ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2018, Volume 9, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http : //publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 515
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

SULTANAEV YAUDAT TALGATOVICH

(to the 70th birthday)

On 19th July 2018 was 70th birthday of Yaudat Talgatovich Sultanaev, doctor of science (1990), professor (1991), honorary scientist of the Russian Federation, laureate of State award of the Republic of Bashkortostan in the field of science and technology, professor of the Bashkir State Pedagogical University, member of the Editorial Board of the Eurasian Mathematical Journal.

Ya.T. Sultanaev was born in the sity of Orsk. In 1971 he graduated from the Bashkir State University and then completed his postgraduate studies in the Moscow State University. Ya.T. Sultanaev's scientific supervisors were distinguished mathematicians A.G. Kostyuchenko and B.M. Levitan.

Ya.T. Sultanaev is a famous specialist in the spectral theory of differential operators and the qualitative theory of ordinary differential equations.

He obtained bilateral Tauberian theorems of Keldysh type, completely solved the problem on spectral assymptotics for semi-bounded ordinary differential operators, suggested a new method of investigation of assymptotic behaviour of solutions to singular differential equations which allowed him to essentially weaken the conditions on coefficients.

Jointly with V.A. Sadovnichii and A.M. Akhtyamov, he investigated inverse spectral problems with non-separated boundary conditions.

He published more than 70 papers in leading mathematical journals.

Among pupils of Ya.T. Sultanaev there are more than 20 candidates of science and one doctor of science.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yaudat Talgatovich on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 9, Number 4 (2018), 22 – 60

ON THE MEAN CONVERGENCE OF MULTIPLE FOURIER SERIES AND THE ASYMPTOTICS OF THE DIRICHLET KERNEL OF SPHERICAL MEANS

K.I. Babenko

Communicated by E.D. Nursultanov

Key words: Lebesgue constants, Fourier transform, integrability.

AMS Mathematics Subject Classification: 42B15, 42B10.

Abstract. Preprint no. 52 (1971), Institut Prikladnoi Matematiki Akademii Nauk USSR, Moscow (in Russian). Translated into English by E. Liflyand. See comments to this paper by E. Liflyand in this volume of the Eurasian Mathematical Journal.

DOI: https://doi.org/10.32523/2077-9879-2018-9-4-22-60

Introduction

In the theory of multiple Fourier series, problems of convergence of the so-called spherical partial sums have been studied for a long time. A multiple Fourier series can be interpreted as an expansion in eigenfunctions of the Laplace operator, and, moreover, the natural way of ordering of the eigenfunctions which corresponds to the increasing order of the eigenvalues leads just to the spherical means. This makes the interest in the spherical means understandable.

As is well known, spherical means differ drastically in their properties from the rectangular means, and first of all, in such problems as convergence in C and in L^p, $p \geq 1$, and almost everywhere convergence. It is clear that in these questions of crucial importance are asymptotic properties of the corresponding Dirichlet kernel, or, if one takes into account the relation with the boundary problem, asymptotic properties of the spectral function.

Let $x \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$, let $\mathbb{Z}^m$ be the lattice grid in $\mathbb{R}^m$, with $n \in \mathbb{Z}^m$, $n = (n_1, ..., n_m)$, n_j are integers; and set $n \cdot x = n_1x_1 + ... + n_mx_m$. Let I be the interval in $\mathbb{R}^m$

$$I = \{x \in \mathbb{R}^m : |x_j| \leq \frac{1}{2}, 1 \leq j \leq m\},$$

and let $f(x)$ be defined in it and Lebesgue integrable. Let us expand $f(x)$ in the multiple Fourier series

$$f(x) \sim \sum_{n\in\mathbb{Z}^m} a_n \exp\{2\pi i n \cdot x\}. \tag{0.5}$$

The spherical means of order α of series (0.5) are of the form

$$\sum_{0\leq|n|\leq N}{}' a_n \left(1 - \frac{|n|^2}{N^2}\right)^{\alpha} \exp\{2\pi i n \cdot x\} = S_N^{\alpha}(x; f), \tag{0.6}$$

where $'$ after the sign of the sum means that the summands corresponding to $|n|^2 = N^2$ are taken with the factor $\frac{1}{2}$.

Sum (0.6) is associated with the Dirichlet kernel

$$D_N^\alpha(x) = \sum_{0\le|n|\le N}{}' \left(1-\frac{|n|^2}{N^2}\right)^\alpha \exp\{2\pi i n\cdot x\}, \tag{0.7}$$

asymptotic properties of which are investigated in the first part of the present work. But the asymptotics of the kernel D_N^α is intimately related with the classical and well studied problem of the representation of numbers by the sums of squares. Denote by $r_m(k)$ the number of integer solutions of the Diophantine equation

$$n_1^2+...+n_m^2=k.$$

It is plain to show that

$$D_N^\alpha(x)=\frac{\Gamma(\alpha+1)}{\pi^\alpha}N^{\frac{m}{2}-\alpha}\sum_{n\in\mathbb{Z}^m}\frac{J_{\frac{m}{2}+\alpha}(2\pi N|n+x|)}{|n+x|^{\frac{m}{2}+\alpha}},\quad x\in I, \tag{0.8}$$

where for $\alpha\le\frac{m-1}{2}$ the series is summable by the Riesz means of high enough order. Taking $x=0$, we obtain

$$\begin{aligned} D_N^\alpha(0) &= \sum_{0\le k\le N^2}{}' \left(1-\frac{k}{N^2}\right)^\alpha r_m(k) = \pi^{\frac{m}{2}}\frac{\Gamma(\alpha+1)}{\Gamma(\frac{m}{2}+\alpha+1)}N^m \\ &+\frac{\Gamma(\alpha+1)}{\pi^\alpha}N^{\frac{m}{2}-\alpha}\sum_{k=1}^\infty r_m(k)\frac{J_{\frac{m}{2}+\alpha}(2\pi N\sqrt{k})}{k^{\frac{m}{4}+\frac{\alpha}{2}}}, \end{aligned} \tag{0.9}$$

which, for $\alpha=0$, yields the Hardy-Landau formula [2] if $m=2$ and that of Walfisch [13] and Oppenheim [6] if $m>2$.

Formulas (0.8) and (0.9) are inconvenient, therefore more convenient asymptotic formulas are desirable for the study of summability of multiple Fourier series. It is advisable to establish such a formula for the kernel

$$S_N^\alpha(x) = \sum_{0\le|n|\le N}{}' \left(1-\frac{|n|^2}{N^2}\right)^\alpha |n|^{-\kappa}\exp(2\pi i n\cdot x),$$

with $\kappa\ge0$, rather than for the kernel $D_N^\alpha(x)$. Let T be some quantity subject to condition $T\ge CN$, where C is a constant independent of N and T. The following relation takes place:

$$\begin{aligned} S_N^\alpha(x) &= 2\pi N^{\frac{m}{2}-\kappa+1}|x|^{-\frac{m}{2}+1}\int_0^1(1-u^2)^\alpha J_{\frac{m}{2}-1}(2\pi N|x|u)u^{\frac{m}{2}-\kappa}du \\ &+\frac{\Gamma(\alpha+1)}{\pi^{\alpha+1}}N^{\frac{m-1}{2}-\alpha-\kappa} \\ &\times \mathrm{Re}\sum_{\substack{|n+x|\le\frac{2}{\pi}TN^{-1}\\ n\ne0}}\frac{\exp\{2\pi iN|x|-\omega_m\}}{|n+x|^{\frac{m+1}{2}+\alpha}}\sum_{\nu=0}^q\frac{a_\nu(|n+x|)}{(|n+x|N)^\nu} \\ &+N^{m-\kappa+\varepsilon}T^{-\alpha}\sum_{\frac12N\le|n|\le2N}|n|^{-m-\varepsilon}\Psi_\alpha\left(2T\log\frac{T}{|n|},T\right)\exp(2\pi in\cdot x) \\ &+\sum_{\nu=0}^{\le\frac{\kappa+\varepsilon}{2}}(-1)^\nu N^{-2\nu}\frac{\Gamma(\alpha+1)}{\nu!(\alpha+1-\nu)}\eta(\kappa,2\nu x) \\ &+O\left(\varepsilon^{-1}N^{-\frac{\varepsilon}{2}}+T^{-\frac{m}{2}-\alpha-\varepsilon}N^{m-x}\right). \end{aligned} \tag{0.10}$$

Here $\omega_m = \frac{\pi}{2}(\frac{m+1}{2} + \alpha)$, $q = [\frac{m}{2} + \varepsilon - \alpha] + 2$, and ε is an arbitrary quantity from the interval $[0, 1]$. Further, $a_\nu(u)$, $0 \le u < \infty$, are functions of the complicated concrete form; note only that $a_\nu(u) \in C^l[0, \infty)$ and $a_\nu(u) \equiv 0$ for $u \ge \frac{2}{\pi} NT^{-1}$, $0 \le \nu \le q$. The quantity l can be taken arbitrary but fixed. The function $\Psi_\alpha(u, v)$ is such that $|\Psi_\alpha(u, v)| \le C|u|^{-l}$ and $|\Psi_\alpha(u, v)| \le C_0$ uniformly in $\alpha \ge 0$. Finally, let us define the function $\eta(s, x)$. For $\mathrm{Re}\, s > \mathrm{m}$, introduce the zeta-function

$$\zeta(s, x) = \sum_{\substack{n \in \mathbb{Z}^m \\ n \neq 0}} \frac{\exp(2\pi i n \cdot x)}{|n|^s} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{s}{2}}} \sum_{|n|^2 = k} \exp(2\pi i n \cdot x), \tag{0.11}$$

and if $x \neq 0$, $x \in I$, it is easy to show that $\zeta(s, x)$ is an entire function. By definition,

$$\eta(s, x) = \zeta(s, x) - \pi^{s - \frac{m}{2}} \frac{\Gamma(\frac{m-s}{2})}{\Gamma(\frac{s}{2})} |x|^{s-m}.$$

Formula (0.10) is applicable for any $x \in I$. If $|x| \ge CN^{-1}$, then one can modify this formula by changing the integral on the right-hand side of (0.10) in an appropriate manner. Despite of the awkwardness of (0.10), it is convenient in applications and yields the following results.

Denoting by $\|f\|$ the $L^p(I)$-norm of f, we set

$$\Lambda_p(N, \alpha) = \sup_{\|f\| \le 1} \|S_N^\alpha(x, f)\|.$$

If $p' = \frac{p}{p-1}$, then $\Lambda_{p'}(N, \alpha) = \Lambda_p(N, \alpha)$. It is shown by means of (0.10) that if $p \in [1, \frac{2m}{m+1}]$ and $0 \le \alpha \le \alpha_p = \frac{m}{p} - \frac{m+1}{2}$, then

$$\Lambda_p(N, \alpha) \ge A \left[\frac{N^{p(\alpha_p - \alpha)} - 1}{p(\alpha_p - \alpha)} \right]^{\frac{1}{p}}, \tag{0.12}$$

where the constant A depends only upon m and p. Stein has shown in [9] that $S_N^\alpha(x, f)$ is bounded in the L^p norm, $1 < p \le 2$, if $\alpha > \alpha_p + \frac{1}{p'}$. Thus the question on behavior of the means $S_N^\alpha(x, f)$ in the domain

$$\{\alpha, p : 1 \le p < 2, \alpha \ge 0, \alpha > \alpha_p, \alpha \le \alpha_p + \frac{1}{p'}\}$$

remains open. It is expected that the means $S_N^\alpha(x, f)$ converge to f in $L^p(I)$.

