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SULTANAEV YAUDAT TALGATOVICH

(to the 70th birthday)

On 19th July 2018 was 70th birthday of Yaudat Talgatovich Sultanaev, doctor of science (1990), professor (1991), honorary scientist of the Russian Federation, laureate of State award of the Republic of Bashkortostan in the field of science and technology, professor of the Bashkir State Pedagogical University, member of the Editorial Board of the Eurasian Mathematical Journal.

Ya.T. Sultanaev was born in the sity of Orsk. In 1971 he graduated from the Bashkir State University and then completed his postgraduate studies in the Moscow State University. Ya.T. Sultanaev's scientific supervisors were distinguished mathematicians A.G. Kostyuchenko and B.M. Levitan.

Ya.T. Sultanaev is a famous specialist in the spectral theory of differential operators and the qualitative theory of ordinary differential equations.

He obtained bilateral Tauberian theorems of Keldysh type, completely solved the problem on spectral assymptotics for semi-bounded ordinary differential operators, suggested a new method of investigation of assymptotic behaviour of solutions to singular differential equations which allowed him to essentially weaken the conditions on coefficients.

Jointly with V.A. Sadovnichii and A.M. Akhtyamov, he investigated inverse spectral problems with non-separated boundary conditions.

He published more than 70 papers in leading mathematical journals.

Among pupils of Ya.T. Sultanaev there are more than 20 candidates of science and one doctor of science.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yaudat Talgatovich on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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ON THE CONNECTION BETWEEN FIRST INTEGRALS, INTEGRAL INVARIANTS AND POTENTIALITY OF EVOLUTIONARY EQUATIONS

V.M. Savchin, S.A. Budochkina, Yake Gondo, A.V. Slavko

Communicated by V.I. Burenkov

Key words: first integrals, absolute integral invariants, evolutionary equations.

AMS Mathematics Subject Classification: 49D29, 35A15.

Abstract. Using methods of nonlinear analysis, we have established the connection between first integrals and absolute integral invariants of some evolutionary equations similarly to the case of dynamics of finite dimensional systems.

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1 Introduction

In the case of finite dimensional systems H. Poincaré [4] introduced the notion of integral invariants and found their connection with first integrals of the given equations in variations, Hamilton's equations and others. In the paper some of these questions are investigated for infinite dimensional systems. The development of the qualitative theory for such systems of motion is rather important for specialists in mathematics, mechanics, physics and is being at the initial stage of study. Note that methods of investigation of infinite dimensional systems have been systematically presented in [5], [7]. In what follows we shall use notation and terminology of [1]-[3], [6], [8].

2 Evolutionary equations and their equations in variations

Consider the system of evolutionary equations

$$N^i(u) \equiv \frac{\partial u^i}{\partial t} - X^i(x, t, u_\alpha) = 0, \tag{2.1}$$

$$(x,t) \in Q_T = \Omega \times (0,T), \quad i = \overline{1,n}, \quad |\alpha| = \overline{0,s},$$

where $u(x,t) = (u^1(x,t), u^2(x,t), \ldots, u^n(x,t))$ is an unknown vector-function; Ω is a bounded domain in $\mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega$;
$X^i \in C^{s+1}(\overline{Q}_T \times \mathbb{R}^q)$ $(i = \overline{1,n})$; q is the dimension of the vector $\{u_\alpha\}$, $u_\alpha = D_\alpha u$, $D_\alpha = \partial^{|\alpha|}/(\partial x^1)^{\alpha_1} \ldots (\partial x^n)^{\alpha_n}$.

Assume that there are given the boundary conditions

$$\left.\frac{\partial^\nu u}{\partial n_x^\nu}\right|_{\Gamma_T} = 0, \nu = \overline{0, s-1}, \tag{2.2}$$

where $\Gamma_T = \partial\Omega \times (0,T)$; n_x is the unit vector of the exterior normal to $\partial\Omega$.

Suppose that the domain $D(N)$ of the operator $N = (N^1, \dots, N^n)$ consists of all vector-functions $u \in U = (U^1, ..., U^n), u^i \in U^i = C^{1,s}_{t,x}([0,T] \times \overline{\Omega})$ $(i = \overline{1,n})$ that satisfy conditions (2.2).

