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CONCINNITY OF DYNAMIC INEQUALITIES DESIGNED ON CALCULUS OF TIME SCALES

M.J.S. Sahir, F. Chaudhry

Communicated by L.-E. Persson

Key words: time scales, dynamic inequalities, Kantorovich's ratio, Specht's ratio.

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Abstract. We present some reverse dynamic inequalities of Radon's and Bergström's type on time scales in general form. The extension of Clarkson's dynamic inequality on time scales is also given. Our further investigations explore some dynamic inequalities by using Kantorovich's and Specht's ratios. The calculus of time scales unifies and extends continuous results and their corresponding discrete and quantum analogues.

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1 Introduction

Motivated by the recent developments of the theory and applications of time scales, we will prove some results on time scales. The calculus of time scales was introduced by Stefan Hilger [9]. A time scale is an arbitrary nonempty closed subset of $\mathbb{R}$ of all real numbers. The theory of time scales calculus is applied to harmonize results in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The three popular branches of time scales calculus are delta calculus, nabla calculus and diamond-α calculus. Many dynamic inequalities (see [1, 4, 6, 12, 13, 14, 15]) have been investigated by using this hybrid theory. Basic work on dynamic inequalities is done by Agarwal, Anastassiou, Bohner, Peterson, O'Regan, Saker and several other authors.

In this paper, it is assumed that all integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of the interval $[a, b]$ the given time scale.

2 Preliminaries

We present basic concepts of delta calculus. The results of delta calculus are taken from monographs [4, 5].

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^Δ is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$, such that for all $\epsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be delta differentiable at t, and $f^\Delta(t)$ is called the delta derivative of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous), if it is continuous at each right-dense point and there exists a finite left-sided limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [4, 5].

Definition 1. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [2, 4, 5].

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$ and $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$. A function $f : \mathbb{T}_k \to \mathbb{R}$ is called nabla differentiable at $t \in \mathbb{T}_k$, with nabla derivative $f^\nabla(t)$, if there exists $f^\nabla(t) \in \mathbb{R}$, such that for all $\epsilon > 0$, there is a neighborhood V of t, such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be left-dense continuous (ld-continuous), provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [2, 4, 5].

Definition 2. A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^\nabla(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Next, we present an introduction to the diamond-α derivative, see [1, 16].

Definition 3. Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}$, the diamond-α dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha) f^\nabla(t), \quad 0 \leq \alpha \leq 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

Theorem 2.1 (See [16]). *Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$ and we write $f^\sigma(t) = f(\sigma(t))$, $g^\sigma(t) = g(\sigma(t))$, $f^\rho(t) = f(\rho(t))$ and $g^\rho(t) = g(\rho(t))$. Then*

(i) $f \pm g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with

$$(f \pm g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) \pm g^{\diamond_\alpha}(t).$$

(ii) $fg : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^\sigma(t)g^\Delta(t) + (1-\alpha)f^\rho(t)g^\nabla(t).$$

(iii) For $g(t)g^\sigma(t)g^\rho(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t)g^\sigma(t)g^\rho(t) - \alpha f^\sigma(t)g^\rho(t)g^\Delta(t) - (1-\alpha)f^\rho(t)g^\sigma(t)g^\nabla(t)}{g(t)g^\sigma(t)g^\rho(t)}.$$

Definition 4 (See [16]). Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s, \quad 0 \leq \alpha \leq 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 2.2 (See [16]). *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$-integrable functions on $[a, b]_\mathbb{T}$. Then*

(i) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$;

(ii) $\int_a^t cf(s) \diamond_\alpha s = c\int_a^t f(s) \diamond_\alpha s$;

(iii) $\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s$;

(iv) $\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s$;

(v) $\int_a^a f(s) \diamond_\alpha s = 0$.

We also consider *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0.$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, +\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement of Young's inequality [20] in terms of Kantorovich's ratio holds

$$K^\eta\left(\frac{a}{b}\right)a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q} \tag{2.1}$$

for $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$ and $\eta = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Specht's ratio [7, 17] is defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}} \qquad (h > 0, h \neq 1).$$

We present here some properties of Specht's ratio. See [7, 17, 18] for the proof and details:

(i) $S(1) = 1$ and $S(h) = S\left(\frac{1}{h}\right) > 1$ for all $h > 0$.