It is quite probable that inequality (0.12) cannot be improved in growth of the order. In the case $p = \infty$, it is shown that for $\alpha \le \alpha_1 = \frac{m-1}{2}$,

$$A \frac{N^{\alpha_1 - \alpha} - 1}{\alpha_1 - \alpha} \le \Lambda_\infty(N, \alpha) \le B \frac{N^{\alpha_1 - \alpha} - 1}{\alpha_1 - \alpha}. \tag{0.13}$$

Series (0.11) has remarkable properties. Say, if $p \in [1, \frac{2m}{m+1})$, $\alpha < \alpha_p$, and κ satisfies the inequalities $m - \frac{m}{p} < \kappa < \frac{m-1}{2} - \alpha$, then $\zeta(\kappa, x) \in L^p(I)$, and its spherical means of order α diverge almost everywhere. It would be extremely interesting to figure out the question whether these means diverge everywhere.

There are no analogs of the mentioned theorems for $m = 1$. Concerning the behavior of the means of order $\alpha = \alpha_p$, $1 \le p \le \frac{2m}{m+1}$, the problem is open, but what is expected is an analog of the famous Kolmogorov's example [4]. More precisely, that there exists a function $f \in L^p(I)$, $1 \le p \le \frac{2m}{m+1}$, such that the means of order α_p diverge almost everywhere.

Inequality (0.13) sheds light on the question about sufficient conditions, which ensure the uniform convergence of spherical partial sums. Using (0.10), one may show that if $f(x)$ is representable as

$$f(x) = a_0 + \frac{1}{(2\pi)^\kappa} \int g(\xi)\zeta(\kappa, x-\xi)\, d\xi, \tag{0.14}$$

where $g \in L^p(I)$, $p > \frac{m}{\kappa}$, $0 < \kappa \leq \frac{m-1}{2}$, then the Fourier series of $f(x)$ is uniformly summable by the means of order $\alpha = \frac{m-1}{2} - \kappa$. This result is sharper than the results of B. Levitan [5].

Much earlier, V. Ilyin [3] obtained theorems on the convergence of eigenfunction expansions for the Laplace operator that are close to the formulated above. However, our theorem is more precise for even m.

In the preceding theorem the order of the means $\alpha = \frac{m-1}{2} - \kappa$ cannot be taken smaller, since there exist functions $f(x)$, satisfying the above assumptions, for which

$$\overline{\lim_{N\to\infty}} |S_N^\alpha(0, f)| = \infty$$

if $\alpha < \frac{m-1}{2} - \kappa$.

§ 1

1. For the reader's convenience, we present well-known facts on the theory of zeta-functions. Let $x = (x_1, ..., x_m) \in \mathbb{R}^m$, let

$$f(x) = \sum_{k,l=1}^{m} a_{kl} x_k x_l$$

be a positive definite quadratic form of discriminant d, and let the form $\sum A_{kl} x_k x_l$ be its algebraically reciprocal. Set

$$g(x) = \frac{1}{d} \sum_{k,l=1}^{m} A_{kl} x_k x_l.$$

Let $\mathbb{Z}^m$ be the lattice of integers in $\mathbb{R}^m$, $n \in \mathbb{Z}^m$, $n = (n_1, ..., n_m)$, where n_j are integers. Consider the theta-series

$$\theta(f; t, x) = \theta(t, x) = \sum_{n\in\mathbb{Z}^m} \exp\{-\pi t f(n) + 2\pi n \cdot x\}, \quad \operatorname{Re} t > 0,$$

and

$$v(g; t, x) = v(t, x) = \sum_{n\in\mathbb{Z}^m} \exp\{-\pi t g(n + x)\}, \quad \operatorname{Re} t > 0.$$

It is easy to see that these theta-series are subject to the classical functional equality

$$\theta(t, x) = d^{-\frac{1}{2}} t^{-\frac{m}{2}} v\left(\frac{1}{t}, x\right). \tag{1.1}$$

We associate with the forms f and g the two zeta-series

$$\zeta(f; s, x) = \zeta(s, x) = \sum_{n\in\mathbb{Z}^m,\, n\neq 0} \frac{\exp(2\pi i n \cdot x)}{[f(n)]^{\frac{s}{2}}}$$

and, if $x \notin \mathbb{Z}^m$,

$$\zeta(g; s, x) = \zeta(s \mid x) = \sum_{n\in\mathbb{Z}^m} \frac{1}{[g(n + x)]^{\frac{s}{2}}}.$$

These series converge in the half-plane $\mathrm{Re}\, s > m$ and define there regular functions of variable s.

In what follows, we shall hold, not mentioning this in a special way, that point x belongs to the interval

$$I = \{x \in \mathbb{R}^m : |x_j| \le \frac{1}{2}, 1 \le j \le m\}.$$

If $\mathrm{Re}\, s > m$, it is easy to see that

$$\zeta(s,x) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \int\limits_0^\infty [\theta(\xi,x) - 1]\xi^{\frac{s}{2}-1} d\xi.$$

Splitting the integral into the sum of two integrals, over $[0,1]$ and $[1,\infty]$, applying (1.1) to the function in the first integral and substituting further under the integral sign, we obtain, for $x \ne 0$,

$$\zeta(s,x) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \left\{ \int\limits_1^\infty [\theta(\xi,x) - 1]\xi^{\frac{s}{2}-1} d\xi + d^{-\frac{1}{2}} \int\limits_1^\infty \upsilon(\xi,x)\xi^{\frac{m-s}{2}-1} d\xi^{-\frac{2}{s}} \right\}. \tag{1.2}$$

This formula delivers analytic extension of the function $\zeta(s,x)$ to the whole plane. Thus, $\zeta(s,x)$ is an entire function for $x \ne 0$. The following relation can easily be derived from the last formula:

$$\begin{aligned} \zeta(s,x) - \frac{\pi^{s-\frac{m}{2}}}{d^{\frac{1}{2}}} \frac{\Gamma(\frac{m-s}{2})}{\Gamma(\frac{s}{2})} [g(x)]^{\frac{s-m}{2}} &= \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \left\{ \int\limits_1^\infty [\theta(\xi,x) - 1]\xi^{\frac{s}{2}-1} d\xi \right. \\ &+ d^{-\frac{1}{2}} \int\limits_1^\infty [\upsilon(\xi,x) - \exp(-\pi\xi g(x))]\xi^{\frac{m-s}{2}-1} d\xi^{-\frac{2}{s}} \\ &\left. - d^{-\frac{1}{2}} \int\limits_0^1 \xi^{\frac{m-s}{2}-1} d\xi \exp[-\pi\xi g(x)] d\xi^{-\frac{2}{s}} \right\} \end{aligned} \tag{1.3}$$

provided that $\mathrm{Re}\, s \ge m$.

Introduce the notation

$$\eta(s,x) = \zeta(s,x) - \frac{\pi^{s-\frac{m}{2}}}{d^{\frac{1}{2}}} \frac{\Gamma(\frac{m-s}{2})}{\Gamma(\frac{s}{2})} [g(x)]^{\frac{s-m}{2}}. \tag{1.4}$$

Observing that

$$\begin{aligned} &\int\limits_0^1 \xi^{\frac{m-s}{2}-1} \exp\{-\pi\xi g(x)\}\, d\xi \\ &= \frac{2}{m-s} \exp\{-\pi g(x)\} + \frac{2\pi g(x)}{m-s} \int\limits_0^1 \xi^{\frac{m-s}{2}} \exp\{-\pi\xi g(x)\}\, d\xi, \end{aligned}$$

we derive from (1.3) the inequality

$$|\eta(s,x)| \le C \exp\left(\frac{\pi}{2}|t|\right), \quad s = \sigma + it, \tag{1.5}$$

provided $-1 \le \sigma \le m+1$, $|t| \ge 1$. In the half-plane $\mathrm{Re} s < 0$, we transform the first integral in formula (1.2) first by means of relation (1.1) and then integrating by parts under the integral sign. This yields

$$\zeta(s,x) = \frac{\pi^{\frac{s}{2}}}{d^{\frac{1}{2}}\Gamma(\frac{s}{2})} \int_0^\infty \upsilon(\xi,x)\xi^{\frac{m-s}{2}-1} d\xi, \quad x \ne 0,$$

from which the next functional equation follows in a trivial manner:

$$\zeta(s,x) = \frac{\pi^{s-\frac{m}{2}}\Gamma(\frac{m-s}{2})}{d^{\frac{1}{2}}\Gamma(\frac{s}{2})}\zeta(m-s \mid x). \tag{1.6}$$

In what follows, we shall denote by ε an arbitrary quantity from the interval $[0,1]$.

Proposition 1.1. *In the strip $-\varepsilon \le \mathrm{Re}\, s \le m+\varepsilon$, the estimate*

$$|\eta(s,x)| \le B\varepsilon^{-1}(|t|+1)^{\frac{m+\varepsilon-\sigma}{2}}, \quad s = \sigma + it, \tag{1.7}$$

holds, where the constant B depends only upon m.

Proof. The function

$$\chi(s,x) = \frac{s(m-s)\Gamma(\frac{s}{4})}{(s+1)(m+1-s)\Gamma(\frac{m+\varepsilon}{2} - \frac{s}{4})}\eta(s,x)$$

is regular in the strip $-\varepsilon \le \mathrm{Re}\, s \le m+\varepsilon$. It follows from the asymptotic formula for the gamma-function that for $\mathrm{Im}\, a = 0$

$$\left|\frac{\Gamma(a-s)}{\Gamma(s)}\right| = |t|^{a-2\sigma}(1+O(t^{-1})), \tag{1.8}$$

with a constant in O depending only on a. Therefore, (1.6) yields

$$|\chi(-\varepsilon+it,x)| \le C_1 \sum_{n\in\mathbb{Z}^m, n\ne 0} |n|^{-m-\varepsilon} \le C_2\varepsilon^{-1}.$$

On the other hand, it is easy to see that

$$|\chi(m+\varepsilon+it,x)| \le C_3\varepsilon^{-1},$$

where the constants C_2 and C_3 depend only on m. In the considered strip, (1.5) and (1.8) imply the inequality

$$|\chi(s,x)| \le C_4 \exp(C_5|t|), \quad s = \sigma+it,$$

which yields, by the Phragmén-Lindelöf theorem,

$$|\chi(s,x)| \le \varepsilon^{-1}\max(C_2,C_3).$$

From the last inequality (1.7) follows, because of (1.8). □

2. Let us give a few simple facts of auxiliary nature. Let $\varphi \in C^\infty[a,b]$, and $h(t) \in C^l[a,b]$. Set

$$I_k = \int_a^b t^{k-1}h(t)\exp\{i\varphi(t)\}\, dt, \quad k > 0.$$

Proposition 1.2. *If* $h^{(\nu)}(a) = h^{(\nu)}(b) = 0$, $0 \le \nu \le l-1$, *and* $\varphi(t) \equiv \kappa t$, *then*

$$|I_1| \le \kappa^{-l} \int_a^b |h^{(l)}(t)|\, dt = C_l \kappa^{-l}. \tag{1.9}$$

Proof. Integrating by parts, (1.9) follows. □

Proposition 1.3. *If* $h(t)$ *satisfies the assumptions of the preceding proposition, and* $\varphi(t) = \kappa\varphi_0(t)$, *with* $|\varphi_0'(t)| \ge 1$ *for* $t \in [a,b]$, *then for* $b - a < \infty$,

$$|I_1| \le D_l \kappa^{-l}. \tag{1.10}$$

Proof. By assumption, $\varphi_0(t)$ is a monotone function. Introducing a new variable $\xi = \varphi_0(t)$ under the integral sign, we arrive at the situation of Proposition 1.2, and thus (1.10) follows from (1.9). □

Let $h(t)$ satisfy the assumptions of Proposition 1.2 and, besides that, the condition

$$|h^{(\nu)}(t)| \le C_\nu t^{-\nu}, \quad 0 \le \nu \le l, \quad t \in [a,b], a > 0. \tag{1.11}$$

Proposition 1.4. *Suppose that* $h(t)$ *satisfies the above conditions, and* $\varphi(t)$ *is such that*

1. $|\varphi'(t)| \ge \delta > 0$
2. $|\varphi^{(\nu)}(t)| \le \widetilde{C_\nu} t^{-\nu+1}, \quad 0 \le \nu \le l, \quad t \in [a,b]$

Then for $l \ge k+2$,

$$|I_k| \le A_l a^{k-l}, \tag{1.12}$$

where the constant A_l *does not depend on* a *nor on* b.