Let

$$u = u(\lambda; x, t), \quad \lambda \in [0,1], \tag{2.3}$$

be a one-parameter set of elements from $D(N)$ continuously differentiable with respect to λ. This set can be considered as a line on $D(N)$.

Let us introduce the notation

$$\delta u = \frac{\partial u(\lambda; x, t)}{\partial \lambda} d\lambda.$$

Suppose that u and $u + \delta u$ are two infinitesimally closed solutions of system (2.1).

By substituting $u + \delta u$ instead of u into (2.1) and using the equalities

$$X^i(x, t, u_\alpha + \delta u_\alpha) = X^i(x, t, u_\alpha) + \frac{\partial X^i}{\partial u^r_\alpha} \delta u^r_\alpha + o(d\lambda), \quad i = \overline{1,n}, \tag{2.4}$$

we obtain

$$\frac{\partial \delta u^i}{\partial t} = \frac{\partial X^i}{\partial u^r_\alpha} \delta u^r_\alpha + o(d\lambda), \quad i = \overline{1,n}.$$

The summation on indexes of different levels is accepted.

Consider the system

$$\frac{\partial \delta u^i}{\partial t} = \frac{\partial X^i}{\partial u^r_\alpha} \delta u^r_\alpha, \quad i = \overline{1,n}. \tag{2.5}$$

If a particular solution $u = u(x,t)$ of system (2.1) is known, then by substituting it into system of differential equations (2.5), we obtain the system of n linear equations for finding $\delta u = (\delta u^1, \dots, \delta u^n)$. Such equations are called equations in variations for system of evolutionary equations (2.1).

Let us introduce the function

$$\rho(t) = \sqrt{\int\limits_\Omega \sum_{i=1}^n \sum_{|\alpha|=0}^s (\delta u^i_\alpha)^2 (x,t)\, dx}$$

describing the measure of deviation of the basic trajectory $u = u(x,t)$ from the trajectory with the initial value $u(x,0) + \delta u(x,0)$.

3 First integrals and absolute integral invariants

Let us establish the connection between some first integrals of evolutionary equations (2.1) and absolute integral invariants of the first order.

Let $u = u(\lambda; x, t) \in D(N)$ be the set of all solutions to system (2.1), where $\lambda \in \Lambda \subset [0,1]$; Λ is an interval, $V = C([0,T] \times \overline{\Omega})$.

Let an operator $A : D(N) \to W$ be defined on $D(N)$ and, in general, be given a local bilinear form $\int\limits_0^T \langle \cdot, \cdot \rangle_u dt : W \times U \to \mathbb{R}$. By a local bilinear form we mean a two-variable function which is linear in each argument separately and depends, in general, in a non-linear way on u.

Then the integral

$$\int\limits_\Lambda \langle A(u), \delta u \rangle_u \equiv \int\limits_\Lambda \langle A(u), \partial u / \partial \lambda \rangle_u d\lambda \tag{3.1}$$

can depend on t.
Definition 1. The integral

$$\int_{\Lambda} \langle A(u), \delta u\rangle_u \tag{3.2}$$

is called an absolute integral invariant of the first order of system of evolutionary equations (2.1), if for any interval $\Lambda \subset [0,1]$ its value does not depend on t.

Consider the case of

$$\langle A(u), \delta u\rangle = \int_{\Omega} a_i\left(x,t,u_\alpha\right)\delta u^i dx, \tag{3.3}$$

where $a_i \in C^s(\overline{Q}_T \times \mathbb{R}^q), \quad i=\overline{1,n}$.

Theorem 3.1. *The integral*

$$\int_{\Lambda}\int_{\Omega} a_i \delta u^i dx \tag{3.4}$$

is an absolute integral invariant of system (2.1) if and only if the following conditions are satisfied:

$$\frac{\partial a_i}{\partial t} + \frac{\partial a_i}{\partial u^j_\alpha} X^j_\alpha + (-1)^{|\alpha|} D_\alpha \left(a_j \cdot \frac{\partial X^j}{\partial u^i_\alpha}\right) = 0 \quad \forall u \in D\left(N\right), \quad i=\overline{1,n}. \tag{3.5}$$

