ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2018, Volume 9, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http : //publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 515
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

SULTANAEV YAUDAT TALGATOVICH

(to the 70th birthday)

On 19th July 2018 was 70th birthday of Yaudat Talgatovich Sultanaev, doctor of science (1990), professor (1991), honorary scientist of the Russian Federation, laureate of State award of the Republic of Bashkortostan in the field of science and technology, professor of the Bashkir State Pedagogical University, member of the Editorial Board of the Eurasian Mathematical Journal.

Ya.T. Sultanaev was born in the sity of Orsk. In 1971 he graduated from the Bashkir State University and then completed his postgraduate studies in the Moscow State University. Ya.T. Sultanaev's scientific supervisors were distinguished mathematicians A.G. Kostyuchenko and B.M. Levitan.

Ya.T. Sultanaev is a famous specialist in the spectral theory of differential operators and the qualitative theory of ordinary differential equations.

He obtained bilateral Tauberian theorems of Keldysh type, completely solved the problem on spectral assymptotics for semi-bounded ordinary differential operators, suggested a new method of investigation of assymptotic behaviour of solutions to singular differential equations which allowed him to essentially weaken the conditions on coefficients.

Jointly with V.A. Sadovnichii and A.M. Akhtyamov, he investigated inverse spectral problems with non-separated boundary conditions.

He published more than 70 papers in leading mathematical journals.

Among pupils of Ya.T. Sultanaev there are more than 20 candidates of science and one doctor of science.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yaudat Talgatovich on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 9, Number 4 (2018), 91 – 98

ON COMMUTATIVITY OF CIRCULARLY ORDERED C-O-STABLE GROUPS

V.V. Verbovskiy

Communicated by J.A. Tussupov

Key words: circularly ordered group, o-minimality, commutative group, o-stability.

AMS Mathematics Subject Classification: 03B10, 03C52, 03C60, 03C64.

Abstract. A circularly ordered structure is called c-o-stable in λ, if for any subset A of cardinality at most λ and for any cut s there exist at most λ one-types over A that are consistent with s. A theory is called c-o-stable if there exists an infinite λ such that all its models are c-o-stable in λ. In the paper, it is proved that any circularly ordered group, whose elementary theory is c-o-stable, is Abelian.

DOI: https://doi.org/10.32523/2077-9879-2018-9-4-91-98

1 Introduction

If in a linearly ordered set N with the minimal and maximal elements we glue the minimal element with the maximal one, we obtain a circular order, i.e. an order on a circle. So, recall that *a circular order* relation on N is described as a ternary relation K satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x, y, z) \to K(y, z, x))$;
(co2) $\forall x \forall y \forall z (K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;
(co3) $\forall x \forall y \forall z (K(x, y, z) \to \forall t [K(x, y, t) \vee K(t, y, z)])$;
(co4) $\forall x \forall y \forall z (K(x, y, z) \vee K(y, x, z))$.

It is assumed that all x, y, z, $t \in N$. The pair (N, K) is called a *circular ordering.*

The relation $K_0(x, y, z)$ is defined as follows: $K(x, y, z) \wedge x \neq y \wedge y \neq z \wedge z \neq x$.

We say that $K(u_1, \dots, u_n)$ denotes a formula if all subtriples of $u_1, \dots, u_n$ (in increasing order) satisfy K; likewise with K_0 in place of K.

It is possible to connect linear and circular orderings as follows.

Fact 1.1. ([2], Theorem 11.9) *(i) If $\langle M, \leq \rangle$ is a linear ordering and K is a ternary relation obtained from $\leq$ by the rule*

$$K(x, y, z) \triangleq (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x),$$

then K is a circular order relation on M.

(ii) If $\langle N, K \rangle$ is a circular ordering and $a \in N$ then the relation $\leq_a$, defined on $M := N \setminus \{a\}$ by the rule

$$y \leq_a z :\Leftrightarrow K(a, y, z)$$

is a linear ordering. Furthermore, if we extend this linear ordering to the ordering denoted by $\leq'$, on N, adding that $a \leq' b$ for all $b \in M$ then the derived circular order relation is the original circular order relation K.