Proof. Setting $g_0(t) = t^{k-1}h(t)$, we define functions $g_1, ..., g_l$ by means of the recurrent relation

$$g_\nu(t) = \frac{d}{dt}\frac{g_{\nu-1}(t)}{\varphi'(t)}.$$

It can easily be checked by induction in ν that

$$g_\nu(t) = \sum_{\mu=0}^{\nu} g_0^{(\mu)}(t) P_\mu(\varphi),$$

with

$$P_\mu(\varphi) = \sum_{\substack{\mu_1+\ldots+\mu_r=\nu-\mu \\ \mu_j \ge 0}} A^\mu_{\mu_1,\ldots,\mu_r} \varphi^{(\mu_1+1)} \ldots \varphi^{(\mu_r+1)},$$

where the coefficients $A^\mu_{\mu_1,\ldots,\mu_r}$ are of the form $\frac{v}{[\varphi']^s}$, with v integer and s natural. This implies

$$|g_\nu(t)| \le D_\nu t^{k-\nu-1}.$$

From this inequality for $\nu = l$ and obvious relation

$$I_k = i^l \int_a^b g_l(t) \exp[i\varphi(t)]\, dt$$

(1.12) follows. □

Let us find an asymptotics of the integral I_k by means of the Stationary Phase Method. For our goals, it suffices to consider only such functions $\varphi(t)$ that satisfy the conditions:

1. $\varphi'(t)$ has the only zero on $(0,\infty)$ at $t=t_0$.
2. The function $\psi(\xi)=t_0^{-1}\varphi(t_0(1+\xi))$ is regular in the interval $(-1,\infty)$ and is independent of t_0.
3. There exists $\delta>0$ such that

$$\inf|\psi'(\xi)|\geq\delta_0,\quad \xi\in(-1,\infty)\setminus[-\delta,+\delta].$$

Let function $h(t)$ satisfy conditions (1.11) and the conditions

$$h^{(\nu)}(a)=h^{(\nu)}(b)=0,\quad 0\leq\nu\leq[k]+2. \tag{1.13}$$

In what follows, we shall set

$$\tau=t_0\psi''(0).$$

Proposition 1.5. *If the functions $h(t)$ and $\varphi(t)$ satisfy the above mentioned conditions and $l\geq 4([k]+2)$, then*

$$I_k=\sqrt{\frac{2\pi}{|\tau|}}\,t_0^k\exp\{i\varphi(t_0)+\frac{\pi i}{4}\operatorname{sgn}\tau\}\sum_{\nu=0}^{[k]+1}\alpha_\nu\tau^{-\nu}+O(a^{-1}), \tag{1.14}$$

where $\alpha_0=h(t_0)$, and α_ν, $\nu\geq 1$, is a linear combination of the quantities $h^{(\mu)}(t_0)$, $0\leq\mu\leq 2\nu$.

Proof. Let $\theta(t)\in C^\infty(-\infty,\infty)$ be a "cap"-like function. It obeys the conditions:

1. $0\leq\theta(t)\leq 1$
2. $\theta(t)=0$ for $t\in(-\infty,\infty)\setminus[1-\delta,1+\delta]$
3. $\theta(t)=1$ for $t\in[1-\frac{\delta}{2},1+\frac{\delta}{2}]$

Setting $\theta_0(t)=\theta(\frac{t-t_0}{t_0})$ and $\theta_1(t)=1-\theta_0(t)$, we correspondingly have

$$I_k=\int_a^b t^{k-1}\theta_0(t)h(t)\exp(i\varphi(t))\,dt$$
$$+\int_a^b t^{k-1}\theta_1(t)h(t)\exp(i\varphi(t))\,dt=I_{k1}+I_{k2}.$$

The integral I_{k2} can be split into the sum of two integrals, wherein to each of them Proposition 1.4 is applicable, with the quantity $[k]+1$ assumed to be l. We will obtain $I_{k2}=O(a^{-1})$ in this way. Substituting $t=t_0(1+\xi)$ in the integral I_{k1}, we set $\psi(\xi)=\psi(0)+\frac{\psi''(0)}{2}u^2$ then. Without loss of generality, we may consider δ to be such that for $|\xi|\leq\delta$ the function $\psi(\xi)$ is expandable in the convergent power series

$$\psi(\xi)=\xi^2\left(\frac{\psi''(0)}{2}+\frac{\psi'''(0)}{3!}\xi+...\right)$$

and that the function

$$\psi^{\frac{1}{2}}(\xi)=\sqrt{\frac{\psi''(0)}{2}}\,\xi(1+a_2\xi+...)$$

is schlicht in the disk $|\xi| \le \delta$. Therefore, $\xi = \xi(u)$, with $\xi(0) = 0$ and $\xi'(0) = 1$. This substitution will move the interval $[-\delta, \delta]$ of the ξ axis to the interval $[-u_0, u_1]$, $u_0 > 0$, with the orientation preserved. Hence, we obtain

$$I_k = t_0^k \exp(i\varphi(t_0)) \int_{-u_0}^{u_1} \widetilde{\theta}(u) H(u) \exp(\frac{i\tau}{2} u^2)\, du + O(a^{-1}),$$

where

$$H(u) = [1 + \xi(u)]^{k-1} \xi'(u) h[t_0(1 + \xi(u))], \quad \widetilde{\theta}(u) = \theta(\xi(u)).$$

The posed assumptions yield $H(u) \in C^l[-u_0, u_1]$, and, in addition, inequalities (1.11) imply

$$H^{(\nu)}(u) = O(1), \qquad 0 \le \nu \le l,$$

with the constant in O independent of t_0. The Taylor formula gives

$$H(u) = \sum_{l=0}^{q-1} \frac{u^l}{l!} H^{(l)}(0) + u^q H_q(u),$$

where

$$H_q(u) = \frac{1}{(q-1)!} \int_0^1 (1 - v)^{q-1} H^{(q)}(vu)\, dv, \quad q = 2([k] + 2).$$

Integrating by parts, we get for even μ, $\mu = 2s$,

$$\begin{aligned} j_\mu &= \int_{-u_0}^{u_1} \widetilde{\theta}(u) u^\mu \exp\left(\frac{i\tau}{2} u^2\right) du \\ &= \left(\frac{i}{\tau}\right)^s \int_{-u_0}^{u_1} \left[(2s-1)!!\widetilde{\theta}(u) + \sum_{r=1}^{s} \gamma_{sr} u^r \widetilde{\theta}^{(r)}(u)\right] \exp\left(\frac{i\tau u^2}{2}\right) du, \end{aligned}$$

where γ_{sr} are certain constants, which depend only upon s and r. The function

$$\sum_{r=1}^{s} \gamma_{sr} u^r \widetilde{\theta}^{(r)}(u)$$

is identically zero in a neighborhood of the point $u = 0$. Therefore, by Proposition 1.3,

$$\left(\frac{i}{\tau}\right)^s \int_{-u_0}^{u_1} \sum_{r=1}^{s} \gamma_{sr} u^r \widetilde{\theta}^{(r)}(u) \exp\left(\frac{i\tau}{2} u^2\right) du = O(\tau^{-k-1}).$$

Thus,

$$\begin{aligned} j_\mu = \left(\frac{i}{\tau}\right)^s (2s-1)!! \Big\{ &\int_{-\infty}^{\infty} \exp\left(\frac{i\tau}{2} u^2\right) du \\ &+ \int_{-\infty}^{\infty} [\widetilde{\theta}(u) - 1] \exp(\frac{i\tau u^2}{2})\, du \Big\} + O(\tau^{-k-1}). \end{aligned}$$

The integral

$$\int_{-\infty}^{\infty} [\widetilde{\theta}(u) - 1] \exp\left(\frac{i\tau u^2}{2}\right) du \tag{1.15}$$

can be split into two, over the intervals $(-\infty, 0)$ and $(0, \infty)$, and the change of variables $u^2 = v$ can be made in each one. The obtained thus integrals are estimated by means of Proposition 1.2. This leads to the bound $O(\tau^{-k-1})$ for integral (1.15). Finally,

$$j_\mu = (2s-1)!! \left(\frac{i}{\tau}\right)^s \sqrt{\frac{2\pi}{|\tau|}} \exp(\frac{\pi i}{4} \operatorname{sgn} \tau) + O(\tau^{-k-1}).$$

If μ is odd, $\mu = 2s+1$, similar argument will give $j_\mu = O(\tau^{-k-1})$. It is also clear that

$$\int_{-u_0}^{u_1} \widetilde{\theta}(u) u^q H_q(u) \exp\left(\frac{i\tau u^2}{2}\right) du = O(\tau^{-\frac{q}{2}}) = O(\tau^{-[k]-2}).$$

Finally,

$$I_k = t_0^k \sqrt{\frac{2\pi}{|\tau|}} \exp\left(i\varphi(t_0) + \frac{\pi i}{4} \operatorname{sgn} \tau\right) \sum_{s=0}^{[k]+1} \frac{i^s (2s-1)!!}{(2s)!} H^{(rs)}(0) \tau^{-s} + O(a^{-1}),$$

which is equivalent to (1.14). □

3. We are going to derive auxiliary asymptotic formulas for the Riesz means of a multiple trigonometric series. Assume that the form $f(x) = x_1^2 + ... + x_m^2 = |x|^2$. Then $g(x) = |x|^2$ as well. We shall denote the corresponding zeta-functions by $\zeta(s, x)$ and $\zeta(s \mid x)$ as above. Let $\alpha \geq 0$, $m > \kappa \geq 0$, and $x \in I$, $x \neq 0$. Set

$$S_{N,\kappa}^\alpha(x) = S_N^\alpha(x) = \sum_{0 \leq |n| \leq N}{}' \left(1 - \frac{|n|^2}{N^2}\right)^\alpha |n|^{-\kappa} \exp(2\pi i n \cdot x), \tag{1.16}$$

where $'$ after the sign of the sum means that the summands corresponding to $|n|^2 = N^2$ are taken with the factor $\frac{1}{2}$.