Proof. Let (3.4) be an absolute integral invariant of system (2.1). Then, bearing in mind the arbitrariness of the interval Λ, we obtain

$$\frac{d}{dt}\int_{\Omega} a_i \delta u^i dx = 0, \tag{3.6}$$

where the derivatives of δu^i must be defined according to (2.5). From (3.6) we obtain

$$\int_{\Omega}\left(\frac{\partial a_i}{\partial t}\delta u^i + \frac{\partial a_i}{\partial u^j_\alpha} X^j_\alpha \delta u^i + a_i \frac{\partial X^i}{\partial u^j_\alpha}\delta u^j_\alpha\right) dx = 0. \tag{3.7}$$

Integrating by parts and taking into consideration that in accordance with (2.2)

$$\delta u^i_\alpha\big|_{\Gamma_T} = 0, \quad i=\overline{1,n}, \quad |\alpha| = \overline{0,s-1},$$

we get

$$\int_{\Omega}\left[\frac{\partial a_i}{\partial t} + \frac{\partial a_i}{\partial u^j_\alpha} X^j_\alpha + (-1)^{|\alpha|} D_\alpha\left(a_j \cdot \frac{\partial X^j}{\partial u^i_\alpha}\right)\right]\delta u^i dx = 0. \tag{3.8}$$

Since the values of δu^i $\left(i=\overline{1,n}\right)$ can be arbitrary for any fixed t we come to the conclusion that conditions (3.5) are necessary. Their sufficiency can be proved by the reverse reasoning. □
Definition 2. The integral

$$F\left[t,u\right] = \int_{\Omega} \mathcal{F}\left(x,t,u_\alpha\right) dx, \quad \mathcal{F} \in C^{s+1}\left(\overline{Q}_T \times \mathbb{R}^q\right), \tag{3.9}$$

is called a first integral of equations (2.1) under conditions (2.2), if $F\left[t, u\left(x,t\right)\right]$ does not depend on t, when $u\left(x,t\right)$ is a solution to problem (2.1) - (2.2).

Theorem 3.2. *If functional (3.9) is a first integral of equations (2.1) under conditions (2.2), then*

$$\int\limits_{\Lambda}\int\limits_{\Omega} \frac{\delta F}{\delta u^i} \delta u^i dx \tag{3.10}$$

is an absolute integral invariant of these equations, where

$$\frac{\delta F}{\delta u^i} = (-1)^{|\alpha|} D_\alpha \left(\frac{\partial \mathfrak{F}}{\partial u^i_\alpha} \right)$$

is the functional derivative of F with respect to u^i.

Proof. We have

$$\left. \frac{dF}{dt} \right|_{(2.1),(2.2)} = \int\limits_{\Omega} \left(\frac{\partial \mathfrak{F}}{\partial t} + \frac{\delta F}{\delta u^j} X^j \right) dx, \tag{3.11}$$

where the lower index (2.1), (2.2) means the value of $\frac{dF}{dt}$ along solutions to problem (2.1)–(2.2).

It follows

$$\frac{\delta}{\delta u^i} \left(\left. \frac{dF}{dt} \right|_{(2.1),(2.2)} \right) = \frac{\partial}{\partial t} \frac{\delta F}{\delta u^i} + (-1)^{|\alpha|} D_\alpha \left[\left(\frac{\partial}{\partial u^i_\alpha} \frac{\delta F}{\delta u^j} \right) X^j \right] +$$

$$+ (-1)^{|\alpha|} D_\alpha \left(\frac{\delta F}{\delta u^j} \frac{\partial X^j}{\partial u^i_\alpha} \right), \quad i = \overline{1, n}. \tag{3.12}$$