A subset A of a circularly ordered structure $\mathcal{N} = \langle N, =, K, \ldots \rangle$ is said to be *convex* if for any elements a and $b \in A$ either any element of $K(a, N, b)$ is contained in A or any element of $K(b, N, a)$ is contained in A. A maximal convex subset of a set A is said to be *a convex component* of the set A.

Recall that a group G having a linear order relation $<$ is said to be *linearly ordered* if for any elements a, b and c the inequality $a < b$ implies both $ac < bc$ and $ca < cb$. We say that a group G is *linearly orderable* if there exists a linear ordering of the set of elements of G with regards to which G is a linearly ordered group. If a group G has a circular order relation K then it is said to be *circularly ordered* if for any elements a, b, c and d the relation $K(a, b, c)$ implies both $K(ad, bd, cd)$ and $K(da, db, dc)$. It is easy to see that a linearly ordered group is a circularly ordered group if a circular ordering is defined as in Fact 1.1.

Natural examples of cyclically ordered non-linearly ordered groups are non-zero subgroups of the multiplicative group $\mathbf{S}^1 = \langle \{z \in \mathbb{C} \mid |z| = 1\}, K, \cdot, 1 \rangle$ of complex numbers moduluses of which are equal to 1 containing elements of finite order. Indeed, the multiplication in the group is a turning of the unit circumference, and any turning cannot change a mutual location of three elements. Since a linearly ordered group is a torsion free group, the considered groups are not linearly orderable.

Recall that a linearly ordered structure $\mathcal{M} = \langle M, <, \ldots \rangle$ is *weakly o-minimal* if any parametrically definable set is a finite union of convex sets. In [5] it had been proved that weakly o-minimal ordered groups are Abelian and divisible.

The following notion has been introduced and originally studied in [3]. A circularly ordered structure $\mathcal{M} = \langle M, K, \ldots \rangle$ is *weakly circularly minimal* if any parametrically definable set is a finite union of convex sets. Recall that such a structure $\mathcal{M} = \langle M, K, \ldots \rangle$ is *circularly minimal* if any parametrically definable set is a finite union of intervals and points [6]. Thus, the weak circular minimality is a generalization of the circular minimality.

In [6] D. Macpherson and Ch. Steinhorn described circularly minimal circularly ordered groups. In [3] B. Kulpeshov and D. Macpherson started studying weakly circularly minimal circularly ordered structures. In [4] B. Kulpeshov and V. Verbovskiy proved that any weakly circularly minimal circularly ordered group is Abelian and that they need not be divisible. Here our aim is to study c-o-stable circularly ordered groups and to prove that they are commutative.

Let $\varphi(x; y, z) \triangleq K_0(y, x, z)$. A complete φ-type over a circularly ordered structure is called *a cut* in this structure.

Definition 1. A circularly ordered structure $\mathcal{M} = (M, K, \ldots)$ is said to be *c-o-stable in a cardinality* λ if for any subset $A \subseteq M$ with $|A| \le \lambda$ and any cut s in M there exist at most λ one-types over the set A which are consistent with the cut s.

A theory is *c-o-stable in* λ if each of its model is.

A theory is *c-o-stable* if there exists an infinite cardinal λ in which the theory is c-o-stable.

This definition is similar to the next one.

Definition 2. [1] A linearly ordered structure $\mathcal{M} = (M, <, \ldots)$ is said to be *o-stable in a cardinality* λ if for any subset $A \subseteq M$ with $|A| \le \lambda$ and any cut s in M there exist at most λ one-types over the set A which are consistent with the cut s.

A theory is *o-stable in* λ if each of its model is.

A theory is *o-stable* if there exists an infinite cardinal λ in which the theory is o-stable.

These two definitions are partial cases of definition of a stable up to Δ theory, which was introduced in [9].

Definition 3. [9] Let $\mathcal{M}$ be an arbitrary structure, $A \subseteq M$. Let Δ and ∇ be sets of formulae of the form $\varphi(x; \bar{y})$.

1. The model $\mathcal{M}$ is *stable up to* Δ in (λ, ∇) if for all $A \subseteq M$ with $|A| \leq \lambda$, for any Δ-type p over M there are at most λ ∇-types over A which are consistent with p, i.e. $|S^1_{\nabla,p}(A)| \leq \lambda$.