We will derive a formula for $S_N^\alpha(x)$ by means of contour integrals. This argument is well known in the theory of zeta-function and can be found in Chapter 3 of monograph [10]. However, we shall insert in it a technical novelty, due to which the problem of the estimate of the remainder term will be trivialized. Introduce a function

$$\Psi(s) = \frac{\Gamma(\alpha+1)\Gamma(s)}{\Gamma(s+\alpha+1)}.$$

It is known that for $\operatorname{Re} s > 0$,

$$\Psi(s) = \int_0^\infty e^{-vs} (1 - e^{-v})^\alpha \, dv, \tag{1.17}$$

or

$$\Psi(s) = \int_0^1 u^{s-1} (1-u)^\alpha \, du. \tag{1.18}$$

It follows from (1.17) that

$$\Psi(s) = \sum_{\nu=0}^{r-1} \frac{b_{\nu_\alpha}}{s^{\alpha+\nu+1}} + O(s^{-\alpha-r-1}), \tag{1.19}$$

where $b_{0_\alpha} = \Gamma(\alpha+1)$ and $b_{\nu_0} = 0$ if $\nu \geq 1$. In the last formula, the branch of the function s^α is taken which is positive for $s > 0$. Formula (1.17) after simple calculations reduces to the relation

$$\Psi^{(\nu)}(\sigma + it) = O(|t|^{-\nu-\alpha-1}), \quad \nu \geq 0. \tag{1.20}$$

Now, let us introduce a smoothing function. Let l be a natural number considered to be large enough as compared with m. Introduce parameter T, and assume that $T \geq C_0 N$. Set

$$\chi(u) = A_l \int_0^{2-u} v^l (1-v)^l \, dv, \tag{1.21}$$

where the constant A_l is determined from the condition $\chi(1) = 1$. Let $\theta(t) = 0$ for $|t| \geq 2T$, $\theta(t) = 1$ for $|t| \leq T$, and $\theta(t) = \chi(\frac{|t|}{T})$ for $T \leq |t| \leq 2T$. It is clear that $\theta(t) \in C^l(-\infty, \infty)$ and

$$|\theta^{(\nu)}(t)| \leq B_\nu T^{-\nu}, \quad 0 \leq \nu \leq l, \tag{1.22}$$

where the constants B_ν are independent of T.

For $\sigma > 0$, by Mellin's theorem,

$$I(y) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Psi(s) y^s ds = \begin{cases} (1-y^{-1})^\alpha, & \text{if } y > 1 \\ 0, & \text{if } 0 < y < 1 \end{cases} \tag{1.23}$$

On the other hand, we can represent this integral as

$$I(y) = \frac{1}{2\pi} \int_{-2T}^{2T} \theta(t) \Psi(\sigma + it) y^{\sigma+it} dt + I_1(y),$$

where

$$I_1(y) = \frac{y^\sigma}{2\pi} \int_{-\infty}^{\infty} \Phi(t) y^{it} dt,$$

with $\Phi(t) = [1 - \theta(t)]\Psi(\sigma + it)$. It is easy to see that

$$I_1(y) = \frac{y^\sigma}{2\pi} \int_T^\infty [1 - \theta(t)] \left\{ \Psi(\sigma + it) y^{it} + \Psi(\sigma - it) y^{-it} \right\} dt.$$

Based on (1.19), after simple calculations the relation follows

$$I_1(y) = \frac{y^\sigma}{2\pi} \sum_{\nu=0}^{r-1} a_{\nu_\alpha} \int_T^\infty [1 - \theta(t)] \frac{\sin(t \log y - \frac{\pi}{2}(\alpha + \nu))}{t^{\alpha+\nu+1}} dt + O(y^\sigma T^{-\alpha-r}).$$

Setting $r = m + 2$ and introducing

$$\Psi_\alpha(z, T) = \frac{1}{\pi} \sum_{\nu=0}^{r-1} \frac{a_{\nu_\alpha}}{T^\nu} \int_1^\infty [1 - \theta(Tt)] \frac{\sin(zt - \frac{\pi}{2}(\alpha + \nu))}{t^{\alpha+\nu+1}} dt,$$

we then have

$$I_1(y) = \frac{y^\sigma}{T^\alpha}\Psi_\alpha(T\log y, T) + O\left(\frac{y^\sigma}{T^{\alpha+r}}\right). \tag{1.24}$$

By Proposition 1.2,

$$|\Psi_\alpha(z,T)| \le C|z|^{-l}. \tag{1.25}$$

A routine calculation shows that uniformly in α, for $\alpha \ge 0$,

$$|\Psi_\alpha(z,T)| \le C_0. \tag{1.26}$$

Defining σ_0 from the relation $\kappa + 2\sigma_0 = m + \varepsilon$, where $\varepsilon > 0$ and $\varepsilon < 1$, and taking $\sigma = \sigma_0$ in (1.23), we derive from this formula

$$S_N^\alpha(x) = \sum_{0\ne n\in\mathbb{Z}^m} |n|^{-\kappa} I\left(\frac{N^2}{|n|^2}\right)\exp(2\pi i n\cdot x).$$

This yields

$$S_N^\alpha(x) = \frac{1}{2\pi}\int_{-2T}^{2T} \theta(t)\Psi(s)\zeta(\kappa+2s,x)N^{2s}dt + R_N(x), \tag{1.27}$$

where $s = \sigma + it$. By (1.24) and condition $r = m+2$,

$$\begin{aligned} R_N(x) = &\sum_{0\ne n\in\mathbb{Z}^m} |n|^{-\kappa-2\sigma_0}\Psi_\alpha\left(2T\log\frac{N}{|n|}, T\right)\exp(2\pi i n\cdot x) \\ &+ O(\varepsilon^{-1}N^{-1-\kappa}T^{-\alpha}). \end{aligned}$$

Note that in accordance with definition of $S_N^\alpha(x)$, we have

$$\Psi_0 = \Psi_0(0,T) = -\sum_{\nu=1}^{r-1}\frac{a_{\nu 0}}{\pi\nu}\frac{\sin\frac{\pi\nu}{2}}{T^\nu}.$$

Using inequality (1.25), we can simplify a little the last formula for $R_N(x)$ by moving a part of the summands to the remainder. Thus,

$$\begin{aligned} R_N(x) = &\frac{N^{2\sigma_0}}{T^\alpha}\sum_{\frac{1}{2}N\le|n|\le 2N} |n|^{-\kappa-2\sigma_0}\Psi_\alpha\left(2T\log\frac{N}{|n|}, T\right)\exp(2\pi i n\cdot x) \\ &+ O\left(\frac{1}{\varepsilon}N^{-1-\kappa}T^{-\alpha}\right). \end{aligned}$$

Set

$$J(x) = \frac{1}{2\pi}\int_{-2T}^{2T} \theta(t)\Psi(s)\zeta(\kappa+2s,x)N^{2s}dt \tag{1.28}$$

and

$$R_N^\alpha(x) = N^{m-\kappa+\varepsilon}T^{-\alpha}\sum_{\frac{1}{2}N\le|n|\le 2N} |n|^{-m-\varepsilon}\Psi_\alpha\left(2T\log\frac{N}{|n|}, T\right)\exp(2\pi i n\cdot x).$$

Then

$$S_N^\alpha(x) = J(x) + R_N^\alpha(x) + O(\varepsilon^{-1}N^{-1-\kappa}T^{-\alpha}). \tag{1.29}$$

By this, derivation of asymptotic formulas for the means of high order trivializes. Consider an analogue of the de la Vallée-Poussin means of the function $\zeta(\kappa, x)$.

Let a function $\Phi(x) \in C^\infty(-\infty,\infty)$ and satisfy the conditions: a) $\Phi(x) \equiv 1$ for $x \le \frac{1}{2}$ and b) $\Phi(x) \equiv 0$ for $x \ge 1$. Denote by $\widetilde{\Phi}(s)$ the Mellin transform of function $\Phi(x)$. Clearly,

$$\widetilde{\Phi}(s) = \int_0^1 x^{s-1}\Phi(x)\,dx = -\frac{1}{s}\int_{\frac{1}{2}}^1 x^s\Phi'(x)\,dx,$$

which implies that $s\widetilde{\Phi}(s)$ is an entire function of exponential type, and $\widetilde{\Phi}(s)$ is meromorphic with the unique pole of first order at $s = 0$ and residue equal 1. It is clear that $\widetilde{\Phi}(s)$ decreases faster than any power of $|t|^{-1}$ as $|t| \to \infty$. Therefore,

$$\begin{aligned} V_N(x) &= \sum_{0\le|n|\le N} |n|^{-\kappa}\Phi\left(\frac{|n|^2}{N^2}\right)\exp(2\pi i n\cdot x) \\ &= \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \widetilde{\Phi}(s)\zeta(\kappa+2s,x)N^{2s}ds. \end{aligned}$$

This yields

$$\begin{aligned} V_N(x) &= \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \widetilde{\Phi}(s)\eta(\kappa+2s,x)N^{2s}ds \\ &+ \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \widetilde{\Phi}(s)h(s,x)N^{2s}ds, \end{aligned} \tag{1.30}$$

where

$$h(s,x) = \pi^{\kappa-\frac{m}{2}+2s}\frac{\Gamma(\frac{m-\kappa}{2}-s)}{\Gamma(\frac{\kappa}{2}+s)}|x|^{2s-m+\kappa}. \tag{1.31}$$

Denoting the integrals on the right-hand side of (1.30) by $\widetilde{J}_0(x)$ and $\widetilde{J}_1(x)$, we have

$$V_N(x) = \widetilde{J}_0(x) + \widetilde{J}_1(x). \tag{1.32}$$

4. Transform integral (1.28) by shifting the path of integration to the left in such a way that using the functional equation becomes possible. However, if we wish to obtain an asymptotic formula operable for arbitrary $x \in I$, $x \ne 0$, then it is better to apply the mentioned transformation to the function $\eta(\kappa+2s,x)$ rather than to $\zeta(\kappa+2s,x)$. Setting

$$f(s) = \Psi(s)\eta(\kappa+2s,x)N^{2s} \tag{1.33}$$

and

$$g(s) = \Psi(s)\pi^{\kappa-\frac{m}{2}+2s}\frac{\Gamma(\frac{m-\kappa}{2}-s)}{\Gamma(\frac{\kappa}{2}+s)}|x|^{-m+\kappa+2s}N^{2s}, \tag{1.34}$$

we obtain

$$J(x) = \frac{1}{2\pi}\int\limits_{-2T}^{2T} \theta(t) f(s)\, dt + \frac{1}{2\pi}\int\limits_{-2T}^{2T} \theta(t) g(s)\, dt = J_0(x) + J_1(x), \tag{1.35}$$

with $s = \sigma_0 + it$ in the last formula. Observing that

$$\begin{aligned} J_0(x) &= \frac{1}{2\pi i}\int\limits_{\sigma_0-2iT}^{\sigma_0-iT} \chi(-\frac{s-\sigma_0}{iT}) f(s)\, ds + \frac{1}{2\pi i}\int\limits_{\sigma_0-iT}^{\sigma_0+iT} f(s)\, ds \\ &+ \frac{1}{2\pi i}\int\limits_{\sigma_0+iT}^{\sigma_0+2iT} \chi(\frac{s-\sigma_0}{iT}) f(s)\, ds = J_{01}(x) + J_{02}(x) + J_{03}(x), \end{aligned}$$

and shifting the path of integration to the left, we transform each of the integrals $J_{0k}(x)$, $1 \le k \le 3$, to have lain on the line $\mathrm{Re}\, s = \sigma_1$, where

$$\sigma_1 = -\frac{\kappa+\varepsilon}{2}.$$

Setting $A_m = \pi^{\frac{m}{2}} \frac{\Gamma(\alpha+1)\Gamma(\frac{m-\kappa}{2})}{\Gamma(\alpha+1+\frac{m-\kappa}{2})\Gamma(\frac{m}{2})}$, we derive that $A_m N^{m-\kappa}$ is the residue of $f(s)$ at the pole $s = \frac{m-\kappa}{2}$, and hence, by the Cauchy theorem,

$$\begin{aligned} J_{02}(x) &= \sum_{\nu=0}^{[\frac{\kappa+\varepsilon}{2}]} (-1)^\nu N^{-2\nu} \frac{\Gamma(\alpha+1)}{\nu!\Gamma(\alpha+1-\nu)} \eta(\kappa-2\nu, x) + A_m N^{m-\kappa} \\ &+ \frac{1}{2\pi i}\int\limits_{\sigma_1-iT}^{\sigma_1+iT} f(s)\, ds + \frac{1}{2\pi i}\int\limits_{\sigma_1+iT}^{\sigma_0+iT} f(s)\, ds - \frac{1}{2\pi i}\int\limits_{\sigma_1-iT}^{\sigma_0+iT} f(s)\, ds. \end{aligned} \tag{1.36}$$

