nu}\right).$$

Denote by

$$T\left(x, \mathbb{D}\right) = \sum_{(\alpha:\nu)<s} b_\alpha(x) D^\alpha, \tag{3.1}$$

a differential form containing only lower order terms, where the same notations are used as in Section 2 and $b_\alpha \in C^{k-s,\nu}\left(\mathbb{R}^m\right)$.

For a positive-valued function q such that $\frac{1}{q(x)} \rightrightarrows 0$ as $|x| \to \infty$ it is easy to check that $T\left(x, \mathbb{D}\right)$ originates a bounded linear operator, acting from $H_q^{k,\nu}(\mathbb{R}^m)$ to $H_q^{k-s,\nu}(\mathbb{R}^m)$.

Lemma 3.3. *Let $q \in \widetilde{Q}$ be such that $\frac{1}{q(x)} \rightrightarrows 0$ as $|x| \to \infty$. Then the operator $T\left(x, \mathbb{D}\right)$ is a compact operator acting from $H_q^{k,\nu}(\mathbb{R}^m)$ to $H_q^{k-s,\nu}(\mathbb{R}^m)$.*

Denote

$$\widetilde{P}(x, \mathbb{D}) = P(x, \mathbb{D}) + T\left(x, \mathbb{D}\right).$$

Theorem 3.1. *Let $q \in \widetilde{Q}$ be such that $\frac{1}{q(x)} \rightrightarrows 0$ as $|x| \to \infty$. Let operators $\left(P; H^{k,\nu}\right)$ and $\left(\widetilde{P}; H^{k,\nu}\right)$ be Noetherian operators, and operator $\left(P; H_q^{k,\nu}\right)$ be normally solvable. Then the following equality holds:*

$$ind\left(\widetilde{P}; H^{k,\nu}\right) = ind\left(P; H^{k,\nu}\right).$$

Proof. It is easy to see that Lemma 3.2 can be applied to $\left(P; H^{k,\nu}\right)$, and we get that $\left(P; H_q^{k,\nu}\right)$ is also a Noetherian operator with $ind\left(P; H^{k,\nu}\right) = ind\left(P; H_q^{k,\nu}\right)$.

From Lemma 3.3 we have that $T\left(x, \mathbb{D}\right)$, acting from $H_q^{k,\nu}(\mathbb{R}^m)$ to $H_q^{k-s,\nu}(\mathbb{R}^m)$ is a compact operator. So we have that $\left(\widetilde{P}; H_q^{k,\nu}\right)$ is also a Noetherian operator and $ind\left(\widetilde{P}; H_q^{k,\nu}\right) = ind\left(P; H_q^{k,\nu}\right)$ (see [8] 8.5.20). Taking into consideration the Noethericity of $\left(\widetilde{P}; H^{k,\nu}\right)$ Lemma 3.2 can be applied to it. So we have:

$$ind\left(\widetilde{P}; H^{k,\nu}\right) = ind\left(\widetilde{P}; H_q^{k,\nu}\right) = ind\left(P; H_q^{k,\nu}\right) = ind\left(P; H^{k,\nu}\right).$$

□

Remark 1. *In general, lower order terms of differential operator can affect Noethericity. In the case, when the Noethericity is preserved, the index of an operator, perturbed by lower order terms, can also change. Thus, the conditions in Theorem 3.1 are essential.*

The following example demonstrates it.

Let a be a positive real number. Consider the operators $P_1 u = u'' + au$ and $P_2 u = u'' - au$ acting from $H^2(\mathbb{R}^1)$ to $L_2(\mathbb{R}^1)$. It is easy to check that $P_2 : H^2(\mathbb{R}^1) \to L_2(\mathbb{R}^1)$ is Noetherian with $\dim Ker(P_2) = \dim coker(P_2) = ind(P_2) = 0$ and $P_1 : H^2(\mathbb{R}^1) \to L_2(\mathbb{R}^1)$ is not Noetherian ($\dim Ker(P_1) = \dim Ker(P_1^*) = 0$, but $Im(P_1) \neq \overline{Im(P_1)}$).

This shows that lower order terms of differential operator can affect the Noethericity.

Let $\delta \in \mathbb{R}_+$. Denote $r_\delta(x) = e^{-\delta\sqrt{1+x^2}} \in Q$.

Consider

$$P_{1,r_\delta} u = e^{-\delta\sqrt{1+x^2}} P_1 \left(e^{\delta\sqrt{1+x^2}} u \right) = u'' + 2\delta \frac{x}{\sqrt{1+x^2}} u' + \\ + \left(\delta^2 \frac{x^2}{1+x^2} + \delta \frac{1}{(1+x^2)^{3/2}} + a \right) u.$$

It can be checked that the bounded linear operator $P_{1,r_\delta} : H^2(\mathbb{R}^1) \to L_2(\mathbb{R}^1)$ is a Noetherian operator (see [9], theorem 4.1) and the following equalities hold:

$$Ker\,(P_{1,r_\delta}) = Span\{\cos(\sqrt{a}x) e^{-\delta\sqrt{1+x^2}}; \sin(\sqrt{a}x) e^{-\delta\sqrt{1+x^2}}\},\ \dim Ker\,(P_{1,r_\delta}) = 2,$$

$$Ker\,\left(P^*_{1,r_\delta}\right) = \{0\},\ \dim coker\,(P_{1,r_\delta}) = 0.$$

So it is obtained that $ind\,(P_{1,r_\delta}) = 2$.