2. The theory T is *stable up to* Δ in (λ, ∇) if every model of T is. Sometimes we write that T is (λ, ∇)-stable up to Δ.

3. If $\nabla = \mathcal{L}$ we omit it and write that T is stable in λ or λ-stable up to Δ.

4. T is *stable up to* Δ if there exists a λ in which T is stable up to Δ. We write that T is *stable up to* φ meaning that T is stable up to $\Delta = \{\varphi\}$.

Lemma 1.1. *An o-stable ordered structure is a c-o-stable circularly ordered structure under the circular order* $K(x, y, z) \triangleq (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x)$.

Proof. Let $\mathcal{M} = (M, <, \dots)$ be an o-stable in λ ordered structure.

Let s_K be a cut in the sense of K, that is let $s_K(x)$ be a complete Δ-type over M, where $\Delta = \{K_0(x, y, z)\}$. Note that here we use K_0 because we are interested only in non-algebraic types. Assume that for some $a < b$ in M it holds that $K_0(x, b, a) \in s_K$. Then by definition $K_0(x, b, a)$ is equivalent to

$$(x < b < a) \vee (a < x < b) \vee (b < a < x),$$

which can be reduced to $a < x < b$, because the other parentheses are false.

Now we define a cut in the sense of $\leq$:

$$s_{\leq}(x) = \{c < x,\ x < d : c, d \in M,\ c < d,\ K(x, d, c) \in s_K\}$$

We show that $s_{\leq}(x)$ is a complete Δ-type over M, where $\Delta = \{x < z,\ y < x\}$.

If $c < a$, then obviously $c < x \in s_{\leq}(x)$. If $b < c$, then $x < b \in s_{\leq}(x)$. Let $c \in (a, b)$. By definition of K_0 it holds that $K_0(a, c, b)$ is true. By Axiom (co4) either $K_0(x, c, a) \in s_K(x)$, or $K_0(x, b, c) \in s_K(x)$, because $s_K(x)$ is complete. Note that $K_0(x, c, a)$ is equivalent to $a < x < c$, and $K_0(x, b, c)$ is equivalent to $c < x < b$. So, either $x < c$ belongs to $s_{\leq}(x)$, or $c < x$ belongs to $s_{\leq}(x)$.

Moreover, $s_{\leq}(x) \vdash s_K(x)$ as well as $s_K(x) \vdash s_{\leq}(x)$. So, any one-type over a set A of cardinality at most λ, which is consistent with $s_K(x)$, is also consistent with $s_{\leq}(x)$. Since $\mathcal{M}$ is o-stable in λ, the cardinality of the set of all one-types over A, which are consistent with $s_{\leq}(x)$ is at most λ. Then $(M, K, \dots)$ is c-o-stable in λ.

Now assume that for any $a < b$ the formula $K_0(x, b, a)$ does not belong to $s_K(x)$. By Axiom (co4) $K_0(x, a, b)$ belongs to $s_K(x)$. By definition it is equivalent to

$$(x < a < b) \vee (b < x < a) \vee (a < b < x),$$

which can be reduced to $(x < a) \vee (b < x)$.

Now let $s_+(x)$ be the cut $+\infty$, that is for any $a \in M$ the formula $a < x$ belongs to $s_+(x)$, and let $s_-(x)$ be the cut $-\infty$, that is for any $a \in M$ the formula $x < a$ belongs to $s_-(x)$.

In this case the set of all realizations of $s_K(x)$ in any elementary extension is equal to the union of the sets of all realizations of $s_-(x)$ and $s_+(x)$. Since each of $s_-(x)$ and $s_+(x)$ has at most λ complete one-type over A which are consistent with them, so $s_K(x)$ has at most $\lambda + \lambda = \lambda$ complete one-type over A which are consistent with it, that is why $(M, K, \dots)$ is c-o-stable in λ. $\square$

The proof of the next lemma is similar to the proof of Lemma 1.1: one can easily show that each cut in the sense of $\leq$ defines a unique cut in the set of K, so the number of types which are consistent with a cut in the sense of $\leq$ is equal to the number of types which are consistent with the corresponding cut in the sense of K.