(ii) $S(h)$ is a monotone increasing function on $(1, +\infty)$ and monotone decreasing function on $(0, 1)$.

The following inequality is due to Furuichi [8] and provides a refinement for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{\eta}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q} \tag{2.2}$$

for $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$ and $\eta = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

3 Main results

In this section, we give the following extension of reverse Radon's inequality on time scales.

Theorem 3.1. *Let $w, f, g \in C\left([a, b]_{\mathbb{T}}, \mathbb{R} \setminus \{0\}\right)$ be $\diamond_\alpha$-integrable functions. If $\beta > 0$, $\gamma \geq 1$ and $0 < m \leq \left(\frac{|f(x)|}{|g(x)|}\right)^{\beta+\gamma} \leq M < \infty$ on the set $[a, b]_{\mathbb{T}}$, then*

$$\int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x \leq \left(\frac{M}{m}\right)^{\frac{\beta+\gamma-1}{\beta+\gamma}} \frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma-1}}. \tag{3.1}$$

Proof. Let $p = \beta + \gamma$ and $q = \frac{\beta+\gamma}{\beta+\gamma-1}$. Consider the conditions $0 < m \leq \frac{|f(x)|^p}{|g(x)|^q} \leq M$, therefore

$$|g(x)| \geq M^{-\frac{1}{q}} |f(x)|^{\frac{p}{q}} \Rightarrow |f(x)g(x)| \geq M^{-\frac{1}{q}} |f(x)|^p, \quad \forall x \in [a, b]_{\mathbb{T}}.$$

Thus, we have

$$\left(\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x\right)^{\frac{1}{p}} \geq M^{-\frac{1}{pq}} \left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}}. \tag{3.2}$$

On the other hand, we have that

$$|f(x)| \geq m^{\frac{1}{p}} |g(x)|^{\frac{q}{p}} \Rightarrow |f(x)g(x)| \geq m^{\frac{1}{p}} |g(x)|^q, \quad \forall x \in [a, b]_{\mathbb{T}}.$$

Thus,

$$\left(\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x\right)^{\frac{1}{q}} \geq m^{\frac{1}{pq}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}. \tag{3.3}$$

Multiplying (3.2) and (3.3), we obtain

$$\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x. \tag{3.4}$$

Replacing $|f(x)|$ by $\frac{|f(x)|}{|g(x)|^{\frac{1}{q}}}$ and $|g(x)|$ by $|g(x)|^{\frac{1}{q}}$ in inequality (3.4), simultaneously, we obtain

$$\left(\int_a^b \frac{|w(x)||f(x)|^p}{|g(x)|^{\frac{p}{q}}} \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \int_a^b |w(x)||f(x)| \diamond_\alpha x. \tag{3.5}$$

Hence (3.5) takes the form

$$\int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta+\gamma-1}} \diamond_\alpha x \leq \left(\frac{M}{m}\right)^{\frac{\beta+\gamma-1}{\beta+\gamma}} \frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\beta+\gamma-1}}. \tag{3.6}$$

Replacing $|w(x)|$ by $|w(x)||g(x)|^{\gamma-1}$ in inequality (3.6), we obtain (3.1). □

Next, we give extended reverse Bergström's inequality on time scales.

Corollary 3.1. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R}\setminus\{0\})$ be $\diamond_\alpha$-integrable functions. If $0 < m \leq \left(\frac{|f(x)|}{|g(x)|}\right)^2 \leq M < \infty$ on the set $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b \frac{|w(x)||f(x)|^2}{|g(x)|} \diamond_\alpha x \leq \left(\frac{M}{m}\right)^{\frac{1}{2}} \frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^2}{\int_a^b |w(x)||g(x)| \diamond_\alpha x}. \tag{3.7}$$

Proof. Putting $\beta = \gamma = 1$ in Theorem 3.1, we get (3.7). □

Remark 1. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k > 0$ and $g(k) = y_k > 0$ for $k \in \{1, 2, \ldots, n\}$. If $\gamma = 1$, then (3.1) reduces to

$$\sum_{k=1}^{n} \frac{x_k^{\beta+1}}{y_k^{\beta}} \leq \left(\frac{M}{m}\right)^{\frac{\beta}{\beta+1}} \frac{\left(\sum_{k=1}^{n} x_k\right)^{\beta+1}}{\left(\sum_{k=1}^{n} y_k\right)^{\beta}}. \tag{3.8}$$

Inequality (3.8) is just the reverse of the classical inequality

$$\frac{\left(\sum_{k=1}^{n} x_k\right)^{\beta+1}}{\left(\sum_{k=1}^{n} y_k\right)^{\beta}} \leq \sum_{k=1}^{n} \frac{x_k^{\beta+1}}{y_k^{\beta}}. \tag{3.9}$$

The inequality from (3.9) is called, in literature, Radon's inequality *[10]*.