Thus, this example shows that lower order terms can affect the index value of Noetherian operators, acting in Sobolev spaces of functions defined on $\mathbb{R}^m$.

Remark 2. *If the index value is preserved under perturbation by lower order terms of a differential operator, the dimensions of kernel and cokernel can be affected.*

Consider the operator $P_3 u = u'' + \frac{2 - x^2\sqrt{1+x^2}}{4(1+x^2)^{\frac{3}{2}}} u$, acting from $H^2(\mathbb{R}^1)$ to $L_2(\mathbb{R}^1)$. It can be shown that $P_3 : H^2(\mathbb{R}^1) \to L_2(\mathbb{R}^1)$ is a Noetherian operator and

$$Ker\,(P_3) = Span\{e^{-\frac{1}{2}\sqrt{1+x^2}}\},\ \dim Ker\,(P_3) = \dim coker\,(P_3) = 1, \text{so } ind\,(P_3) = 0.$$

For $P_2 : H^2(\mathbb{R}^1) \to L_2(\mathbb{R}^1)$ from the previous example we have that $\dim Ker(P_2) = \dim coker(P_2) = ind(P_2) = 0$.

So $ind\,(P_3) = ind(P_2) = 0$, but the dimensions of kernel and cokernel are affected by lower order terms of differential operator.

Let

$$L_s\,(\mathbb{D}) = \sum_{(\alpha:\nu)=s} a_\alpha D^\alpha, \tag{3.2}$$

where coefficients a_α are real numbers and the same notations are used as in Section 2.

Consider $L(x, \mathbb{D}) = L_s(\mathbb{D}) + T(x, \mathbb{D})$ (see (3.1)). It generates a bounded linear operator acting from $H^{k,\nu}(\mathbb{R}^m)$ to $H^{k-s,\nu}(\mathbb{R}^m)$ (it is denoted by $\left(L; H^{k,\nu}\right)$). For a positive-valued function q, which satisfies $\frac{1}{q(x)} \rightrightarrows 0$ as $|x| \to \infty$, $L(x, \mathbb{D})$ originates a bounded linear operator acting from $H^{k,\nu}_q(\mathbb{R}^m)$ to $H^{k-s,\nu}_q(\mathbb{R}^m)$ (denoted by $\left(L; H^{k,\nu}_q\right)$).

Theorem 3.2. *Let $q \in \widetilde{Q}$ be such that $\frac{1}{q(x)} \rightrightarrows 0$ as $|x| \to \infty$. Let $\left(L; H^{k,\nu}\right)$ be a semi-elliptic Noetherian operator and $\left(L; H^{k,\nu}_q\right)$ be normally solvable. Then $\left(L; H^{k,\nu}\right)$ is a Fredholmian operator.*

Proof. By applying Lemma 3.2 we have that $\left(L; H_q^{k,\nu}\right)$ is also a Noetherian operator and $ind\left(L; H^{k,\nu}\right) = ind\left(L; H_q^{k,\nu}\right)$. Then due to the semi-ellipticity of $L(x,\mathbb{D})$ and the fact that the coefficients a_α of its principal part are real-valued constants, there exists such c_0 that $L(x,\mathbb{D})$ can be represented as

$$L(x,\mathbb{D}) = L^1(\mathbb{D}) + L^2(x,\mathbb{D}),$$

where $L^1(\mathbb{D}) = L_s(\mathbb{D}) + c_0, L^1(\xi) \neq 0,$ for all $\xi \in \mathbb{R}^m$ and $L^2(x,\mathbb{D}) = T(x,\mathbb{D}) - c_0$.

From Lemma 3.3 we get that $\left(L^2; H_q^{k,\nu}\right)$ is a compact operator. It follows that $\left(L^1; H_q^{k,\nu}\right)$ is a Noetherian operator and $ind\left(L; H_q^{k,\nu}\right) = ind\left(L^1; H_q^{k,\nu}\right)$ (see [8], 8.5.20). In [5] it is proven that $L^1(\mathbb{D}) : H^{k,\nu}(\mathbb{R}^m) \to H^{k-s,\nu}(\mathbb{R}^m)$ is a Noetherian operator and $ind\left(L^1; H^{k,\nu}\right) = 0$. Lemma 3.2 can be applied to $L^1(\mathbb{D})$ and we get

$$ind\left(L^1; H_q^{k,\nu}\right) = ind\left(L^1; H^{k,\nu}\right) = 0.$$

So we obtain

$$ind\left(L; H^{k,\nu}\right) = ind\left(L; H_q^{k,\nu}\right) = ind\left(L^1; H_q^{k,\nu}\right) = ind\left(L^1; H^{k,\nu}\right) = 0.$$

□

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