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SULTANAEV YAUDAT TALGATOVICH

(to the 70th birthday)

On 19th July 2018 was 70th birthday of Yaudat Talgatovich Sultanaev, doctor of science (1990), professor (1991), honorary scientist of the Russian Federation, laureate of State award of the Republic of Bashkortostan in the field of science and technology, professor of the Bashkir State Pedagogical University, member of the Editorial Board of the Eurasian Mathematical Journal.

Ya.T. Sultanaev was born in the sity of Orsk. In 1971 he graduated from the Bashkir State University and then completed his postgraduate studies in the Moscow State University. Ya.T. Sultanaev's scientific supervisors were distinguished mathematicians A.G. Kostyuchenko and B.M. Levitan.

Ya.T. Sultanaev is a famous specialist in the spectral theory of differential operators and the qualitative theory of ordinary differential equations.

He obtained bilateral Tauberian theorems of Keldysh type, completely solved the problem on spectral assymptotics for semi-bounded ordinary differential operators, suggested a new method of investigation of assymptotic behaviour of solutions to singular differential equations which allowed him to essentially weaken the conditions on coefficients.

Jointly with V.A. Sadovnichii and A.M. Akhtyamov, he investigated inverse spectral problems with non-separated boundary conditions.

He published more than 70 papers in leading mathematical journals.

Among pupils of Ya.T. Sultanaev there are more than 20 candidates of science and one doctor of science.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yaudat Talgatovich on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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ON RELATIVE SEPARABILITY IN HYPERGRAPHS OF MODELS OF THEORIES

B.Sh. Kulpeshov, S.V. Sudoplatov

Communicated by J.A. Tussupov

Key words: hypergraph of models, elementary theory, separability, relative separability, quite o-minimal theory.

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Abstract. In the paper, notions of relative separability for hypergraphs of models of a theory are defined. Properties of these notions and applications to ordered theories are studied: characterizations of relative separability both in a general case and for almost ω-categorical quite o-minimal theories are established.

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1 Preliminaries

Hypergraphs of models of a theory are related to derivative objects allowing to obtain an essential structural information both on theories themselves and related semantical objects including graph ones [11, 12, 13, 1, 14, 5, 15, 6].

In the present paper, notions of relative separability for hypergraphs of models of a theory are defined. Properties of these notions and applications to ordered theories are studied: characterizations of relative separability both in a general case and for almost ω-categorical quite o-minimal theories are established.

Recall that a *hypergraph* is a pair of sets (X, Y), where Y is a subset of the Boolean set $\mathcal{P}(X)$ of the set X.

Let $\mathcal{M}$ be a model of a complete theory T. Following [14], we denote by $H(\mathcal{M})$ the family of all subsets N of the universe M of $\mathcal{M}$ that are universes of elementary submodels $\mathcal{N}$ of the model $\mathcal{M}$: $H(\mathcal{M}) = \{N \mid \mathcal{N} \preccurlyeq \mathcal{M}$, i.e. $\mathcal{N}$ is an elementary submodel of $\mathcal{M}\}$. The pair $(M, H(\mathcal{M}))$ is called the *hypergraph of elementary submodels* of the model $\mathcal{M}$ and denoted by $\mathcal{H}(\mathcal{M})$.

For a cardinality λ by $H_\lambda(\mathcal{M})$ and $\mathcal{H}_\lambda(\mathcal{M})$ are denoted restrictions for $H(\mathcal{M})$ and $\mathcal{H}(\mathcal{M})$ respectively on the class of all elementary submodels $\mathcal{N}$ of models $\mathcal{M}$ such that $|N| < \lambda$.

By $\mathcal{H}_p(\mathcal{M})$ we denote the restriction of the hypergraph $\mathcal{H}_{\omega_1}(\mathcal{M})$ on the class of all elementary submodels $\mathcal{N}$ of the model $\mathcal{M}$ that are prime over finite sets. Similarly by $H_p(\mathcal{M})$, is denoted the corresponding restriction for $H_{\omega_1}(\mathcal{M})$.

Definition 1. [14, 3]. Let (X, Y) be a hypergraph, x_1, x_2 be distinct elements of X. We say that the element x_1 is *separated* or *separable* from the element x_2, or T_0*-separable* if there is $y \in Y$ such that $x_1 \in y$ and $x_2 \notin y$. The elements x_1 and x_2 are called *separable*, T_2*-separable*, or *Hausdorff separable* if there are disjoint $y_1, y_2 \in Y$ such that $x_1 \in y_1$ and $x_2 \in y_2$.

Recall that for a set A in a structure $\mathcal{M}$, an element b is called *algebraic* (respectively, *definable*) over A, if it satisfies $\mathcal{M} \models \varphi(b, \overline{a}) \wedge \exists^{=n} x\varphi(x, \overline{a})$ for some formula $\varphi(x, \overline{y})$, $\overline{a} \in A$,

$n \in \omega$ ($n = 1$). The set of all algebraic (definable) elements over A is called the *algebraic* (*definable*) closure of A and denoted by $\mathrm{acl}(A)$ ($\mathrm{dcl}(A)$). If $A = \{a\}$ the algebraic (definable) closure is denoted by $\mathrm{acl}(a)$ ($\mathrm{dcl}(a)$).

Theorem 1.1. [14]. *Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, a and b be elements of $\mathcal{M}$. The following conditions are equivalent:*
(1) *the element a is separable from the element b in $\mathcal{H}(\mathcal{M})$;*
(2) *the element a is separable from the element b in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*
(3) $b \notin \mathrm{acl}(a)$.