According to the Leibniz formula

$$(-1)^{|\alpha|} D_\alpha \left[\left(\frac{\partial}{\partial u^i_\alpha} \frac{\delta F}{\delta u^j} \right) X^j \right] = \sum_{|\beta|=0}^{s} (-1)^{|\alpha|} \binom{\alpha}{\beta} D_{\alpha-\beta} \left(\frac{\partial}{\partial u^i_\alpha} \frac{\delta F}{\delta u^j} \right) X^j_\beta, \tag{3.13}$$

$$i = \overline{1, n}.$$

Since the operator of the functional derivative $\frac{\delta}{\delta u^i}$ is potential on the given domain $D(N)$ with respect to the classical bilinear form

$$\Phi(v, g) = \int\limits_0^T \int\limits_\Omega \sum_{i=1}^n v^i(x,t) g^i(x,t) dx dt,$$

then the following conditions are satisfied [5, p. 108]:

$$(-1)^{|\alpha|} \binom{\alpha}{\beta} D_{\alpha-\beta} \left(\frac{\partial}{\partial u^i_\alpha} \frac{\delta F}{\delta u^j} \right) = \frac{\partial}{\partial u^j_\beta} \frac{\delta F}{\delta u^i} \quad \forall u \in D(N), \quad i, j = \overline{1, n}, \quad |\beta| = \overline{0, s}. \tag{3.14}$$

Taking into consideration (3.13), from (3.12) we obtain

$$\frac{\partial}{\partial t} \frac{\delta F}{\delta u^i} + \frac{\partial}{\partial u^j_\alpha} \left(\frac{\delta F}{\delta u^i} \right) X^j_\alpha + (-1)^{|\alpha|} D_\alpha \left(\frac{\delta F}{\delta u^j} \frac{\partial X^j}{\partial u^i_\alpha} \right) = 0 \quad \forall u \in D(N), \quad i = \overline{1, n}. \tag{3.15}$$

Under $a_i = \delta F / \delta u^i$ ($i = \overline{1, n}$) these relations are the same as conditions (3.5). Thus from that the validity of the theorem follows. □

Theorem 3.3. *If*

$$\int\limits_{\Lambda}\int\limits_{\Omega}\frac{\delta F}{\delta u^i}\delta u^i dx \tag{3.16}$$

is an absolute integral invariant of system (2.1), then functional of kind (3.9) is a first integral of these equations under conditions (2.2).

Proof. We obtain

$$0=\frac{\partial}{\partial t}\frac{\delta F}{\delta u^i}+\frac{\partial}{\partial u^j_\alpha}\left(\frac{\delta F}{\delta u^i}\right)X^j_\alpha+(-1)^{|\alpha|}D_\alpha\left(\frac{\delta F}{\delta u^j}\frac{\partial X^j}{\partial u^i_\alpha}\right)=\frac{\delta}{\delta u^i}\left(\left.\frac{dF}{dt}\right|_{(2.1),(2.2)}\right).$$

Then it follows that (see [3])

$$\left.\frac{d}{dt}F\right|_{(2.1),(2.2)}=\int\limits_{\Omega} div\mathcal{R}dx, \tag{3.17}$$

where $\mathcal{R}=(\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3)$ is a vector-function, depending on x, t, u_α, $\mathcal{R}_i|_{\partial\Omega}=0, i=\overline{1,3}$.

From (3.17) we obtain that the functional

$$F_1[t,u]\equiv F[t,u]-\int\limits_0^t\int\limits_{\Omega} div\mathcal{R}dxdt \tag{3.18}$$

which is a first integral of problem (2.1) - (2.2). □

4 Linear integral invariant of the first order

In some cases the method of construction of integral invariants can be based on the use of Lagrangians of given systems.

Let us consider a density $\mathcal{L}=\mathcal{L}(t,x,u_\alpha,\dot{u}_\alpha)$ of the Lagrangian

$$L=\int\limits_{\Omega}\mathcal{L}dx, \tag{4.1}$$

where $x\in\Omega\subset\mathbb{R}^m$, $|\alpha|=\overline{0,s}$, $\left.\frac{\partial^\nu u}{\partial n^\nu_x}\right|_{\Gamma_T}=\varphi^\nu(t,x)$ $\left(\nu=\overline{0,s-1}\right)$; $\varphi^\nu(t,x)$ are some given functions.