Lemma 1.2. *Assume that $\mathcal{M} = (M, K, \dots)$ is a c-o-stable circularly ordered structure. Then for any element $a \in M$ the structure $(M, \leq, \dots)$ is linearly ordered for the relation $x \leq y$ defined by the formula $K(a, x, y)$ and as a linearly ordered structure is o-stable. If K is also a linear order, then $(M, \leq, \dots)$ is o-stable.*

Let T be a theory of a language $\mathcal{L}$, and $\mathcal{M} \prec \mathcal{N}$ two models of T such that $\mathcal{N}$ is $|M|^+$-saturated. For any formula $\varphi(\bar{x}, \bar{\alpha})$ with the parameters $\bar{\alpha}$ in N we add a new relational symbol $P_{\phi(\bar{x},\bar{\alpha})}(\bar{x})$ interpreted by $P_{\phi(\bar{x},\bar{\alpha})}(M) = \phi(N, \bar{\alpha}) \cap M^k$ in order to form language $\mathcal{L}^*$. The set $\phi(N, \bar{\alpha}) \cap M^k$ is said to be *externally definable.*

Fact 1.2. [8] *Let T be an o-stable theory of a language $\mathcal{L}$, and $\mathcal{M} \prec \mathcal{N}$ two models of T such that $\mathcal{N}$ is $|M|^+$-saturated. Then the elementary theory T^* of the expansion $\mathcal{M}^*$ of $\mathcal{M}$ is o-stable.*

As a direct corollary of Lemma 1.2 and of Fact 1.2 we obtain the following theorem.

Theorem 1.1. *Let T be an c-o-stable theory of a language $\mathcal{L}$, and $\mathcal{M} \prec \mathcal{N}$ two models of T such that $\mathcal{N}$ is $|M|^+$-saturated. Then the elementary theory T^* of the expansion $\mathcal{M}^*$ of $\mathcal{M}$ is c-o-stable.*

Since the union of an increasing chain of groups is a group, we can determine the subgroup G^c of G as the union of all proper convex subgroups of G. Observe that the subgroup G^c is linearly orderable. Also we can observe that the subgroup G^c is not necessarily definable. Indeed, take the multiplicative group $\mathbf{S}^1$ of complex numbers moduluses of which are equal to 1 and realize a type of infinitesimal elements. Then the infinitesimal elements will form the subgroup G^c which obviously is not definable.

Fact 1.3. [4] *If $G = G^c$ then G is linearly orderable. Moreover, it will be linearly ordered by the following ordering: $x \leq y \triangleq P(x^{-1}y)$, where $P(x) \triangleq K(1, x, x^2)$.*

Now we recall the basic fact on o-stable ordered groups, that they are commutative.

Fact 1.4. [8] *Any ordered group, whose elementary theory is o-stable, is Abelian.*

Lemma 1.3. *Let $\mathcal{G} = (G, K, \cdot, \dots)$ be a circularly ordered group, whose elementary theory is c-o-stable. If $G = G^c$, then G is Abelian.*

Proof. If $G = G^c$ then by Fact 1.3 the group G is ordered relatively the relation $x \leq y \triangleq P(x^{-1}y)$, where $P(x) \triangleq K(1, x, x^2)$. By Lemma 1.2 G is o-stable. So, by Fact 1.4 it is Abelian. □

From now on we consider a circularly ordered group $\mathcal{G}$ whose elementary theory is c-o-stable.

It follows from Lemma 1.3 that if we assume that G is not Abelian then $G \neq G^c$, so G is not linearly orderable and G^c is a maximal convex linearly orderable subgroup.

Let $a \notin G^c$. It is easy to see that G^c is linearly ordered by $x \leq y \triangleq K(a, x, y)$. So, the elementary theory of G^c with the full induced structure is o-stable by Lemma 1.2. Then G^c is Abelian.

We note that the property of being the maximal convex proper subgroup is preserved under group automorphisms preserving circular ordering, so for any inner automorphism τ it holds that $\tau(G^c) = G^c$. Recall that $\tau(g) = h^{-1}gh$ for some $h \in G$. So, the subgroup G^c is a normal subgroup of G.

Now we may consider the quotient group G/G^c. Obviously, this group does not contain non-trivial convex subgroups. The following assertion follows from the results in [7]:

Lemma 1.4. *The quotient-group G/G^c as a cyclically ordered group is isomorphically embedded into the multiplicative group of complex numbers moduluses of which are equal to 1. As a corollary, it is Abelian.*

So, for the moment we have proved that G is metabelian.