Theorem 1.2. [14]. *Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, a and b be elements of $\mathcal{M}$. The following conditions are equivalent:*
(1) *the elements a and b are separable in $\mathcal{H}(\mathcal{M})$;*
(2) *the elements a and b are separable in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*
(3) $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \varnothing$.

Corollary 1.1. [14]. *Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, a and b be elements of $\mathcal{M}$, and there exists a prime model over a. The following conditions are equivalent:*
(1) *the element a is separable from the element b in $\mathcal{H}(\mathcal{M})$;*
(2) *the element a is separable from the element b in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*
(3) *the element a is separable from the element b in $\mathcal{H}_p(\mathcal{M})$;*
(4) $b \notin \mathrm{acl}(a)$.

Corollary 1.2. [14]. *Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, a and b be elements of $\mathcal{M}$, and there exist prime models over a and b respectively. The following conditions are equivalent:*
(1) *the elements a and b are separable in $\mathcal{H}(\mathcal{M})$;*
(2) *the elements a and b are separable in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*
(3) *the elements a and b are separable in $\mathcal{H}_p(\mathcal{M})$;*
(4) $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \varnothing$.

Definition 2. [14]. Let (X, Y) be a hypergraph, X_1, X_2 be disjoint nonempty subsets of the set X. We say that the set X_1 is *separated* or *separable* from the set X_2, or *T_0-separable* if there is $y \in Y$ such that $X_1 \subseteq y$ and $X_2 \cap y = \varnothing$. The sets X_1 and X_2 are called *separable*, *T_2-separable*, or *Hausdorff separable* if there are disjunct $y_1, y_2 \in Y$ such that $X_1 \subseteq y_1$ and $X_2 \subseteq y_2$.

By using the proofs of Theorems 1.1 and 1.2, the following generalizations of these theorems are established.

Theorem 1.3. [14] *Let $\mathcal{M}$ be a λ-saturated model of a complete theory T, $\lambda \geq \max\{|\Sigma(T)|, \omega\}$, A and B be nonempty sets in $\mathcal{M}$ having the cardinalities $< \lambda$. The following conditions are equivalent:*
(1) *the set A is separable from the set B in $\mathcal{H}(\mathcal{M})$;*
(2) *the set A is separable from the set B in $\mathcal{H}_\lambda(\mathcal{M})$;*
(3) $\mathrm{acl}(A) \cap B = \varnothing$.

Theorem 1.4. [14] *Let $\mathcal{M}$ be a λ-saturated model of a complete theory T, $\lambda \geq \max\{|\Sigma(T)|, \omega\}$, A и B be nonempty sets in $\mathcal{M}$ having the cardinalities $< \lambda$. The following conditions are equivalent:*
(1) *the sets A and B are separable in $\mathcal{H}(\mathcal{M})$;*
(2) *the sets A and B are separable in $\mathcal{H}_\lambda(\mathcal{M})$;*
(3) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$.

Recall that a complete theory T of a countable language is said to be *small* if it has countably many types over the empty set: $|S(\emptyset)| = \omega$.

We have by analogy with Corollaries 1.1 and 1.2

Corollary 1.3. [14]. *Let $\mathcal{M}$ be an ω-saturated model of a small theory T, A and B be finite nonempty sets in $\mathcal{M}$. The following conditions are equivalent:*

(1) *the set A is separable from the set B in $\mathcal{H}(\mathcal{M})$;*

(2) *the set A is separable from the set B in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*

(3) *the set A is separable from the set B in $\mathcal{H}_p(\mathcal{M})$;*

(4) $\mathrm{acl}(A) \cap B = \varnothing$.

Corollary 1.4. [14]. *Let $\mathcal{M}$ be an ω-saturated model of a small theory T, A and B be finite nonempty sets in $\mathcal{M}$. The following conditions are equivalent:*

(1) *the sets A and B are separable in $\mathcal{H}(\mathcal{M})$;*

(2) *the sets A and B are separable in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*

(3) *the sets A and B are separable in $\mathcal{H}_p(\mathcal{M})$;*

(4) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$.

The following proposition extends Theorem 1.4 with an additional criterion.

Proposition 1.1. *Let T be a theory, $\mathcal{M} \models T$, $\emptyset \neq A \subseteq M$, $\emptyset \neq B \subseteq M$, $\mathcal{M}$ be $|A \cup B|^+$-saturated. Then A and B are separable from each other in $\mathcal{H}(\mathcal{M})$ if and only if the following conditions hold:*

(1) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \emptyset$;

(2) *for any isolated type $p \in S_1(\emptyset)$, $p(\mathcal{M}) \setminus \mathrm{acl}(A) \neq \emptyset$ and $p(\mathcal{M}) \setminus \mathrm{acl}(B) \neq \emptyset$.*

Proof. If A and B are separable from each other in $\mathcal{H}(\mathcal{M})$ then by Theorem 1.4 we have $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \emptyset$. If there is an isolated type $p \in S_1(\emptyset)$ such that $p(M) \subseteq \mathrm{acl}(A)$ then there is $\mathcal{M}_2 \prec \mathcal{M}$ with $B \subseteq M_2$ and $p(\mathcal{M}) \cap M_2 = \emptyset$, i.e. p is not realized in $\mathcal{M}_2$. Similarly, $p(\mathcal{M}) \not\subseteq \mathrm{acl}(A)$.

If the conditions (1), (2) hold then A and B are separable from each other in $\mathcal{H}(\mathcal{M})$ by Theorem 1.4. $\square$

Recall that a subset A of a linearly ordered structure M is called *convex* if for any $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters) subset of the structure M is a union of finitely many convex sets in M.