Then the variation takes the form

$$\delta\int\limits_{t_0}^{t_1}Ldt=\int\limits_{t_0}^{t_1}\int\limits_{\Omega}\left(\frac{\partial\mathcal{L}}{\partial\dot{u}_\alpha}\delta\dot{u}_\alpha+\frac{\partial\mathcal{L}}{\partial u_\alpha}\delta u_\alpha\right)dxdt. \tag{4.2}$$

Integrating by parts, from (4.2) we get

$$\begin{aligned}\delta\int\limits_{t_0}^{t_1}Ldt &= \int\limits_{t_0}^{t_1}\int\limits_{\Omega}\left[(-1)^{|\alpha|}D_\alpha\left(\frac{\partial\mathcal{L}}{\partial\dot{u}_\alpha}\right)\delta\dot{u}+(-1)^{|\alpha|}D_\alpha\left(\frac{\partial\mathcal{L}}{\partial u_\alpha}\right)\delta u\right]dxdt=\\ &= \int\limits_{t_0}^{t_1}\int\limits_{\Omega}\left[\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{u}}\cdot\delta u\right)-\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{u}}\right)\cdot\delta u+\frac{\delta L}{\delta u}\delta u\right]dxdt,\end{aligned}$$

where $(t_0, t_1) \subset (0, T)$.

Consequently

$$\delta \int_{t_0}^{t_1} L dt = \int_{\Omega} \frac{\delta L}{\delta \dot{u}} \delta u \bigg|_{t=t_1} dx - \int_{\Omega} \frac{\delta L}{\delta \dot{u}} \delta u \bigg|_{t=t_0} dx + \int_{t_0}^{t_1} \int_{\Omega} \left(\frac{\delta L}{\delta u} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{u}} \right) \right) \delta u dx dt.$$

Thus, along the real trajectories we have

$$\delta \int_{t_0}^{t_1} L dt = \int_{\Omega} \frac{\delta L}{\delta \dot{u}} \delta u \bigg|_{t=t_1} dx - \int_{\Omega} \frac{\delta L}{\delta \dot{u}} \delta u \bigg|_{t=t_0} dx.$$

Introducing the density of the generalized impulse $p = \frac{\delta L}{\delta \dot{u}}$, from here we obtain

$$\delta \int_{t_0}^{t_1} L dt = \int_{\Omega} p \delta u|_{t=t_1} dx - \int_{\Omega} p \delta u|_{t=t_0} dx.$$

Let the initial state u_0 of the given system depends on a parameter $\lambda \in (\lambda_1, \lambda_2)$ and $u_0(x, \lambda_1) = u_0(x, \lambda_2)$.

Then

$$\int_{\Lambda} \int_{\Omega} p \delta u|_{t=t_0} dx = \int_{\Lambda} \int_{\Omega} p \delta u|_{t=t_1} dx.$$

Thus

$$\int_{\Lambda} \int_{\Omega} p \delta u dx \tag{4.3}$$

is a first order linear integral invariant of the system described by Lagrangian (4.1).

5 The infinite dimensional conservative systems

Let us consider the evolutionary problem

$$\begin{cases} N(u) \equiv \frac{\partial^2 u}{\partial t^2} - K(x, u_\alpha, \dot{u}_\alpha) = 0, \ u \in D(N), & (x,t) \in Q_T = \Omega \times (0,T), \\ \frac{\partial^\nu u}{\partial n_x^\nu} \Big|_{\Gamma_T} = 0, \quad |\alpha| = \overline{0, s}, \ \nu = \overline{0, s-1}, & \end{cases} \tag{5.1}$$

where K is a sufficiently smooth function. Suppose that there exists the energy first integral.

Theorem 5.1. *Problem (5.1) has the first integral of the kind*

$$I[u] = \int_{\Omega} \left(\frac{1}{2} u_t^2 + f(x, u_\alpha) \right) dx \tag{5.2}$$

if and only if K does not depend on $\dot{u}_\alpha$, that is

$$K = K(x, u_\alpha).$$

Moreover,

$$V[u] \equiv \int_{\Omega} f(x, u_\alpha) dx = - \int_{\Omega} \int_0^1 K(x, \lambda u_\alpha) u d\lambda dx.$$