Now we need some facts from [9].

Let Δ be a family of formulae of the form $\psi(x;\bar{y})$.

Definition 4. [9] A formula $\varphi(\bar{x};\bar{y})$ has *the order property over B inside a partial type $s(\bar{x})$* over a set A if there are sequences $\bar{a}_n$ and $\bar{b}_n$ for $n \in \omega$ such that $\bar{a}_n \models s(\bar{x})$ and $\bar{b}_n \in B$ for each $n < \omega$ and $\varphi(\bar{a}_n, \bar{b}_m)$ holds if and only if $n \leq m$.

A theory T has *the order property in spite of* Δ if there is a model $\mathcal{M}$ of T and a Δ-type $s(\bar{x})$ over M such that some formula $\varphi(\bar{x};\bar{y})$ has the order property over M inside the type s.

Similarly one can define *the strict order property inside a partial type* and *the strict order property in spite of* Δ, *the independence property inside a partial type*, and *the independence order property in spite of* Δ.

Fact 1.5. [9] *A theory T is stable up to Δ if and only if T does not have the order property in spite of Δ.*

In the further reasoning we shall use the following two facts from [4].

Fact 1.6. *Let G be a cyclically ordered group, and the subgroup G^c is abelian. Suppose that there are $g \in G$ and a positive integer n such that $g^n \in G^c$. Then the centralizer $C(g)$ contains G^c and, as a corollary, $g^k G^c$ for any integer k.*

Fact 1.7. *Let G be a cyclically ordered group, and the subgroup G^c is abelian. Suppose that there are $g_1, g_2 \in G$ such that both $g_1 G^c$ and $g_2 G^c$ have a finite order in the quotient-group G/G^c. Then $g_1 g_2 = g_2 g_1$.*

Now we prove a simple lemma.

Lemma 1.5. *Let H and $K \leq G$ be such that some coset aH is a subset of K. Then $H \leq K$.*

Proof. Let $aH \subseteq K$ and $h \in H$. Then ah and $a \in K$, so $h \in K$. $\square$

Lemma 1.6. *Let G be a circularly ordered group whose elementary theory is c-o-stable. If the centralizer $C(g)$ of some element $g \in G$ has non-empty intersections with infinitely many cosets of G^c, then $G^c \subseteq C(g)$.*

Proof. Note that the element gG^c of the quotient group G/G^c has the infinite order, so without loss of generality we may assume that there exists a sequence $\langle b_n : n < \omega \rangle$ of elements in $C(g)$ such that $K(e, b_0, b_1, \ldots, b_n)$ holds for each n and $b_i \notin b_j G^c$ for any $i < j < \omega$. Let the cut $s(x)$ be defined as $\sup\{b_n : n < \omega\}$ taking $x \leq y$ as $K(e, x, y)$. Then in some saturated model $\mathcal{N}$ it holds that:

1) if $s(a)$ holds, then both $s(ag)$ and $s(ga)$ hold for any element $g \in G^c$;

2) $s(\mathcal{N})$ contains elements as from $C(g)$ as not from $C(g)$, because of Lemma 1.5.

Let a formula $F(x, h)$ say that there exists an element $y \in C(g)$, such that $K(1, xy^{-1}, h)$, that is $x \in \bigcup_{y \in [1,h]} yC(g)$.

Recall that G^c is an ordered group, where the order is definable by $K(x, y, a)$ for some $a \in G \setminus G^c$, so we use order terminology working with G^c.

First we prove that the intersection $C(g) \cap G^c$ is unbounded in G^c. Indeed, otherwise there exists a positive element $h \in G^c$ such that $C(g) \cap G^c \subset (e, h)$. Since G^c is ordered, so $h^n \notin C(g) \cap G^c$ for any positive integer n. Then $h^{n+1}C(g) \not\subseteq \bigcup_{x \in [e, h^n]} xC(g)$. This implies that the formula $F(x, h)$ has the strict order property inside the cut s, for a contradiction by Fact 1.5.