If we proceed in the same way to the integrals J_{01} and J_{03} and then sum up the obtained expressions, the integrals over the horizontal intervals will interfere, which make their contribution to move to the remainder. Indeed, taking, for example, the integral J_{01}, we have

$$\begin{aligned} J_{01}(x) &= \frac{1}{2\pi i}\int\limits_{\sigma_1-2iT}^{\sigma_1-iT} \chi(-\frac{s-\sigma_0}{iT}) f(s)\, ds + \frac{1}{2\pi i}\int\limits_{\sigma_1-iT}^{\sigma_0-iT} \chi(-\frac{s-\sigma_0}{iT}) f(s)\, ds \\ &- \frac{1}{2\pi i}\int\limits_{\sigma_1-2iT}^{\sigma_0-2iT} \chi(\frac{s-\sigma_0}{iT}) f(s)\, ds. \end{aligned} \tag{1.37}$$

By Proposition 1.1, the function in the last integral of formula (1.37) will be an $O(\varepsilon^{-1}N^{2\sigma}T^{\beta-\sigma})$ quantity, where $\beta = -l - 1 - \alpha + \frac{m-\kappa+\varepsilon}{2}$. Hence, considering $T \ge N^\delta$, $\delta > 0$, we can choose l in such a way that this integral be $O(\varepsilon^{-1}N^{-1})$. The difference of the last integral in (1.36) and that before the last one in (1.37) gives

$$\frac{1}{2\pi i}\int\limits_{\sigma_1-iT}^{\sigma_0-iT} \left[\chi(-\frac{s-\sigma_0}{iT}) - 1\right] f(s)\, ds,$$

and since $\chi(-\frac{s-\sigma_0}{iT}) = 1 + O(T^{-l})$ for $s = \sigma - iT$, $\sigma_1 \le \sigma \le \sigma_0$, the last integral is $O(\varepsilon^{-1}N^{-1})$ as well. Thus,

$$
\begin{aligned}
J_0(x) = j(x) &+ \sum_{\nu=0}^{[\frac{\kappa+\varepsilon}{2}]} (-1)^\nu N^{-2\nu} \frac{\Gamma(\alpha+1)}{\nu!\Gamma(\alpha+1-\nu)} \eta(\kappa-2\nu, x) \\
&+ A_m N^{m-\kappa} + O(\varepsilon^{-1}N^{-1}),
\end{aligned}
\tag{1.38}
$$

where

$$
j(x) = \frac{1}{2\pi i} \int_{\sigma_1 - iT}^{\sigma_1 + iT} \xi(s) f(s)\, ds,
$$

with the function $\xi(s)$ defined by

$$
\xi(s) = \xi(\sigma + it) = \begin{cases} \chi(-\frac{s-\sigma_0}{iT}), & \text{if } -2T < t < -T \\ 1, & \text{if } \quad |t| < T \\ \chi(\frac{s-\sigma_0}{iT}), & \text{if } \quad T < t < 2T. \end{cases}
$$

If $x \in I$ and

$$
|x| \ge C_0 N^{-1}, \tag{1.39}
$$

where C_0 is a constant, then it worth transforming the integral $J_1(x)$ in (1.35) in an analogous manner as well. On the basis of the above arguments, we obtain, for l large enough,

$$
\begin{aligned}
J(x) = j(x) &+ \frac{1}{2\pi i} \int_{\sigma_1 - 2iT}^{\sigma_1 + 2iT} \xi(s) f(s)\, ds \\
&+ \sum_{\nu=0}^{[\frac{\kappa+\varepsilon}{2}]} (-1)^\nu N^{-2\nu} \frac{\Gamma(\alpha+1)}{\nu!\Gamma(\alpha+1-\nu)} \zeta(\kappa-2\nu, x) + O(\varepsilon^{-1}N^{-1}).
\end{aligned}
\tag{1.40}
$$

5. The study of integral (1.38) and its estimation are fulfilled in a different way accordingly to whether the quantity $T_0 = N^\beta$, $\beta = \frac{\kappa+\varepsilon}{m+2\varepsilon-2\alpha}$, is greater than T, or smaller than T. Using functional equation (1.6), we get

$$
f(s) = \Psi(s) \pi^{2s+\kappa-\frac{m}{2}} \frac{\Gamma(\frac{m-\kappa}{2} - s)}{\Gamma(\frac{\kappa}{2} + s)} N^{2s} \left[\zeta(m-\kappa-2s \mid x) - |x|^{s-\frac{m-\kappa}{2}} \right].
$$

Since $x \in I$, it follows from (1.8) and (1.19) that

$$
\begin{aligned}
|f(\sigma_1 + it)| &< C(|t|^{\frac{m}{2}+\varepsilon-\alpha-1} + 1) N^{-\kappa-\varepsilon} \sum_{0 \ne n \in \mathbb{Z}^m} |n|^{-m-\varepsilon} \\
&\le \frac{C_1}{\varepsilon}(|t|^{\frac{m}{2}+\varepsilon-\alpha-1} + 1) N^{-\kappa-\varepsilon}.
\end{aligned}
$$

Therefore, if $T_0 \ge T$, then

$$
|j(x)| \le \frac{1}{2\pi} \int_{-2T_0}^{2T_0} C_1 \varepsilon^{-1} N^{-\kappa-\varepsilon} \left(|t|^{\frac{m}{2}+\varepsilon-\alpha-1} + 1 \right) dt \le C_2 \varepsilon^{-1} N^{-\frac{\kappa+\varepsilon}{2}}. \tag{1.41}
$$

In the case where $T_0 < T$ we introduce a partition of unity by means of the function $\Phi_0(t) \in C[-\infty, \infty]$, subject to conditions: a) $0 \le \Phi_0(t) \le 1$, b) $\Phi_0 \equiv 0$ for $t \le \frac{1}{2}$, and c) $\Phi_0 \equiv 1$ for $t \ge 1$. Further, set $\Phi(t) = \Phi_0(\frac{t}{T_0})$ and $\widetilde{\Phi}(t) = 1 - \Phi(t)$. Then

$$j(x) = \frac{1}{2\pi} \int\limits_{-2T}^{2T} \Phi(t)\xi(\sigma_1 + it) f(\sigma_1 + it)\, dt + \frac{1}{2\pi} \int\limits_{-T_0}^{T_0} \widetilde{\Phi}(t)\xi(\sigma_1 + it) f(\sigma_1 + it)\, dt.$$

Taking into account the preceding estimates, we obtain

$$j(x) = \frac{1}{2\pi} \int\limits_{-2T}^{2T} \Phi(t)\xi(\sigma_1 + it) f(\sigma_1 + it)\, dt + O(\varepsilon^{-1} N^{-\frac{\kappa+\varepsilon}{2}})$$

or

$$j(x) = \operatorname{Re} \left\{ \frac{1}{\pi} \int\limits_{T_0}^{2T} \Phi(t)\xi(\sigma_1 + it) f(\sigma_1 + it)\, dt \right\} + O(\varepsilon^{-1} N^{-\frac{\kappa+\varepsilon}{2}}).$$

Due to the classical asymptotic formula for the gamma-function, if $|\arg s| \le \pi - \delta$, $\delta > 0$, and $|s|$ is large,

$$\log \Gamma(s) = (s - \frac{1}{2}) \log s - s + \frac{1}{2} \log 2\pi + \sum_{\nu=1}^{q_0} \frac{(-1)^{\nu-1}}{(2\nu - 1)2\nu} B_\nu s^{-2\nu+1} + O(s^{-2q_0-1}),$$

where $\log s$ is the branch of the logarithm positive for $s > 1$, and B_j, $1 \le j \le q_0$, are the Bernoulli numbers. By a simple calculation, we derive from this

$$\begin{aligned}\Psi(s) &= \pi^{2s+\kappa-\frac{m}{2}} \frac{\Gamma(\alpha+1)\Gamma(s)\Gamma(\frac{m-\kappa}{2} - s)}{\Gamma(\alpha+1+s)\Gamma(\frac{\kappa}{2}+s)} \\ &= \pi^{2s+\kappa-\frac{m}{2}} \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \exp\Big\{ -\frac{m-1}{2}\pi i + (2+\pi i)(\frac{\kappa}{2}+s) \\ &+ (\frac{m}{2} - \kappa - 2s) \log(\frac{\kappa}{2}+s) \Big\} \left[\sum_{\nu=0}^{q-1} \frac{B_\nu}{s^\nu} + O(s^{-q}) \right].\end{aligned}$$

Setting $s = \sigma_1 + it$ for $t >> 1$, we get after simple calculations

$$\Psi(\sigma_1 + it) = At^{\gamma-1} \exp\left\{ 2it + 2it \log \frac{\pi}{t} \right\} \left[\sum_{\nu=0}^{q-1} \frac{\alpha_\nu}{t^\nu} + O(t^{-q}) \right], \tag{1.42}$$

where

$$A = \pi^{-\frac{m}{2}-\varepsilon} \Gamma(\alpha+1) \exp\left\{ -\frac{\pi i}{2} \left(\frac{m}{2} + \alpha \right) \right\} \tag{1.43}$$

and

$$\gamma = \frac{m}{2} + \varepsilon - \alpha. \tag{1.44}$$

Note also that $\alpha_0 = 1$.