Remark 2. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k > 0$ and $g(k) = y_k > 0$ for $k \in \{1, 2, \ldots, n\}$. Then inequality given in (3.7) reduces to

$$\sum_{k=1}^{n} \frac{x_k^2}{y_k} \leq \left(\frac{M}{m}\right)^{\frac{1}{2}} \frac{\left(\sum_{k=1}^{n} x_k\right)^2}{\sum_{k=1}^{n} y_k}. \tag{3.10}$$

Inequality (3.10) is just the reverse of the classical inequality

$$\frac{\left(\sum_{k=1}^{n} x_k\right)^2}{\sum_{k=1}^{n} y_k} \leq \sum_{k=1}^{n} \frac{x_k^2}{y_k}. \tag{3.11}$$

Inequality (3.11) is called Bergström's or Titu Andreescu's inequality or also Engel's inequality in literature as given in *[3]* with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \ldots = \frac{x_n}{y_n}$.

The following result is an extension of dynamic Clarkson's type inequality on time scales.

Theorem 3.2. *Let* $p \geq 1$, $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$. *If* $1 < m \leq \frac{|f(x)|}{|g(x)|} \leq M < \infty$, $\forall x \in [a,b]_{\mathbb{T}}$, *then*

$$\int_a^b |w(x)| \left(|f(x)|^p + |g(x)|^p\right) \diamond_\alpha x$$
$$\leq \Lambda \int_a^b |w(x)| \left(|f(x)| + |g(x)|\right)^p \diamond_\alpha x + \Omega \int_a^b |w(x)| \left(|f(x)| - |g(x)|\right)^p \diamond_\alpha x, \quad (3.12)$$

where $\Lambda = \frac{M^p(m+1)^p + (M+1)^p}{2(M+1)^p(m+1)^p}$ *and* $\Omega = \frac{1+m^p}{2(m-1)^p}$.

Proof. By using the given condition $\frac{|f(x)|}{|g(x)|} \leq M$, we have

$$(M+1)^p |f(x)|^p \leq M^p \left(|f(x)| + |g(x)|\right)^p, \quad \forall x \in [a,b]_{\mathbb{T}}.$$

Therefore

$$\int_a^b |w(x)||f(x)|^p \diamond_\alpha x \leq \left(\frac{M}{M+1}\right)^p \int_a^b |w(x)| \left(|f(x)| + |g(x)|\right)^p \diamond_\alpha x. \quad (3.13)$$

On the other hand, we have that

$$\left(1 + \frac{1}{m}\right)^p |g(x)|^p \leq \left(\frac{1}{m}\right)^p \left(|f(x)| + |g(x)|\right)^p, \quad \forall x \in [a,b]_{\mathbb{T}}.$$

Thus,

$$\int_a^b |w(x)||g(x)|^p \diamond_\alpha x \leq \left(\frac{1}{m+1}\right)^p \int_a^b |w(x)| \left(|f(x)| + |g(x)|\right)^p \diamond_\alpha x. \quad (3.14)$$

Adding (3.13) and (3.14), we obtain

$$\int_a^b |w(x)| \left(|f(x)|^p + |g(x)|^p\right) \diamond_\alpha x$$
$$\leq \left\{ \left(\frac{M}{M+1}\right)^p + \left(\frac{1}{m+1}\right)^p \right\} \int_a^b |w(x)| \left(|f(x)| + |g(x)|\right)^p \diamond_\alpha x. \quad (3.15)$$