Let H be the greatest convex subgroup of $C(g) \cap G^c$. Recall that G^c is commutative, so it is a normal subgroup. Consider the quotient group G^c/H. Since H is convex, we can define an order on G^c/H as $aH \leq bH$ if and only if $a \leq b$.

Now we define the following subset A of G^c as: $a \in A$ if and only if $H < a$ and $a < b$ for any $b \in C(g) \cap G^c \cap (\sup H, +\infty)$.

We claim that A/H is finite. If it is infinite, we may find an infinite increasing sequence $\langle a_i : i \in I\rangle$ of elements of the set A such that $a_iH < a_jH$ whenever $i < j$. Then $a_jC(g) \not\subseteq \bigcup_{x\in[e,c_i]} xC(g)$ and we obtain that the formula $F(x,h)$ has the strict order property inside the cut s, which contradicts Fact 1.5.

Moreover, A is empty. Assume the contrary. Let the set A be not empty. Then the quotient group G/H is discretely ordered and there are an injective homomorphism $\tau : \mathbb{Z} \to G/H$, such that $\tau(1)$ is the least positive coset of H, and there is a natural number n such that any representative of $\tau(n)$ in G is in $C(g)$.

Observe that $\tau(\mathbb{Z})$ is a subgroup of the center $Z(G/H)$ of the quotient group G/H. Indeed, if $\tau(1) \notin Z(G/H)$, then there is an element $c \in G/H$ such that $\tau(1)c \neq c\tau(1)$, say, $\tau(1)c < c\tau(1)$. Since $\tau(1)$ is positive, so $c < \tau(1)c < c\tau(1)$. Eliminating c we obtain that $\tau(0) = c^{-1}c < c^{-1}\tau(1)c < \tau(1)$, which contradicts to the fact that $\tau(1)$ is the least positive element in G/H.

Let b be a representative in G of the coset $\tau(1)$. Since $b \notin C(g)$, so $bg \neq gb$. On the other hand, the element bH is central in G/H. Hence, $[b,g] \in H \leq C(g) \cap G^c$. Since the subgroup G^c is Abelian, the elements b and $[b,g]$ commute. By easy calculations

$$b^n g = b^{n-1}(bg) = b^{n-1}gb[b,g] = b^{n-2}gb[b,g]b[b,g] = b^{n-2}gb^2[b,g]^2 = \cdots = gb^n[b,g]^n,$$

we obtain that $e = [b^n, g] = [b,g]^n$, because the element $b^n \in C(g)$ as a representative in G of the coset $\tau(n)$. This yields a contradiction, because any ordered group is torsion-free.

Since $A = \emptyset$, so $C(g)/H$ is dense in G^c/H.

Note that in an ordered group both functions $f_a(x) = ax$ and $g_a(x) = xa$ are continuous. Indeed, $ax_0\varepsilon^{-1} < ax < ax_0\varepsilon$ if and only if $x_0\varepsilon^{-1} < x < x_0\varepsilon$.

Then since $C(g) \cap G^c/H$ is dense in G^c/H we obtain that $C(g) \cap G^c/H = G^c/H$. Indeed, Let b be an arbitrary element of G^c. Since $C(g)/H$ is dense in G^c/H, there is a sequence $\{c_\alpha\}$ of elements from $C(g)/H$, which converges to bH. Recall also, that $H \leq C(g)$, so $gH = Hg$. Then

$$gbH = g \cdot \lim c_\alpha H = \lim gc_\alpha H = \lim c_\alpha gH = \lim c_\alpha Hg = (\lim c_\alpha H) \cdot g = bHg = bgH.$$

Assume that there exists a positive element $b \in G^c$ such that $b \notin C(g)$. By above consideration $[g,b] \in H$. Let $c = [g,b]$. As we did it before $[g, b^n] = [g,b]^n = c^n$.