In the following definitions M is a weakly o-minimal structure, $A, B \subseteq M$, M be $|A|^+$-saturated, $p, q \in S_1(A)$ be non-algebraic types.

Definition 3. [2] We say that p is not *weakly orthogonal* to q ($p \not\perp^w q$) if there exist an A-definable formula $H(x, y)$, $\alpha \in p(M)$ and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M, \alpha)$ and $\beta_2 \notin H(M, \alpha)$.

Definition 4. [8] We say that p is not *quite orthogonal* to q ($p \not\perp^q q$) if there exists an A-definable bijection $f : p(M) \to q(M)$. We say that a weakly o-minimal theory is *quite o-minimal* if the notions of weak and quite orthogonality of 1-types coincide.

In the work [9] the countable spectrum for quite o-minimal theories with non-maximal number of countable models has been described:

Theorem 1.5. *Let T be a quite o-minimal theory with non-maximal number of countable models. Then T has exactly $3^k \cdot 6^s$ countable models, where k and s are natural numbers. Moreover, for any $k, s \in \omega$ there exists a quite o-minimal theory T having exactly $3^k \cdot 6^s$ countable models.*

Realizations of these theories with a finite number of countable models are natural generalizations of Ehrenfeucht examples obtained by expansions of dense linear orderings by a countable set of constants, and they are called theories of *Ehrenfeucht type*. Moreover, these realizations are representative examples for hypergraphs of prime models [11, 13, 14].

2 Relative separability in hypergraphs of models of theories

Observe that since by Theorem 1.4 and Corollary 1.4 separability of sets A and B in hypergraphs $H(\mathcal{M})$ is possible only when $\mathrm{acl}(A)\cap\mathrm{acl}(B)=\varnothing$, such a separability doesn't hold when $\mathrm{acl}(\emptyset)\neq\emptyset$. Thus, it is natural to consider the following notions of *relative* separability.

Definition 5. Let (X,Y) be a hypergraph, x_1,x_2 be distinct elements of X, $Z\subset X$, $x_2\notin Z$. We say that the element x_1 is *Z-separated* or *Z-separable* from the element x_2, or *(T_0,Z)-separable* if there is $y\in Y$ such that $x_1\in y\cup Z$ and $x_2\notin y$. In this case the set y is called *Z-separating* x_1 from x_2. At the additional condition $x_1\notin Z$ the elements x_1 and x_2 are called *Z-separable*, *(T_2,Z)-separable*, or *Hausdorff Z-separable* if there are $y_1,y_2\in Y$ such that $(y_1\cap y_2)\setminus Z=\varnothing$, $x_1\in y_1$ and $x_2\in y_2$.

Let X_1,X_2 be nonempty subsets of the set X, $(X_1\cap X_2)\setminus Z=\emptyset$, $X_2\not\subseteq Z$. We say that the set X_1 is *Z-separated* or *Z-separable* from the set X_2, or *(T_0,Z)-separable* if there is $y\in Y$ such that $X_1\subseteq y\cup Z$ and $(X_2\cap y)\setminus Z=\varnothing$. At the additional condition $X_1\not\subseteq Z$ the sets X_1 and X_2 are called *Z-separable*, *(T_2,Z)-separable*, or *Hausdorff Z-separable* if there are $y_1,y_2\in Y$ such that $(y_1\cap y_2)\setminus Z=\varnothing$, $X_1\subseteq y_1\cup Z$ and $X_2\subseteq y_2\cup Z$.

Note 2.1. 1. The notions of separability given in Section 1 correspond to Z-separability for $Z=\varnothing$, $X_1\neq\varnothing$, $X_2\neq\varnothing$.

2. If $X_2\subseteq Z$ then the set X_2 can also be assumed Z-separable from X_1, although there is no reason to say on real separability of elements of the set X_2 from X_1.

For a tuple $\bar{a}$ and a set Z we denote by $\bar{a}Z$ the union of the set Z with the set of all elements contained in $\bar{a}$.

The following theorem modifies Theorem 1.1, and it is a generalization of the theorem for $\mathrm{acl}(\varnothing)=\varnothing$.

Theorem 2.1. *Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, Z be the algebraic closure of some finite set in $\mathcal{M}$, a and b be elements of $\mathcal{M}$, $b\notin Z$. The following conditions are equivalent:*

(1) *the element a is Z-separable from the element b in $H(\mathcal{M})$ by some set y from $H(\mathcal{M})$ containing Z;*

(2) *the element a is Z-separable from the element b in $H_{\omega_1}(\mathcal{M})$ by some set y from $H_{\omega_1}(\mathcal{M})$ containing Z;*

(3) $b\notin\mathrm{acl}(aZ)$.

Proof. The implications (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (3) are obvious (clearly, if $b\in\mathrm{acl}(Z\cup\{a\})$ then b belongs to any model $\mathcal{N}\preccurlyeq\mathcal{M}$ containing $Z\cup\{a\}$).

To prove the implication (3) $\Rightarrow$ (2) we need the following lemma.

Lemma 2.1. *Let $\bar{a}$ be a tuple, B be a finite set for which $(\mathrm{acl}(\bar{a}Z)\cap B)\setminus Z=\varnothing$, and $\varphi(x,\bar{a})$ be some consistent formula. Then there is an element $c\in\varphi(\mathcal{M},\bar{a})$ such that $(\mathrm{acl}(\bar{a}cZ)\cap B)\setminus Z=\varnothing$.*

Proof. If $\varphi(\mathcal{M},\bar{a})\cap Z\neq\varnothing$ then there is nothing to prove since as c we can take an arbitrary element of $\varphi(\mathcal{M},\bar{a})\cap Z$.