By given hypothesis, we have

$$m - 1 \leq \frac{|f(x)|}{|g(x)|} - 1 \Rightarrow |g(x)| \leq \frac{|f(x)| - |g(x)|}{m-1},$$

where $\forall x \in [a,b]_{\mathbb{T}}$. Thus,

$$\int_a^b |w(x)||g(x)|^p \diamond_\alpha x \leq \left(\frac{1}{m-1}\right)^p \int_a^b |w(x)| \left(|f(x)| - |g(x)|\right)^p \diamond_\alpha x. \quad (3.16)$$

On the other hand, we have that

$$1 - \frac{1}{m} \leq 1 - \frac{|g(x)|}{|f(x)|} \Rightarrow |f(x)| \leq \frac{m}{m-1} \left(|f(x)| - |g(x)|\right), \quad \forall x \in [a,b]_{\mathbb{T}}.$$

Thus,

$$\int_a^b |w(x)||f(x)|^p \diamond_\alpha x \leq \left(\frac{m}{m-1}\right)^p \int_a^b |w(x)| \left(|f(x)| - |g(x)|\right)^p \diamond_\alpha x. \quad (3.17)$$

Adding (3.16) and (3.17), we obtain

$$\int_a^b |w(x)| \left(|f(x)|^p + |g(x)|^p\right) \diamond_\alpha x \\ \leq \left\{ \left(\frac{m}{m-1}\right)^p + \left(\frac{1}{m-1}\right)^p \right\} \int_a^b |w(x)| \left(|f(x)| - |g(x)|\right)^p \diamond_\alpha x. \quad (3.18)$$

Adding (3.15) and (3.18), we get the desired inequality (3.12). □

Remark 3. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k > 0$ and $g(k) = y_k > 0$ for $k \in \{1, 2, \ldots, n\}$. Then (3.12) reduces to

$$\sum_{k=1}^{n} (x_k^p + y_k^p) \leq \Lambda \sum_{k=1}^{n} (x_k + y_k)^p + \Omega \sum_{k=1}^{n} (x_k - y_k)^p. \quad (3.19)$$

Our next result concerning extended Young's inequality with Kantorovich's ratio on time scales is investigated.

Theorem 3.3. *Let $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $f \equiv 0$ nor $g \equiv 0$. If $0 < m \leq \left|\frac{f(x)}{g(x)}\right| \leq M < \infty$, $\forall x \in [a,b]_{\mathbb{T}}$, then*

$$\int_a^b K^\eta\left(\frac{|f(x)|^p}{|g(x)|^q}\right) |w(x)||f(x)g(x)| \diamond_\alpha x \leq \Lambda \int_a^b |w(x)| \left(|f(x)|^p + |g(x)|^p\right) \diamond_\alpha x \\ + \Omega \int_a^b |w(x)| \left(|f(x)|^q + |g(x)|^q\right) \diamond_\alpha x, \quad (3.20)$$

where $\Lambda = \frac{2^{p-1}M^p}{p(M+1)^p}$, $\Omega = \frac{2^{q-1}}{q(m+1)^q}$, $\eta = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $K(.)$ is Kantorovich's ratio.

Proof. By using the given hypothesis, we have that

$$\frac{|f(x)|}{|g(x)|} \leq M \Rightarrow (M+1)|f(x)| \leq M(|f(x)| + |g(x)|), \quad \forall x \in [a,b]_{\mathbb{T}}.$$

Therefore

$$\int_a^b |w(x)||f(x)|^p \diamond_\alpha x \leq \left(\frac{M}{M+1}\right)^p \int_a^b |w(x)|(|f(x)| + |g(x)|)^p \diamond_\alpha x. \quad (3.21)$$

On the other hand, we have that

$$m \leq \frac{|f(x)|}{|g(x)|} \Rightarrow (m+1)|g(x)| \leq |f(x)| + |g(x)|, \quad \forall x \in [a,b]_{\mathbb{T}}.$$

Thus,

$$\int_a^b |w(x)||g(x)|^q \diamond_\alpha x \leq \left(\frac{1}{m+1}\right)^q \int_a^b |w(x)|(|f(x)| + |g(x)|)^q \diamond_\alpha x. \quad (3.22)$$

Now, using Young's inequality (2.1), we have

$$K^\eta\left(\frac{|f(x)|^p}{|g(x)|^q}\right) |f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q, \quad \forall x \in [a,b]_{\mathbb{T}}. \quad (3.23)$$