Setting $q = [\gamma] + 2$ in formula (1.42), we will use that formula for transforming the integral $j(x)$. After simple calculations, we obtain

$$j(x) = N^{2\sigma_1}\mathrm{Re}\Big\{A \sum_{0\neq n\in\mathbb{Z}^m} |n+x|^{-m-\varepsilon}\frac{1}{\pi}\int\limits_{T_0}^{2T} \Phi(t)\xi(\sigma_1+it)t^{\gamma-1}$$
$$\times \exp(2i\varphi_n(t))\sum_{\nu=0}^{q-1}\frac{\alpha_\nu}{t^\nu}\,dt\Big\} + O(\varepsilon^{-1}N^{-\frac{\kappa+\varepsilon}{2}}), \tag{1.45}$$

where

$$\varphi_n(t) = t + t\log\frac{t_n}{t}, \qquad t_n = \pi|n+x|N.$$

On the interval $[T, 2T]$, we introduce the auxiliary polynomials

$$P_j(t) = \sum_{\nu=0}^{r}\frac{\chi^{(\nu)}(\frac{\sigma_1-\sigma_0}{lT}+1+j)}{\nu!}\left(\frac{t-T-jT}{T}\right)^\nu + j - 1, \quad j = 0, 1,$$

and then extend the functions $P_j(t)$ to $(0,\infty)$ by letting them be zero for $t \in (0,\infty)\setminus[T, 2T]$. We shall denote the obtained functions by $P_j(t)$ as well. By means of the function $\Phi_0(t)$, defined in the beginning of this subsection, we introduce on $(0,\infty)$ the function

$$\xi_*(t) = \xi(\sigma_1+it) - \left[1-\Phi_0\left(2\frac{t-T}{T}\right)\right]P_0(t) - \left[1-\Phi_0\left(2\frac{2T-t}{T}\right)\right]P_1(t).$$

The following properties of $\xi_*(t)$ are obvious: 1) $\xi_* \in C^2(0,\infty)$, 2)

$$\xi_*^{(\nu)}(t) = O(T^{-\nu}), \qquad \nu = 0, 1, ..., r. \tag{1.46}$$

It is easy to check that

$$P_j(t) = O(T^{r-l}), \qquad j = 0, 1. \tag{1.47}$$

Choose l so large that $r = [\frac{l}{2}] >> 4([\gamma]+2)$. If we replace the function $\xi(\sigma_1+it)$ by $\xi_*(t)$ in the integral in (1.45), the error we make will not exceed $O(\varepsilon^{-1}N^{-\frac{\kappa+\varepsilon}{2}})$ by (1.47). Applying then Proposition 1.5 to the obtained integral and setting $\omega_m = \frac{\pi}{2}(\frac{m+1}{2}+\alpha)$, we will find that

$$j(x) = \frac{\Gamma(\alpha+1)}{\pi^{\alpha+1}}N^{\frac{m-1}{2}-\alpha-\kappa}$$
$$\times\,\mathrm{Re}\sum_{\substack{|n+x|\leq\frac{2}{\pi}\frac{T}{N}\\ n\neq 0}}\frac{\exp\{2\pi iN|n+x|-i\omega_m\}}{|n+x|^{\frac{m+1}{2}+\alpha}}\sum_{\nu=0}^{q-1}\frac{\alpha_\nu(|n+x|)}{(|n+x|N)^\nu}$$
$$+\,O(\varepsilon^{-1}N^{-\frac{\kappa+\varepsilon}{2}}), \tag{1.48}$$

where $\alpha_\nu(|n+x|)$ is a linear combination of the quantities

$$\xi_*^{(\mu)}(\pi|n+x|N), \qquad \mu = 0, ..., 2\nu,$$

with

$$\alpha_0(|n+x|) = \xi_*(\pi|n+x|N). \tag{1.49}$$

Recall that

$$q = [\frac{m}{2} + \varepsilon - \alpha] + 2. \tag{1.50}$$

Formulas can be given for calculating $\alpha_\nu(|n+x|)$ but they are cumbersome enough if $\pi|n+x|N > T$, and we will not dwell on them.

Let us now proceed to transforming the integral $\widetilde{J}_0(x)$ in (1.32). The Cauchy theorem yields

$$\widetilde{J}_0(x) = \widetilde{\Phi}\left(\frac{m-\kappa}{2}\right) A_m N^{m-\kappa} + \eta(\kappa, x) + \frac{1}{2\pi i} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \widetilde{\Phi}(s)\eta(\kappa + 2s, x) N^{2s} ds.$$

Denoting the last integral by $I(x)$ and applying the functional equality, we obtain

$$I(x) = \sum_{0 \neq n \in \mathbb{Z}^m} \frac{\pi^{\kappa - \frac{m}{2}}}{|n+x|^{m-\kappa}} \frac{1}{2\pi i} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \widetilde{\Phi}(s) \frac{\Gamma(\frac{m-\kappa}{2} - s)}{\Gamma(\frac{\kappa}{2} + s)} (\pi^2 |n+x|^2 N^2)^s \, ds.$$

Since $\widetilde{\Phi}(s)$ decreases faster than any power of $|t|^{-1}$, the line of integration can be shifted to the left arbitrarily far away. Therefore,

$$|I(x)| \leq C_\mu N^{-\mu},$$

where any arbitrary positive quantity can be taken as μ, and C_μ is a constant depending only on μ. Thus,

$$\widetilde{J}_0(x) = \widetilde{\Phi}(\frac{m-\kappa}{2}) A_m N^{m-\kappa} + \eta(\kappa, x) + O(N^{-\mu}). \tag{1.51}$$

6. Let us find the asymptotics of the integral $J_1(x)$ and of the integral

$$j_1(x) = \frac{1}{2\pi i} \int\limits_{\sigma_1 - 2iT}^{\sigma_1 + 2iT} \xi(s) g(s) \, ds.$$

In virtue of the calculations of Subsection 5 for $t_0 \geq T_0$, where $t_0 = \pi|x|N$,

$$\begin{aligned} j_1(x) = {} & \frac{\Gamma(\alpha+1) N^{\frac{m-1}{2} - \alpha - \kappa}}{\pi^{\alpha+1} |x|^{\frac{m+1}{2} + \alpha}} \operatorname{Re} \exp\{2\pi i |x| N - i\omega_m\} \sum_{\nu=0}^{q-1} \frac{\alpha_\nu(|x|)}{(|x|N)^\nu} \\ & + O(\Omega(N, x)), \end{aligned} \tag{1.52}$$

with $\Omega(N, x) = \varepsilon^{-1} N^{-\frac{\kappa+\varepsilon}{2}}$. If $1 << t_0 < T_0$, then only Ω will change in the last formula. To prove this, it suffices to observe that taking $\frac{t_0}{2}$ in place of T_0 in the calculations of Subsection 5 and, in accordance with this, setting $\Phi(t) = \Phi_0(\frac{2t}{t_0})$ and $\widetilde{\Phi}(t) = 1 - \Phi(t)$, we get that for the integral

$$\frac{1}{2\pi} \int\limits_{-2T}^{2T} \Phi(t) \xi(\sigma_1 + it) g(\sigma_1 + it) \, dt$$

asymptotic formula (1.52) holds, with

$$\Omega(N, x) = N^{\frac{m-1}{2} - \alpha - \kappa - q - 1} |x|^{-\frac{m+1}{2} - \alpha - q - 1}.$$

It is easy to see that

$$\frac{1}{2\pi}\int\limits_{-\frac{t_0}{2}}^{\frac{t_0}{2}} \widetilde{\Phi}(t)g(\sigma_1+it)\,dt = O(N^{-\kappa-\varepsilon}|x|^{-m-\varepsilon}). \tag{1.53}$$

Indeed, for this, it suffices to observe that

$$\frac{d^\nu}{dt^\nu}\left\{\widetilde{\Phi}(t)\Psi(\sigma_1+it)\frac{\Gamma(\frac{m-\kappa}{2}-\sigma_1-it)}{\Gamma(\frac{\kappa}{2}+\sigma_1+it)}\right\} = O\left[(|t|+1)^{\gamma-\nu-1}\right],$$

where γ is given by formula (1.44). Therefore, from Proposition 1.2 estimate (1.53) follows, and taking q so that

$$q+1\geq\frac{m-1}{2}+\varepsilon,$$

we conclude in the general case that

$$\Omega(N,x) = \varepsilon^{-1}N^{-\frac{\kappa+\varepsilon}{2}} + N^{-\kappa-\varepsilon}|x|^{-m-\varepsilon}. \tag{1.54}$$

Consider now the integral $J_1(x)$. It is clear that

$$J_1(x) = \frac{1}{2\pi i}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} g(s)\,ds - A_m N^{m-\kappa} + J_{11}(x),$$

where the quantity σ_2 satisfies the inequalities $0<\sigma_2<\frac{m-\kappa}{2}$ and

$$J_{11}(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}[\theta(t)-1]g(\sigma_0+it)\,dt.$$

If $\frac{m}{4}-\frac{\kappa}{2}-\frac{1}{4}<\mathrm{Re}s<\frac{m-\kappa}{2}$, then the classical Weber formula in the theory of Bessel functions gives

$$\int\limits_0^\infty J_{\frac{m}{2}-1}(t)t^{\frac{m}{2}-\kappa-2s}dt = 2^{\frac{m}{2}-\kappa-2s}\frac{\Gamma(\frac{m-\kappa}{2}-s)}{\Gamma(\frac{\kappa}{2}+s)},$$

and hence the convolution theorem for the Mellin transform yields

$$\begin{aligned} J_1(x) = {} & \frac{(2\pi)^{\kappa-\frac{m}{2}}}{|x|^{m-\kappa}}\int\limits_0^{2\pi|x|N}\left(1-\frac{t^2}{(2\pi|x|N)^2}\right)^\alpha J_{\frac{m}{2}-1}(t)t^{\frac{m}{2}-\kappa}\,dt \\ & - A_m N^{m-\kappa} + J_{11}(x). \end{aligned} \tag{1.55}$$

It is easy to see that for $|t|\geq T$,

$$\frac{d^\nu}{dt^\nu}\left\{[\theta(t)-1]\Psi(\sigma_0+it)\frac{\Gamma(\frac{m-\kappa}{2}-\sigma_0-it)}{\Gamma(\frac{\kappa}{2}+\sigma_0+it)}\right\} = O(|t|^{-\beta-\nu-1}),$$

where $\beta=\frac{m}{2}+\alpha+\varepsilon$. Therefore, by Proposition 1.2,

$$J_{11}(x) = O\left\{N^{m-\kappa+\varepsilon}|x|^\varepsilon T^{-\beta-\nu}|\log(\pi|x|N)|^{-\nu}\right\},$$

which yields, by choosing an appropriate ν,

$$J_{11}(x) = O\left\{N^{m-\kappa}\frac{T^{-\frac{m}{2}-\alpha-\varepsilon}}{(1+T|\log(\pi N|x|)|)^m}\right\}. \tag{1.56}$$

The final formula for the integral $J(x)$ will be obtained on the basis of (1.35), (1.38), (1.48), (1.55) and (1.56). Moreover, it is advisable to specify the case where condition (1.39) is valid. Set

$$\begin{aligned}\mathcal{L}_N^\alpha(x) = {}& \frac{\Gamma(\alpha+1)}{\pi^{\alpha+1}}N^{\frac{m-1}{2}-\alpha-\kappa} \\ & \times \operatorname{Re}\sum_{|n+x|\le\frac{2}{\pi}TN^{-1}}\frac{\exp\{2\pi iN|n+x|-i\omega_m\}}{|n+x|^{\frac{m+1}{2}+\alpha}}\sum_{\nu=0}^{q}\frac{\alpha_\nu(|n+x|)}{(|n+x|N)^\nu}\end{aligned}$$

and

$$\widetilde{\mathcal{L}}_N^\alpha(x) = \mathcal{L}_N^\alpha(x) - \frac{\Gamma(\alpha+1)N^{\frac{m-1}{2}-\alpha-\kappa}}{\pi^{\alpha+1}|x|^{\frac{m+1}{2}+\alpha}}\operatorname{Re}\left\{\exp[2\pi iN|x|-i\omega_m]\sum_{\nu=0}^{q}\frac{\alpha_\nu(|x|)}{(|x|N)^\nu}\right\}.$$