Suppose that $\varphi(\mathcal{M},\bar{a})\cap Z=\varnothing$. By compactness and using consistent formulas $\varphi'(x,\bar{a})$ with the condition $\varphi'(x,\bar{a})\vdash\varphi(x,\bar{a})$ instead of $\varphi(x,\bar{a})$, it suffices to prove that for any $d\in B\setminus Z$ and a finite set of formulas $\psi_1(x,\bar{a},y),\ldots,\psi_n(x,\bar{a},y)$ with the condition

$$\psi_i(x,\bar{a},y)\vdash\varphi'(x,\bar{a})\wedge\forall x\left(\varphi'(x,\bar{a})\to\exists^{=k_i}y\psi_i(x,\bar{a},y)\right)$$

for some natural k_i, $i = 1, \ldots, n$, there is an element $c \in \varphi'(\mathcal{M}, \bar{a})$ such that

$$\models \bigwedge_{i=1}^{n} \neg\psi_i(c, \bar{a}, d).$$

Assume to the contrary that for any $c \in \varphi'(\mathcal{M}, \bar{a})$ there is i such that $\models \psi_i(c, \bar{a}, d)$. Then the formula $\chi(x, \bar{a}, y) \rightleftharpoons \bigvee_{i=1}^{n} \psi_i(x, \bar{a}, y)$ satisfies the following condition: for any $c \in \varphi'(\mathcal{M}, \bar{a})$, $\models \chi(c, \bar{a}, d)$ and $\chi(c, \bar{a}, y)$ has finitely many, no more than $m = k_1 + \ldots + k_n$ solutions. Consequently, the formula

$$\theta(\bar{a}, y) \rightleftharpoons \exists x(\chi(x, \bar{a}, y) \wedge \forall z((\varphi'(z, \bar{a}) \to (\chi(x, \bar{a}, y) \wedge \chi(z, \bar{a}, y)))$$

satisfies d and has no more than m solutions.

This fact contradicts the condition $d \notin \mathrm{acl}(\bar{a}Z)$. $\square$

Continuation of the proof of Theorem 2.1.

Assuming that $b \notin \mathrm{acl}(aZ)$, we construct by induction a countable model $\mathcal{N} \preccurlyeq \mathcal{M}$ such that $\mathrm{acl}(aZ) \subset N$, $b \notin N$, and $N = \bigcup_{n\in\omega} A_n$ for a chain of some sets A_n.

In the initial step we consider the set $A_0 = \mathrm{acl}(aZ)$ and renumber all consistent formulas of the form $\varphi(x, \bar{a})$, $\bar{a} \in A_0$: $\Phi_0 = \{\varphi_{0,m}(x, \bar{a}_m) \mid m \in \omega\}$. According to this numeration we construct at most a countable set $A_1 = \bigcup_{m\in\omega\cup\{-1\}} A_{1,m} \supset A_0$ with the condition $b \notin \mathrm{acl}(A_1)$. Let $A_{1,-1} \rightleftharpoons A_0$. If the set $A_{1,m-1}$ had been already defined and $\varphi_{0,m}(\mathcal{M}, \bar{a}_m) \cap A_{1,m-1} \neq \varnothing$ then we put $A_{1,m} \rightleftharpoons A_{1,m-1}$; if $\varphi_{0,m}(\mathcal{M}, \bar{a}_m) \cap A_{1,m-1} = \varnothing$ we choose by Lemma 2.1 an element $c_m \in \varphi_m(\mathcal{M}, \bar{a}_m)$ such that $b \notin \mathrm{acl}(c_m A_{1,m-1})$, and put $A_{1,m} \rightleftharpoons \mathrm{acl}(c_m A_{1,m-1})$.

If at most a countable set A_n had been already constructed, we renumber all consistent formulas of the form $\varphi(x, \bar{a})$, $\bar{a} \in A_n$: $\Phi_n = \{\varphi_{n,m}(x, \bar{a}_m) \mid m \in \omega\}$. According to this enumeration we construct at most a countable set $A_{n+1} = \bigcup_{m\in\omega\cup\{-1\}} A_{n+1,m} \supset A_n$ with the condition $b \notin \mathrm{acl}(A_{n+1})$. Let $A_{n+1,-1} \rightleftharpoons A_n$. If the set $A_{n+1,m-1}$ had been already defined and $\varphi_{n,m}(\mathcal{M}, \bar{a}_m) \cap A_{n+1,m-1} \neq \varnothing$ then put $A_{n+1,m} \rightleftharpoons A_{n+1,m-1}$; if $\varphi_{n,m}(\mathcal{M}, \bar{a}_m) \cap A_{n+1,m-1} = \varnothing$, we choose by Lemma 2.1 an element $c_m \in \varphi_{n,m}(\mathcal{M}, \bar{a}_m)$ such that $b \notin \mathrm{acl}(c_m A_{n+1,m-1})$ and put $A_{n+1,m} \rightleftharpoons \mathrm{acl}(c_m A_{n+1,m-1})$.

By constructing the set $\bigcup_{n\in\omega} A_n$ forms a required universe N of a countable model $\mathcal{N} \preccurlyeq \mathcal{M}$ such that $\mathrm{acl}(Z \cup \{a\}) \subseteq N$ and $b \notin N$. $\square$

Applying Lemma 2.1, we obtain the following lemma.