Proof. Let $u(x,t)$ be a solution to (5.1). Then

$$\frac{dI\left[u(x,t)\right]}{dt}\bigg|_{(5.1)} = \int\limits_{\Omega} \left(u_t \cdot u_{tt} + \frac{\partial f}{\partial u_\alpha}\dot{u}_\alpha\right)dx\bigg|_{(5.1)} =$$

$$= \int\limits_{\Omega} \left[u_t \cdot K\left(x, u_\alpha, \dot{u}_\alpha\right) + \frac{\delta V}{\delta u}u_t\right]dx = \int\limits_{\Omega} \left[K\left(x, u_\alpha, \dot{u}_\alpha\right) + \frac{\delta V}{\delta u}\right] u_t dx \equiv 0$$

$$\forall t \in [0, T].$$

That condition is fulfilled if and only if

$$\left[K\left(x, u_\alpha, \dot{u}_\alpha\right) + \frac{\delta V}{\delta u}\right] u_t \equiv 0.$$

If u_t is not identical zero it follows that K defines a potential operator [2], which does not depend on $\dot{u}_\alpha$ and thus

$$K\left(x, u_\alpha\right) = -\frac{\delta V}{\delta u}$$

and $V[u] = -\int\limits_{\Omega}\int\limits_{0}^{1} K\left(x, \lambda u_\alpha\right) u d\lambda dx + const.$ □

Since $\frac{1}{2}u_t^2 \geq 0$, then the inequality $V\left[u\right] \leq h$ is always valid in evolution.

Let us define the domain of motion opportunity by

$$M^h = \left\{u\left(x,t\right) \in D\left(N\right) \colon V[u\left(x,t\right)] \leq h\right\}.$$

According to Theorem 5.1, equation (5.1) can be written as follows:

$$u_{tt} = -\frac{\delta V\left[u\right]}{\delta u}. \tag{5.3}$$

The critical points of the functional V - the energy potential - have the clear dynamical sence — each of them is a state of equilibrium. The solution $u \equiv u^*$ is admissible if and only if $\frac{\delta V[u^*]}{\delta u} = 0$.

The energy value $h^* = V\left[u^*\right]$ corresponding to u^* is a critical value of the functional V.

If the value of h changes, the domain M^h also changes.

6 Example

Consider the following partial differential equation

$$N\left(u\right) \equiv \frac{\partial^2 u}{\partial t^2} - a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2k^2\frac{\partial u}{\partial t} = 0, \tag{6.1}$$

describing the motion of a membrane.

Here $u = u\left(x, y, t\right)$ is an unknown function; a, k are constants, $(x, y, t) \in Q_T = (0, l^1) \times (0, l^2) \times (0, T)$.

We set

$$D\left(N\right) = \left\{u \in U = C^2\left(\overline{Q}_T\right) : \quad u|_{x=0} = u|_{y=0} = 0,\, u|_{x=l^1} = u|_{y=l^2} = 0\right\}. \tag{6.2}$$

Let us denote $V = C\left(\overline{Q}_T\right)$ and determine the bilinear form by setting

$$\Phi(v, h) = \int_0^T \int_0^{l^2} \int_0^{l^1} v\,(x, y, t) \cdot h\,(x, y, t)\, dxdydt. \tag{6.3}$$

Since the condition $\Phi(N'_u h, g) = \Phi(N'_u g, h)\ \forall u \in D(N), \forall h, g \in D(N'_u)$ is not true, then operator N (6.1) is not potential [2] on $D\,(N)$ (6.2) with respect to bilinear form (6.3). It is easy to find a variational multiplier M for equation (6.1) in the form

$$M = exp\left(2k^2 t\right).$$

Then the equivalent equation

$$\tilde{N}\,(u) \equiv e^{2k^2 t} \cdot N\,(u) = 0 \tag{6.4}$$

admits the variational formulation with the Hamiltonian action $F\,[u] = \int_0^T L dt$, with the Lagrangian

$$L\,[u] = \frac{1}{2} \int_0^{l^2} \int_0^{l^1} e^{2k^2 t} \left\{ \left(\frac{\partial u}{\partial t}\right)^2 - a^2 \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right) \right\} dxdy. \tag{6.5}$$

Introducing the density of generalized impulse

$$p = \frac{\delta L}{\delta u_t},$$

we get

$$p = e^{2k^2 t} \frac{\partial u}{\partial t}. \tag{6.6}$$

In accordance with (4.3) we obtain the following first order linear integral invariant

$$\int_\Lambda \int_0^{l^2} \int_0^{l^1} p \delta u dxdy$$

of equation (6.1).

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