In this notation

$$\begin{aligned}J(x) = {}& 2\pi N^{\frac{m}{2}-\kappa+1}|x|^{-\frac{m}{2}+1}\int_0^1(1-u^2)^\alpha J_{\frac{m}{2}-1}(2\pi N|x|u)u^{\frac{m}{2}-\kappa}\,du \\ & +\widetilde{\mathcal{L}}_N^\alpha(x) + \sum_{\nu=0}^{[\frac{\kappa+\varepsilon}{2}]}(-1)^\nu N^{-2\nu}\frac{\Gamma(\alpha+1)}{\nu!\Gamma(\alpha+1-\nu)}\zeta(\kappa-2\nu,x) \\ & + O\{\varepsilon^{-1}N^{-\frac{\varepsilon}{2}} + N^{m-\kappa}T^{-\frac{m}{2}-\alpha-\varepsilon}\}.\end{aligned} \tag{1.57}$$

If condition (1.39) is valid, that is,

$$N|x| \ge C_0,$$

then, by (1.40), (1.48), (1.57) and (1.60),

$$\begin{aligned}J(x) = {}& \widetilde{\mathcal{L}}_N^\alpha(x) + \sum_{\nu=0}^{[\frac{\kappa+\varepsilon}{2}]}(-1)^\nu N^{-2\nu}\frac{\Gamma(\alpha+1)}{\nu!\Gamma(\alpha+1-\nu)}\zeta(\kappa-2\nu,x) \\ & + O\left\{\varepsilon^{-1}N^{-\frac{\varepsilon}{2}} + N^{-\kappa-\varepsilon}|x|^{-m-\varepsilon} + N^{m-\kappa}T^{-\frac{m}{2}-\alpha-\varepsilon}\right\}.\end{aligned} \tag{1.58}$$

Now, (1.29) and (1.57) imply the basic relation

$$\begin{aligned}S_{N,\kappa}^\alpha(x) = {}& 2\pi N^{\frac{m}{2}-\kappa+1}|x|^{-\frac{m}{2}+1}\int_0^1(1-u^2)^\alpha J_{\frac{m}{2}-1}(2\pi N|x|u)u^{\frac{m}{2}-\kappa}\,du \\ & +\widetilde{\mathcal{L}}_N^\alpha(x) + R_N^\alpha(x) + \sum_{\nu=0}^{[\frac{\kappa+\varepsilon}{2}]}(-1)^\nu N^{-2\nu}\frac{\Gamma(\alpha+1)}{\nu!\Gamma(\alpha+1-\nu)}\zeta(\kappa-2\nu,x) \\ & + O\left\{\varepsilon^{-1}N^{-\frac{\varepsilon}{2}} + \frac{N^{m-\kappa}T^{-\frac{m}{2}-\alpha-\varepsilon}}{(1+T|\log(\pi N|x|)|)^m}\right\}.\end{aligned} \tag{1.59}$$

If here $|x| \ge C_0N^{-1}$, then

$$\begin{aligned}S_{N,\kappa}^\alpha(x) = {}& \widetilde{\mathcal{L}}_N^\alpha(x) + R_N^\alpha(x) \\ & + \sum_{\nu=0}^{[\frac{\kappa+\varepsilon}{2}]}(-1)^\nu N^{-2\nu}\frac{\Gamma(\alpha+1)}{\nu!\Gamma(\alpha+1-\nu)}\zeta(\kappa-2\nu,x) \\ & + O\{\varepsilon^{-1}N^{-\frac{\varepsilon}{2}} + N^{m-\kappa}T^{-\frac{m}{2}-\alpha-\varepsilon} + N^{-\kappa-\varepsilon}|x|^{-m-\varepsilon}\}.\end{aligned} \tag{1.60}$$

7. The integral $\widetilde{J}_1(x)$ in (1.32) is calculated in a trivial way. Indeed,

$$\widetilde{J}_1(x) = -\widetilde{\Phi}\left(\frac{m-\kappa}{2}\right) A_m N^{m-\kappa} - \frac{1}{2\pi i} \int\limits_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \widetilde{\Phi}(s) h(s,x) N^{2s} ds.$$

Applying, as above, the Weber formula and the convolution theorem, we obtain

$$\widetilde{J}_1(x) = 2\pi \frac{N^{\frac{m}{2}-\kappa+1}}{|x|^{\frac{m}{2}-1}} \int\limits_0^1 J_{\frac{m}{2}-1}(2\pi N|x|t) t^{\frac{m}{2}-\kappa} \Phi(t^2)\, dt - \widetilde{\Phi}\left(\frac{m-\kappa}{2}\right) A_m N^{m-\kappa}.$$

In virtue of (1.32), (1.51) and the last formula, it turns out that

$$V_N(x) = 2\pi \frac{N^{\frac{m}{2}-\kappa+1}}{|x|^{\frac{m}{2}-1}} \int\limits_0^1 J_{\frac{m}{2}-1}(2\pi N|x|t) t^{\frac{m}{2}-\kappa} \Phi(t^2)\, dt + \eta(\kappa, x) + O(N^{-\mu}). \tag{1.61}$$

If $|x| \geq C_0 N^{-1}$, then the integral $\widetilde{J}_1(x)$ can be processed in the same way as the integral $\widetilde{J}_0(x)$. Hence,

$$V_N(x) = \zeta(\kappa, x) + O(|x|^{\kappa-\mu} N^{m-\mu}), \tag{1.62}$$

where any arbitrarily large fixed quantity can be taken as μ.

§ 2

1. Let us consider the question of the boundedness in the L^p metrics of the Riesz means $S_N^\alpha(x)$.

We shall need a few simple statements of auxiliary nature concerning the asymptotics of the number of integer solutions of the Diophantine equation

$$n_1^2 + \ldots + n_m^2 = k,$$

denoted above by $r_m(k)$.

Proposition 2.1. *For an arbitrary small quantity $\delta > 0$,*

$$r_m(k) \leq C_{m,\delta} k^{\frac{m}{2}-1+\delta}. \tag{2.1}$$

Proof. By a classical result,

$$r_2(k) = 4 \sum_{d|k} (-1)^{\frac{d-1}{2}},$$

where the sum is taken over the odd divisors of k. If $\tau(k)$ denotes the number of divisors, it follows from this that

$$r_2(k) \leq 4\tau(k).$$

The next argument concerning the estimate of $\tau(k)$ can be found in [12]. Let $p_1, \ldots, p_\nu$ be the prime divisors of k, so that $k = p_1^{\alpha_1} \ldots p_\nu^{\alpha_\nu}$. Then, as is known, $\tau(k) = (\alpha_1+1)\ldots(\alpha_\nu+1)$, and

$$\frac{\tau(k)}{k^\delta} = \prod \frac{\alpha_j + 1}{p_j^{\delta\alpha_j}} \leq C_\delta,$$

since for $p > 2^{\frac{1}{\delta}}$, the inequality holds

$$\frac{\alpha+1}{p^{\alpha\delta}} \leq 1,$$

while for an arbitrary p, there holds $\frac{\alpha+1}{p^{\alpha\delta}} \leq \frac{1}{\delta}$. Thus, (2.1) holds true for $m = 2$, where from the general case can easily be derived by induction. Indeed, for $m > 2$,

$$\begin{aligned} r_m(k) &= \sum_{|\nu|\leq\sqrt{k}} r_{m-1}(k-\nu^2) \leq C_{m-1,\delta} \sum_{|\nu|\leq\sqrt{k}} (k-\nu^2)^{\frac{m-1}{2}-1+\delta} \\ &\leq C_{m-1,\delta} k^{\frac{m-1}{2}-1+\delta} \sum_{|\nu|\leq\sqrt{k}} 1 \leq C_{m,\delta} k^{\frac{m}{2}-1+\delta}, \end{aligned}$$

where $3C_{m-1,\delta}$ can be taken as $C_{m,\delta}$. □

Remark. The bound (2.1) is can be easily proved for $m > 4$, since it is well known, [14], that

$$r_m(k) = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} k^{\frac{m}{2}-1} S_m(k) + O(k^{\frac{m}{4}}), \quad m \geq 4,$$

where $S_m(k)$ is a so-called special series, and if $m > 4$, then $S_m(k) \leq C_m$.

Proposition 2.2. *Let $\tau > 0$, $0 < \gamma < 1$ and $\upsilon \geq 2$. Then*

$$\sum_{\frac{1}{4}N^2\leq k\leq 4N^2} \frac{r_m^\gamma(k)}{(\tau|N^2-k|+1)^\upsilon} \leq C(1+\tau^{-1}) N^{2\gamma(\frac{m}{2}-1+\delta)}. \tag{2.2}$$

Proof. The following chain of inequalities holds, each step of which is obvious:

$$\begin{aligned} \sum_{\frac{1}{4}N^2\leq k\leq 4N^2} \frac{r_m^\gamma(k)}{(\tau|N^2-k|+1)^\upsilon} &\leq C_1 N^{2\gamma(\frac{m}{2}-1+\delta)} \sum_{k=-\infty}^{\infty} \frac{1}{(\tau|N^2-k|+1)^\upsilon} \\ &\leq C_1 N^{2\gamma(\frac{m}{2}-1+\delta)} \left[2 + 2\sum_{1}^{\infty} \frac{1}{(k\tau+1)^\upsilon}\right] \\ \leq 2C_1 N^{2\gamma(\frac{m}{2}-1+\delta)} \left[1 + \int_0^\infty \frac{dx}{(\tau x+1)^\upsilon}\right] &\leq 2C_1 \left(1 + \frac{1}{\tau(\upsilon-1)}\right) N^{2\gamma(\frac{m}{2}-1+\delta)}, \end{aligned}$$

□

Proposition 2.3. *The relation holds*

$$\sum_{0\leq k\leq N^2}{}' r_m(k) = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)} N^m + O\left(N^{\frac{(m-1)m}{m+1}+\delta}\right). \tag{2.3}$$

Proof. We will make use of formula (1.59) in § 1, with $\alpha = \kappa = 0$, by setting $x = 0$ in it. Elementary calculations yield then

$$\begin{aligned} \sum_{0\leq k\leq N^2}{}' r_m(k) &= \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)} N^m + \widetilde{\mathcal{L}}_N^0(0) + R_N^0(0) \\ &\quad + O(1 + \varepsilon^{-1} N^{-\frac{\varepsilon}{2}} + N^m T^{-\frac{m}{2}-\varepsilon}). \end{aligned} \tag{2.4}$$

It is easy to see that

$$
\begin{aligned}
|\widetilde{\mathcal{L}}_N^0(0)| &\le C_1 N^{\frac{m-1}{2}} \sum_{0<|n|\le TN^{-1}} |n|^{-\frac{m+1}{2}} \\
&\le C_2 N^{\frac{m-1}{2}} \int\limits_{|x|\le 2TN^{-1}} \frac{dx}{|x|^{\frac{m+1}{2}}} \le C_3 T^{\frac{m-1}{2}}.
\end{aligned} \tag{2.5}
$$