Lemma 2.2. *Let $\mathcal{M}$ be an ω-saturated model of a complete theory T, $\bar{a}, \bar{b} \in M$, Z be the algebraic closure of some finite set in $\mathcal{M}$. If $(\mathrm{acl}(\bar{a}Z) \cap \mathrm{acl}(\bar{b}Z)) \setminus Z = \varnothing$ and $\varphi(x, \bar{a}')$ is a consistent formula, $\bar{a}' \in \bar{a}Z$, then there is $c \in \varphi(\mathcal{M}, \bar{a}')$ such that $(\mathrm{acl}(\bar{a}cZ) \cap \mathrm{acl}(\bar{b}Z)) \setminus Z = \varnothing$.*

Theorem 2.2. *Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, Z be the algebraic closure of some finite set in $\mathcal{M}$, a and b be elements of $\mathcal{M}$, $a, b \notin Z$. The following conditions are equivalent:*

(1) *the elements a and b are Z-separable in $H(\mathcal{M})$ by some sets y and z from $H(\mathcal{M})$ containing Z;*

(2) *the elements a and b are Z-separable in $H_{\omega_1}(\mathcal{M})$ by some sets y and z from $H_{\omega_1}(\mathcal{M})$ containing Z;*

(3) $(\mathrm{acl}(aZ) \cap \mathrm{acl}(bZ)) \setminus Z = \varnothing$.

Proof. As in the proof of Theorem 2.1 it suffices to prove the implication (3) $\Rightarrow$ (2). Assuming $(\mathrm{acl}(aZ)\cap\mathrm{acl}(bZ))\setminus Z=\varnothing$, we construct by induction countable models $\mathcal{N}_a,\mathcal{N}_b\preccurlyeq\mathcal{M}$ such that $\mathrm{acl}(aZ)\subseteq N_a$, $\mathrm{acl}(bZ)\subseteq N_b$, $(N_a\cap N_b)\setminus Z=\varnothing$, $N_a=\bigcup\limits_{n\in\omega}A_n$ for a chain of some sets A_n and $N_b=\bigcup\limits_{n\in\omega}B_n$ for a chain of some sets B_n.

In the initial step we consider the sets $A_0=\mathrm{acl}(aZ)$, $B_0=\mathrm{acl}(bZ)$ and enumerate all consistent formulas of the form $\varphi(x,\bar{a})$, $\bar{a}\in A_0$: $\Phi_0=\{\varphi_{0,m}(x,\bar{a}_m)\mid m\in\omega\}$. According to this enumeration we construct at most a countable set $A_1=\bigcup\limits_{m\in\omega\cup\{-1\}}A_{1,m}\supset A_0$ with the condition $(\mathrm{acl}(A_1)\cap B_0)\setminus Z=\varnothing$. Let $A_{1,-1}\rightleftharpoons A_0$. If the set $A_{1,m-1}$ had been already defined and $\varphi_{0,m}(\mathcal{M},\bar{a}_m)\cap A_{1,m-1}\neq\varnothing$, then put $A_{1,m}\rightleftharpoons A_{1,m-1}$; if $\varphi_{0,m}(\mathcal{M},\bar{a}_m)\cap A_{1,m-1}=\varnothing$ then by Lemma 2.2 we choose an element $c_m\in\varphi_m(\mathcal{M},\bar{a}_m)$ such that $(\mathrm{acl}(c_mA_{1,m-1})\cap\mathrm{acl}(B_0))\setminus Z=\varnothing$ and put $A_{1,m}\rightleftharpoons\mathrm{acl}(c_mA_{1,m-1})$.

If the set A_1 had been already defined, we extend symmetrically the set B_0 to an algebraically closed set B_1 such that $B_1\supseteq Z$, all consistent formulas $\varphi(x,\bar{b})$, $\bar{b}\in B_0$, are realized in B_1 и $(\mathrm{acl}(A_1)\cap\mathrm{acl}(B_1))\setminus Z=\varnothing$.

If at most countable sets A_n and B_n had been already constructed, we renumber all consistent formulas of the form $\varphi(x,\bar{a})$, $\bar{a}\in A_n$: $\Phi_n=\{\varphi_{n,m}(x,\bar{a}_m)\mid m\in\omega\}$. According to this numeration we construct at most a countable set $A_{n+1}=\bigcup\limits_{m\in\omega\cup\{-1\}}A_{n+1,m}\supset A_n$ with the condition $(\mathrm{acl}(A_{n+1})\cap\mathrm{acl}(B_1))\setminus Z=\varnothing$. Let $A_{n+1,-1}\rightleftharpoons A_n$. If the set $A_{n+1,m-1}$ had been already defined and $\varphi_{0,m}(\mathcal{M},\bar{a}_m)\cap A_{n+1,m-1}\neq\varnothing$, then put $A_{n+1,m}\rightleftharpoons A_{n+1,m-1}$; if $\varphi_{0,m}(\mathcal{M},\bar{a}_m)\cap A_{n+1,m-1}=\varnothing$, then by Lemma 2.2 we choose an element $c_m\in\varphi_{n,m}(\mathcal{M},\bar{a}_m)$ such that $(\mathrm{acl}(c_mA_{n+1,m-1})\cap\mathrm{acl}(B_n))\setminus Z=\varnothing$, and put $A_{n+1,m}\rightleftharpoons A_{n+1,m-1}\cup\{c_m\}$.