Consider the following formula: $\varphi(x,g,d) \triangleq d^{-1} < [g,x] < d$. Let $f \in C(g)\cdot b$, that is $f = g_1 b$ for some $g_1 \in C(g)$. Then

$$[g,f] = [g, g_1 b] = g^{-1}b^{-1}g_1^{-1}gg_1b = g^{-1}b^{-1}gb = [g,b]$$

because $g_1^{-1}gg_1 = g$. Thus, $\varphi(G,g,d)$ consists of cosets of $C(g)$. It is easy to see that

$$C(g)\cdot b^n \not\subseteq \varphi(G,g,c^n) \ \text{ and } \ C(g)\cdot b^n \subseteq \varphi(G,g,c^{n+1}).$$

Thus we obtain the strict order property witnessed by $\varphi(x;g,y)$ in the cut s. And this contradicts the c-o-stability of G. □

Lemma 1.7. *Let G be a circularly ordered group whose elementary theory is c-o-stable. Assume that $C(g)$ contains infinitely many cosets of G^c. Then $C(g) = G$, that is g is central.*

Proof. Assume that some element h is not in $C(g)$. Then for any element $a \in G^c$ the elements ah and ha do not belong to $C(g)$ by Lemma 1.6. But the quotient group G/G^c is commutative, so $hgG^c = ghG^c$, that is $c = [g,h]$ is in G^c and then commutes with g, because by Lemma 1.6 $G^c \subseteq C(g)$.

Note that

$$g^n h = g^{n-1}(gh) = g^{n-1}hgc = g^{n-2}hgcgc = g^{n-2}hg^2c^2 = \cdots = hg^n[g,h]^n$$

Thus, $[g^n, h] = [g,h]^n$. As in Lemma 1.6 we consider the following formula $\varphi(x,g,d) \triangleq d^{-1} < [g,x] < d$. Recall that $\varphi(G,g,d)$ consists of cosets of $C(g)$. It is easy to see that

$$C(g) \cdot h^n \nsubseteq \varphi(G,g,c^n) \ \text{ and } \ C(g) \cdot h^n \subseteq \varphi(G,g,c^{n+1}).$$

Note that any element in G/G^c of infinite order generates everywhere dense subgroup in G/G^c. But in this case we obtain the strict order property inside the pre-image under the natural homomorphism of G onto G/G^c of any cut in G/G^c, that contradicts to c-o-stability of G. □

Theorem 1.2. *Let G be a circularly ordered group whose elementary theory is c-o-stable. Then G is Abelian.*

Proof. Let a and $b \in G$. If aG^c has infinite order in G^c, then a is central by Lemma 1.7, so $ab = ba$. If bG^c has infinite order in G^c, then b is central, and the elements a and b commutes. If both aG^c and bG^c are of finite order, then they commute by Fact 1.7. □

Acknowledgments

This work was supported by the grant AP05132688 of the Ministry of Education and Science of the Republic of Kazakhstan.

References

[1] B.S. Baizhanov, V.V. Verbovskiy, *O-stable theories*, Algebra and Logic, 50 (2011), 211–225.

[2] M. Bhattacharjee, H.D. Macpherson, R.G. Moller, P.M. Neumann, *Notes on Infinite Permutation Groups*. Lecture Notes in Mathematics 1698, Springer, 1998, 202 pp.

[3] B.Sh. Kulpeshov, H.D. Macpherson, *Minimality conditions on circularly ordered structures*. Mathematical Logic Quarterly, 51 (2005), 377–399.

[4] B.Sh. Kulpeshov, V.V. Verbovskiy, *On weakly circularly minimal groups*. Mathematical Logic Quarterly, 61 (2015), 82–90.

[5] H.D. Macpherson, D. Marker, Ch. Steinhorn, *Weakly o-minimal structures and real closed fields*, Transactions of The American Mathematical Society, 352 (2000), 5435–5483.

[6] H.D. Macpherson, Ch. Steinhorn, *On variants of o-minimality*. Annals of Pure and Applied Logic, 79 (1996), 165–209.

[7] S. Swierczkowski, *On cyclically ordered groups*. Fundamenta Mathematicae, 47 (1959), 161–166.

[8] V.V. Verbovskiy, *O-stable ordered groups*. Siberian Advances in Mathematics, 22 (2012), 50–74.

[9] V.V. Verbovskiy, *On a classification of theories without the independence property*. Mathematical Logic Quarterly, 61 (2013), 119–124.

Viktor Valerievich Verbovskiy
Center of Mathematics and Cybernetics
The Kazakh-British Technical University
59 Tole bi St
050000, Almaty, Republic of Kazakhstan
E-mail: viktor.verbovskiy@gmail.com

Received: 09.02.2017