Inequality (3.23) takes the form

$$\int_a^b K^\eta\left(\frac{|f(x)|^p}{|g(x)|^q}\right)|w(x)||f(x)g(x)|\diamond_\alpha x \leq \frac{1}{p}\int_a^b |w(x)||f(x)|^p\diamond_\alpha x + \frac{1}{q}\int_a^b |w(x)||g(x)|^q\diamond_\alpha x. \quad (3.24)$$

By using the results from (3.21) and (3.22), inequality (3.24) becomes

$$\begin{aligned}\int_a^b K^\eta\left(\frac{|f(x)|^p}{|g(x)|^q}\right)|w(x)||f(x)g(x)|\diamond_\alpha x &\leq \frac{1}{p}\left(\frac{M}{M+1}\right)^p\int_a^b |w(x)|(|f(x)|+|g(x)|)^p\diamond_\alpha x \\ &\quad + \frac{1}{q}\left(\frac{1}{m+1}\right)^q\int_a^b |w(x)|(|f(x)|+|g(x)|)^q\diamond_\alpha x. \quad (3.25)\end{aligned}$$

Using the elementary inequality

$$(x+y)^\delta \leq 2^{\delta-1}(x^\delta+y^\delta), \quad \delta>1, \quad x,y\geq 0,$$

inequality (3.20) follows from inequality (3.25). □

Our next result concerning extended Young's inequality with Specht's ratio on time scales is explored.

Theorem 3.4. *Let $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $w,f,g\in C([a,b]_{\mathbb{T}},\mathbb{R})$, neither $f\equiv 0$ nor $g\equiv 0$. If $0<m\leq\left|\frac{f(x)}{g(x)}\right|\leq M<\infty$, $\forall x\in[a,b]_{\mathbb{T}}$, then*

$$\begin{aligned}\int_a^b S\left(\left(\frac{|f(x)|^p}{|g(x)|^q}\right)^\eta\right)|w(x)||f(x)g(x)|\diamond_\alpha x &\leq \Lambda\int_a^b |w(x)|\left(|f(x)|^p+|g(x)|^p\right)\diamond_\alpha x \\ &\quad + \Omega\int_a^b |w(x)|\left(|f(x)|^q+|g(x)|^q\right)\diamond_\alpha x, \quad (3.26)\end{aligned}$$

where $\Lambda=\frac{2^{p-1}M^p}{p(M+1)^p}$, $\Omega=\frac{2^{q-1}}{q(m+1)^q}$, $\eta=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $S(.)$ is Specht's ratio.

Proof. Applying (2.2) and the rest of this proof is similar to that of Theorem 3.3. □

Remark 4. Let $\alpha=1$, $\mathbb{T}=\mathbb{Z}$, $a=1$, $b=n+1$, $w\equiv 1$, $f(k)=x_k>0$ and $g(k)=y_k>0$ for $k\in\{1,2,\ldots,n\}$. Then (3.20) reduces to

$$\sum_{k=1}^n K^\eta\left(\frac{x_k^p}{y_k^q}\right)x_k y_k \leq \Lambda\sum_{k=1}^n\left(x_k^p+y_k^p\right)+\Omega\sum_{k=1}^n\left(x_k^q+y_k^q\right) \quad (3.27)$$

and (3.26) reduces to

$$\sum_{k=1}^n S\left(\left(\frac{x_k^p}{y_k^q}\right)^\eta\right)x_k y_k \leq \Lambda\sum_{k=1}^n\left(x_k^p+y_k^p\right)+\Omega\sum_{k=1}^n\left(x_k^q+y_k^q\right). \quad (3.28)$$

4 Conclusion and future work

By using Hölder's reverse fractional integral inequality, weighted Radon's reverse integral inequality [11] was established in continuous form. Inspired by this work, we have presented an extended dynamic reverse Radon's inequality given in Theorem 3.1 on time scales in a more general form. A fractional integral Clarkson-type inequality [11] was also established in continuous form. We have presented Clarkson-type dynamic inequality in the extended form given in Theorem 3.2 on time scales. Motivated by the works of [6, 19], some dynamic inequalities in hybrid and comprehensive forms are established in this research article by using Kantorovich's ratio and Specht's ratio, respectively.

In our future research work, we will continue to find further dynamic inequalities and their reverse versions and applications in extended and generalized forms. It will be interesting to explore dynamic inequalities by using fractional calculus on time scales.

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