If we have the set A_{n+1} then we extend symmetrically the set B_n to at most a countable set B_{n+1} such that all consistent formulas $\varphi(x,\bar{b})$, $\bar{b}\in B_n$, are realized in B_{n+1} и $(\mathrm{acl}(A_{n+1})\cap\mathrm{acl}(B_{n+1}))\setminus Z=\varnothing$.

By constructing the sets $\bigcup\limits_{n\in\omega}A_n$ and $\bigcup\limits_{n\in\omega}B_n$ we form required universes N_a and N_b respectively of Z-separable countable models $\mathcal{N}_a,\mathcal{N}_b\preccurlyeq\mathcal{M}$ such that $a\in N_a$ and $b\in N_b$. $\square$

Combining the proofs of Claims 1.1–1.4 and Theorems 2.1, 2.2, we obtain the following assertions.

Corollary 2.1. *Let $\mathcal{M}$ be an ω-saturated model of a small theory T, Z be the algebraic closure of some finite set in $\mathcal{M}$, a and b be elements of $\mathcal{M}$, $a,b\notin Z$. The following conditions are equivalent:*

(1) *the element a is Z-separable from the element b in $H(\mathcal{M})$ by some set y from $H(\mathcal{M})$ containing Z;*

(2) *the element a is Z-separable from the element b in $H_{\omega_1}(\mathcal{M})$ by some set y from $H_{\omega_1}(\mathcal{M})$ containing Z;*

(3) *the element a is Z-separable from the element b in $H_p(\mathcal{M})$ by some set y from $H_p(\mathcal{M})$ containing Z;*

(4) $b\notin\mathrm{acl}(aZ)$.

Corollary 2.2. *Let $\mathcal{M}$ be an ω-saturated model of a small theory T, Z be the algebraic closure of some finite set in $\mathcal{M}$, a and b be elements of $\mathcal{M}$, $a,b\notin Z$. The following conditions are equivalent:*

(1) *the elements a and b are Z-separable in $H(\mathcal{M})$ by some sets y and z from $H(\mathcal{M})$ containing Z;*

(2) *the elements a and b are Z-separable in $H_{\omega_1}(\mathcal{M})$ by some sets y and z from $H_{\omega_1}(\mathcal{M})$ containing Z;*

(3) *the elements a and b are separable in $H_p(\mathcal{M})$ by some sets y and z from $H_p(\mathcal{M})$ containing Z;*
(4) $(\mathrm{acl}(aZ) \cap \mathrm{acl}(bZ)) \setminus Z = \varnothing$.

Theorem 2.3. *Let $\mathcal{M}$ be a λ-saturated model of a complete theory T, $\lambda \geq \max\{|\Sigma(T)|, \omega\}$, A and B be nonempty sets in $\mathcal{M}$ having cardinalities $< \lambda$, Z be the algebraic closure of some set of cardinality $< \lambda$ in $\mathcal{M}$. The following conditions are equivalent:*
(1) *the set A is Z-separable from the set B in $H(\mathcal{M})$ by some set y from $H(\mathcal{M})$ containing Z;*
(2) *the set A is Z-separable from the set B in $H_\lambda(\mathcal{M})$ by some set y from $H_\lambda(\mathcal{M})$ containing Z;*
(3) $(\mathrm{acl}(A \cup Z) \cap B) \setminus Z = \varnothing$.

Theorem 2.4. *Let $\mathcal{M}$ be a λ-saturated model of a complete theory T, $\lambda \geq \max\{|\Sigma(T)|, \omega\}$, A and B be nonempty sets in $\mathcal{M}$ having cardinalities $< \lambda$, Z be the algebraic closure of some set of cardinality $< \lambda$ in $\mathcal{M}$. The following conditions are equivalent:*
(1) *the sets A and B are Z-separable in $H(\mathcal{M})$ by some sets y and z from $H(\mathcal{M})$ containing Z;*
(2) *the sets A and B are Z-separable in $H_\lambda(\mathcal{M})$ by some sets y and z from $H_\lambda(\mathcal{M})$ containing Z;*
(3) $(\mathrm{acl}(A \cup Z) \cap \mathrm{acl}(B \cup Z)) \setminus Z = \varnothing$.

Corollary 2.3. *Let $\mathcal{M}$ be an ω-saturated model of a small theory T, A and B be finite nonempty sets in $\mathcal{M}$, Z be the algebraic closure of some finite set in $\mathcal{M}$. The following conditions are equivalent:*
(1) *the set A is Z-separable from the set B in $H(\mathcal{M})$ by some set y from $H(\mathcal{M})$ containing Z;*
(2) *the set A is Z-separable from the set B in $H_{\omega_1}(\mathcal{M})$ by some set y from $H_{\omega_1}(\mathcal{M})$ containing Z;*
(3) *the set A is Z-separable from the set B in $H_p(\mathcal{M})$ by some set y from $H_p(\mathcal{M})$ containing Z;*
(4) $(\mathrm{acl}(A \cup Z) \cap B) \setminus Z = \varnothing$.

Corollary 2.4. *Let $\mathcal{M}$ be an ω-saturated model of a small theory T, A and B be finite nonempty sets in $\mathcal{M}$, Z be the algebraic closure of some finite set in $\mathcal{M}$. The following conditions are equivalent:*
(1) *the sets A and B are Z-separable in $H(\mathcal{M})$ by some sets y and z from $H(\mathcal{M})$ containing Z;*
(2) *the sets A and B are Z-separable in $H_{\omega_1}(\mathcal{M})$ by some sets y and z from $H_{\omega_1}(\mathcal{M})$ containing Z;*
(3) *the sets A and B are Z-separable in $H_p(\mathcal{M})$ by some sets y and z from $H_p(\mathcal{M})$ containing Z;*
(4) $(\mathrm{acl}(A \cup Z) \cap \mathrm{acl}(B \cup Z)) \setminus Z = \varnothing$.

3 On separability in hypergraphs of models of ordered theories

Definition 6. [11, 4] Let $p_1(x_1), \ldots, p_n(x_n) \in S_1(T)$. A type $q(x_1, \ldots, x_n) \in S(T)$ is called $(p_1, \ldots, p_n)$*-type* if $q(x_1, \ldots, x_n) \supseteq \bigcup\limits_{i=1}^{n} p_i(x_i)$. The set of all $(p_1, \ldots, p_n)$-types of a theory T is denoted by $S_{p_1,\ldots,p_n}(T)$. A countable theory T is called *almost ω-categorical* if for any types $p_1(x_1), \ldots, p_n(x_n) \in S(T)$ there exist only finitely many types $q(x_1, \ldots, x_n) \in S_{p_1,\ldots,p_n}(T)$.

Theorem 3.1. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M}$ be an ω-saturated model of the theory T, Z be the algebraic closure of some finite set in $\mathcal{M}$, $a, b \in M \setminus Z$. Then the following conditions are equivalent:*

(1) *a is Z-separable from b in $\mathcal{H}(\mathcal{M})$ by some set y from $H(\mathcal{M})$ containing Z;*

(2) *b is Z-separable from a in $\mathcal{H}(\mathcal{M})$ by some set y from $H(\mathcal{M})$ containing Z;*

(3) *the elements a and b are Z-separable in $H(\mathcal{M})$ by some sets y and z from $H(\mathcal{M})$ containing Z;*

(4) $a \notin \mathrm{dcl}(\{bZ\})$;

(5) $b \notin \mathrm{dcl}(\{aZ\})$.

(6) $(\mathrm{dcl}(aZ) \cap \mathrm{dcl}(bZ)) \setminus Z = \varnothing$.

Proof. By Proposition 3.9 [7] Exchange Principle for the algebraic closure of any set holds. By the linear ordering of the model $\mathcal{M}$ $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any $A \subseteq M$. Then by the proofs of Theorems 2.1 and 2.2 we have an equivalence of the conditions (1)–(6). □

Note 3.1. 1. Theorem 3.1 remains true for an arbitrary theory satisfying both Exchange Principle for algebraic closures and the condition $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any $A \subseteq M$.

2. If Exchange Principle for algebraic closures holds and the condition $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any $A \subseteq M$ does not hold, Theorem 3.1 remains true if we replace dcl by acl.

3. If the condition $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any $A \subseteq M$ holds and Exchange Principle for algebraic closures does not hold, Theorem 3.1 splits into three independent statements (1) $\Leftrightarrow$ (5), (2) $\Leftrightarrow$ (4), (3) $\Leftrightarrow$ (6).

Theorem 3.1 immediately implies the following

Corollary 3.1. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M}$ be an ω-saturated model of the theory T, $a, b \in M \setminus dcl(\varnothing)$. Then the following conditions are equivalent:*

(1) *a is separable from b in $\mathcal{H}(\mathcal{M})$;*

(2) *b is separable from a in $\mathcal{H}(\mathcal{M})$;*

(3) $a \notin \mathrm{dcl}(\{b\})$;

(4) $b \notin \mathrm{dcl}(\{a\})$.

Example 1. [10] Let $\mathcal{M} = \langle M; <, P_1^1, P_2^1, f^1\rangle$ be a linearly ordered structure such that M is the disjoint union of interpretations of unary predicates P_1 and P_2 so that $P_1(\mathcal{M}) < P_2(\mathcal{M})$. We identify an interpretation of P_2 with the set of rational numbers $\mathbb{Q}$, ordered as usual, and P_1 with $\mathbb{Q} \times \mathbb{Q}$, ordered lexicographically. The symbol f is interpreted by a partial unary function with $Dom(f) = P_1(\mathcal{M})$ and $Range(f) = P_2(\mathcal{M})$ and is defined by the equality $f((n, m)) = n$ for all $(n, m) \in \mathbb{Q} \times \mathbb{Q}$.

It is known that $\mathcal{M}$ is a countably categorical weakly o-minimal structure, and $Th(\mathcal{M})$ is not quite o-minimal. Take arbitrary $a \in P_1(\mathcal{M})$, $b \in P_2(\mathcal{M})$ such that $f(a) = b$. Then we obtain that a is separable from b in $\mathcal{H}(\mathcal{M})$, but b is not separable from a in $\mathcal{H}(\mathcal{M})$.

Proposition 3.1. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M} \models T$, $A = \{a_1, \ldots, a_{n_1}\}$, $B = \{b_1, \ldots, b_{n_2}\} \subseteq M$ for some positive $n_1, n_2 < \omega$. Then the following conditions are equivalent:*

(1) *A and B are separable from each other in $\mathcal{H}(\mathcal{M})$;*

(2) $\mathrm{dcl}(A) \cap \mathrm{dcl}(B) = \emptyset$.

(3) $\mathrm{dcl}(\{a_i\}) \cap \mathrm{dcl}(\{b_j\}) = \emptyset$ *for any* $1 \le i \le n_1$, $1 \le j \le n_2$.

Proof. (1) $\Rightarrow$ (2) Let A be separable from B in $\mathcal{H}(\mathcal{M})$. This means that there is $\mathcal{M}_1 \prec \mathcal{M}$ such that $A \subseteq M_1$ and $B \cap M_1 = \emptyset$. Then we have: $\mathrm{dcl}(A) \subseteq M_1$, hence we obtain that

$\mathrm{dcl}(A) \cap B = \emptyset$. Similarly, by the condition of separability of B from A in $\mathcal{H}(\mathcal{M})$ it can be established that $\mathrm{dcl}(B) \cap A = \emptyset$.

Assume to the contrary that $\mathrm{dcl}(A) \cap \mathrm{dcl}(B) \neq \emptyset$. Consequently, there is $c \in M$ such that $c \in \mathrm{dcl}(A)$ and $c \in \mathrm{dcl}(B)$. But then by the binarity of $Th(\mathcal{M})$ there exist $a \in A$ and $b \in B$ such that $c \in \mathrm{dcl}(\{a\})$ and $c \in \mathrm{dcl}(\{b\})$. By holding Exchange Principle for algebraic closures we obtain that $b \in \mathrm{dcl}(\{a\})$. This contradicts the condition $\mathrm{dcl}(A) \cap B = \emptyset$.

(2) $\Rightarrow$ (1) In this case we assert that $M_1 := M \setminus \mathrm{dcl}(A)$ and $M_2 := M \setminus \mathrm{dcl}(B)$ are universes of elementary submodels of the model $\mathcal{M}$.

(2) $\Leftrightarrow$ (3) By binarity of $Th(\mathcal{M})$. □

Proposition 3.2. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M} \models T$, $Z = \mathrm{dcl}(\emptyset)$, $A = \{a_1, \ldots, a_{n_1}\}$, $B = \{b_1, \ldots, b_{n_2}\} \subseteq M$ for some positive $n_1, n_2 < \omega$ so that $A \cap Z = B \cap Z = \emptyset$. Then the following conditions are equivalent:*

(1) *A and B are Z-separable in $\mathcal{H}(\mathcal{M})$;*

(2) *$\mathrm{dcl}(A) \cap \mathrm{dcl}(B) = Z$.*

(3) *$\mathrm{dcl}(\{a_i\}) \cap \mathrm{dcl}(\{b_j\}) = Z$ for any $1 \leq i \leq n_1$, $1 \leq j \leq n_2$.*

Proof. (1) $\Rightarrow$ (2) Let A and B be Z-separable in $\mathcal{H}(\mathcal{M})$. Then there exist $\mathcal{M}_1, \mathcal{M}_2 \prec \mathcal{M}$ such that $(\mathcal{M}_1 \cap \mathcal{M}_2) \setminus Z = \emptyset$, $A \subseteq M_1$ and $B \subseteq M_2$. Consequently, $\mathrm{dcl}(A) \cap \mathrm{dcl}(B) \subseteq \mathcal{M}_1 \cap \mathcal{M}_2$. Then $[\mathrm{dcl}(A) \cap \mathrm{dcl}(B)] \setminus Z = \emptyset$, hence $\mathrm{dcl}(A) \cap \mathrm{dcl}(B) = Z$.

(2) $\Rightarrow$ (1) In this case we assert that $M_1 := [M \setminus \mathrm{dcl}(A)] \cup Z$ and $M_2 := [M \setminus \mathrm{dcl}(B)] \cup Z$ are universes of elementary submodels of the model $\mathcal{M}$. □

The arguments for Propositions 1.1 and 3.1 imply the following

Proposition 3.3. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M} \models T$, $\emptyset \neq A, B \subseteq M$, $\mathcal{M}$ be $|A \cup B|^+$-saturated. Then A and B are separable from each other in $\mathcal{H}(\mathcal{M})$ if and only if the following conditions hold:*

(1) *$\mathrm{dcl}(\{a\}) \cap \mathrm{dcl}(\{b\}) = \emptyset$ for any $a \in A$ and $b \in B$;*

(2) *for any isolated type $p \in S_1(\emptyset)$, $p(\mathcal{M}) \setminus \mathrm{dcl}(A) \neq \emptyset$ and $p(\mathcal{M}) \setminus \mathrm{dcl}(B) \neq \emptyset$.*

Corollary 3.2. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M} \models T$, $Z = \mathrm{dcl}(\emptyset)$, A, B be non-empty subsets of M such that $A \cap Z = B \cap Z = \emptyset$, $\mathcal{M}$ be $|A \cup B|^+$-saturated. Then A and B are Z-separable in $\mathcal{H}(\mathcal{M})$ if and only if the following conditions hold:*

(1) *$\mathrm{dcl}(\{a\}) \cap \mathrm{dcl}(\{b\}) = Z$ for any $a \in A$ and $b \in B$;*

(2) *for any isolated type $p \in S_1(\emptyset)$, $p(\mathcal{M}) \setminus \mathrm{dcl}(A) \neq \emptyset$ and $p(\mathcal{M}) \setminus \mathrm{dcl}(B) \neq \emptyset$.*

The arguments for Propositions 1.1 and 3.1 as well as Theorem 2.4 imply the following

Proposition 3.4. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M} \models T$ be λ-saturated, $\lambda \geq \max\{|\Sigma(T)|, \omega\}$, A and B be nonempty sets in $\mathcal{M}$ having cardinalities $< \lambda$, Z be the algebraic closure of some set of cardinality $< \lambda$ in $\mathcal{M}$. Then the following conditions are equivalent:*

(1) *A and B are Z-separable in $\mathcal{H}(\mathcal{M})$;*

(2) *$(\mathrm{dcl}(aZ) \cap \mathrm{dcl}(bZ)) \setminus Z = \emptyset$ for any $a \in A$ and $b \in B$.*

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