Since

$$
|R_N^0(0)| \le 2^{m+\varepsilon} \sum_{\frac14 N^2\le k\le 4N^2} r_m(k) \left|\Psi_0\left(T\log\frac{N^2}{k}, T\right)\right|,
$$

it follows from inequalities (1.25), (1.26) in § 1 and Proposition 2.2 that

$$
|R_N^0(0)| \le CN^{m-2+2\delta}(1+N^2T^{-1}). \tag{2.6}
$$

Setting $T = N^{\frac{2(m+2\delta)}{m+1}}$ and taking into account that δ is arbitrary, we derive (2.3) from (2.4)–(2.6). □

Proposition 2.4. *If $\tau \le N^{-\frac{2}{m+1}-\delta}$ and $\upsilon \ge 2$, then*

$$
\sum_{\frac14 N^2\le k\le 4N^2} \frac{r_m(k)}{(\tau|N^2-k|+1)^{\upsilon}} \le CN^{m-2}\tau^{-1}. \tag{2.7}
$$

Proof. Set

$$
\sigma_\nu = \sideset{}{'}\sum_{N^2-\frac{\nu}{\tau}<k\le N^2-\frac{\nu-1}{\tau}} r_m(k), \qquad \omega_\nu = \sideset{}{'}\sum_{N^2+\frac{\nu-1}{\tau}\le k<N^2+\frac{\nu}{\tau}} r_m(k),
$$

where the prime $'$ after the summation sign means that the summands corresponding to the minimal and to the maximal values of k are taken with the factor $\frac12$. It is easy to see that

$$
\sum_{\frac14 N^2\le k\le 4N^2} \frac{r_m(k)}{(\tau|N^2-k|+1)^{\upsilon}} \le 2\sum_{\nu=1}^{\frac34\tau N^2+1} \frac{\sigma^\nu}{\nu^\upsilon} + 2\sum_{\nu=1}^{\frac34\tau N^2+1} \frac{\omega^\nu}{\nu^\upsilon}. \tag{2.8}
$$

Since, by Proposition 2.3,

$$
\sigma_\nu = O(N^{m-2}\tau^{-1}), \qquad \omega_\nu = O(N^{m-2}\tau^{-1}),
$$

inequality (2.7) follows from (2.8) in an obvious way. □

2. In what follows, we shall consider $1\le p\le 2$, and denote the L^p-norm of the function $f(x)$, defined on the interval I, by $\|f\|$. Let $\mathcal{E}\subseteq I$ be an arbitrary Lebesgue measurable set, $\chi(x)$ be its characteristic function, and $|\mathcal{E}|$ be its measure. By the triangular inequality,

$$
\begin{aligned}
\|\chi R_N^\alpha(x)\| \le 2^{m+\varepsilon} N^{-\kappa} T^{-\alpha} &\sum_{\frac14 N^2\le k\le 4N^2} \left|\Psi_\alpha(T\log\frac{N^2}{k}, T)\right| \\
&\times \|\chi \sum_{|n|^2=k} \exp(2\pi i n\cdot x)\|.
\end{aligned}
$$

Hölder's inequality gives

$$\|\chi \sum_{|n|^2=k} \exp(2\pi i n \cdot x)\| \leq |\mathcal{E}|^{\frac{2-p}{2p}} r_m^{\frac{1}{2}}(k).$$

Applying estimates (1.25) and (1.26) in § 1 and Proposition 2.2, with $\gamma = \frac{1}{2}$, we obtain

$$\|\chi R_N^\alpha\| \leq C N^{\frac{m}{2}+1-\kappa+\delta} T^{-1-\alpha} |\mathcal{E}|^{\frac{2-p}{2p}}. \tag{2.9}$$

3. Let us proceed to the study of the function $\mathcal{L}_N^\alpha(x)$. Let $\psi(x) \in C^\infty(-\infty, \infty)$ satisfy the conditions: a) $\psi(x) \equiv 0$ if $x \leq \frac{1}{2}$ and b) $\psi(x) \equiv 1$ if $x \geq 1$. Let M be a quantity to be specified further, and let

$$\varphi(|n+x|) = \psi\left(\frac{|n+x|}{M}\right) \sum_{\nu=0}^{q} \frac{a_\nu(|n+x|)}{(|n+x|N)^\nu}. \tag{2.10}$$

Set

$$\Phi(x) = \operatorname{Re} \sum_{|n+x| \leq \frac{2T}{\pi N}} \frac{\exp\{2\pi i N|n+x| - i\omega_m\}}{|n+x|^{\frac{m+1}{2}+\alpha}} \varphi(|n+x|) \tag{2.11}$$

and

$$\Psi(x) = \operatorname{Re} \sum_{|n+x| \leq M} \left[1 - \psi\left(\frac{|n+x|}{M}\right)\right] \frac{\exp\{2\pi i N|n+x| - i\omega_m\}}{|n+x|^{\frac{m+1}{2}+\alpha}}$$
$$\times \sum_{\nu=0}^{q} \frac{a_\nu(|n+x|)}{(|n+x|N)^\nu}. \tag{2.12}$$

Then

$$\mathcal{L}_N^\alpha(x) = \frac{\Gamma(\alpha+1)}{\pi^{\alpha+1}} N^{\frac{m-1}{2}-\alpha-\kappa} [\Phi(x) + \Psi(x)]. \tag{2.13}$$

While estimating $\|\mathcal{L}_N^\alpha\|$, essential and the most complicated is the estimate of the integral

$$j = \int_I |\Phi(x)|^2 dx.$$

Expand $\Phi(x)$ in the Fourier series

$$\Phi(x) = \sum_{\nu \in \mathbb{Z}^m} c_\nu \exp(2\pi i \nu \cdot x).$$

In virtue of Parseval's equality,

$$j = \sum_{\nu \in \mathbb{Z}^m} |c_\nu|^2, \tag{2.14}$$

and it remains to estimate the coefficients c_ν. A simple calculation gives

$$c_\nu = \int_{|x| \leq R} \frac{\exp[-2\pi i \nu \cdot x]}{|x|^{\frac{m+1}{2}+\alpha}} \operatorname{Re}\{\exp[2\pi i N|n+x| - i\omega_m]\varphi(|x|)\}\, dx,$$

where $R = \frac{2}{\pi} T N^{-1}$. Let us pass to the polar coordinates in this integral and then, for simplification, make use of the classical integral

$$\int\limits_{\mathbb{S}_{m-1}} \exp\{-2\pi i r\nu \cdot y\}\, d\omega_y = 2\pi \frac{J_{\frac{m}{2}-1}(2\pi r|\nu|)}{(r|\nu|)^{\frac{m}{2}-1}},$$

where $\mathbb{S}_{m-1} = \{y : y \in \mathbb{R}^m, |y| = 1\}$ and $d\omega_y$ is the area element of this sphere. We remind the reader that $J_{\frac{m}{2}-1}$ is the Bessel function of index $\frac{m}{2} - 1$. Thus,

$$c_\nu = \frac{2\pi}{|\nu|^{\frac{m}{2}-1}} \mathrm{Re} \int\limits_{\frac{1}{2}M}^{R} J_{\frac{m}{2}-1}(2\pi|\nu|r)\varphi(r) \exp[2\pi i N r - i\omega_m]\, \frac{dr}{r^{\alpha+\frac{1}{2}}}, \tag{2.15}$$

if $\nu \neq 0$. For $\nu = 0$, the last formula undergoes a trivial modification. It follows from formula (2.10), on the basis of the definition of functions a_ν, that $\varphi \in C^k$, where an arbitrary fixed quantity can be taken as k, provided a natural l is chosen in an appropriate way in formula (1.21) in § 1. It comes out from formula (1.21) in § 1 and definition of the function ψ that

$$\varphi^{(\mu)}(r) = O\left[\left(\frac{N}{T}\right)^{\mu} + M^{-\mu}\right] = O(1), \qquad 0 \le \mu \le k, \tag{2.16}$$

provided that $T \ge CN$.

According to the asymptotic formula for the Bessel functions,

$$J_{\frac{m}{2}-1}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \mathrm{Re}\left\{e^{i[x - \frac{\pi(m-1)}{4}]} \sum_{\nu=0}^{P} \frac{\alpha_\nu}{(2ix)^\nu}\right\} + O(x^{-P-\frac{1}{2}}), \tag{2.17}$$

where α_ν are real-valued quantities, $\alpha_0 = 1$. Note also that

$$\frac{d^\mu}{dr^\mu} J_{\frac{m}{2}-1}(2\pi r|\nu|) = O(|\nu|^{\mu-\frac{1}{2}} r^{-\frac{1}{2}}).$$

Hence, by Proposition 1.2,

$$c_\nu = O\left\{M^{-\alpha}|\nu|^{-\frac{m-1}{2}} \left(\frac{|\nu|}{N}\right)^k \log\frac{R}{M}\right\},$$

and therefore if $|\nu| \le N^{1-\delta}$,

$$c_\nu = O(M^{-\alpha}|\nu|^{\frac{m+1}{2}} N^{-\frac{m}{2}}), \quad c_0 = O(N^{-\frac{m}{2}}). \tag{2.18}$$

If $|\nu| \ge N^{1-\delta}$ and $|\,|\nu| - N| \ge \frac{1}{2}N$, we will use relation (1.23), specifying the quantity P in an appropriate way, and then apply Proposition 1.2. This results in

$$c_\nu = O(M^{-\alpha}|\nu|^{-\frac{m-1}{2}}|\,|\nu| - N|^{-k}). \tag{2.19}$$

It follows from (1.24) and (1.25) that

$$j = \sum_{|\,|\nu|-N| \le \frac{1}{2}N} |c_\nu|^2 + O(N^{-m}M^{-2\alpha}). \tag{2.20}$$

If $||\nu| - N| < \frac{1}{2}N$, then we get a satisfactory bound for c_ν by taking only the leading terms in formulas (1.16) and (1.23). Thus,

$$c_\nu = \frac{2}{|\nu|^{\frac{m-1}{2}}} \operatorname{Re} \int_{\frac{1}{2}M}^{R} \cos\left(2\pi|\nu|r - \frac{\pi(m-1)}{4}\right)$$
$$\times \exp[2\pi i Nr - i\omega_m]\, \psi(\frac{r}{M}) a_0(r) \frac{dr}{r^{1+\alpha}}$$
$$+ O(|\nu|^{-\frac{m+1}{2}} M^{-1-\alpha}).$$

We have, by formula (1.49) in § 1, that $a_0(r) = \xi_*(\pi r N)$, from which, in turn, in virtue of (1.47) in § 1 and definition of the function ξ_*,

$$a_0(r) = \xi(\pi r N) + O(T^{-1}) = \theta(\pi r N) + O(T^{-1}) \tag{2.21}$$

follows. Taking into account that the function θ is real-valued, we conclude that the real part of the integrand will be equal to

$$\cos\left(2\pi|\nu|r - \frac{\pi(m-1)}{4}\right) \cos\left[2\pi N r - \frac{\pi}{2}\left(\frac{m+1}{2} - \alpha\right)\right]$$
$$\times \psi(\frac{r}{M}) \theta(\pi r N) r^{-1-\alpha}$$
$$= \frac{1}{2}\psi(\frac{r}{M}) \theta(\pi r N) r^{-1-\alpha} \Big\{ \sin[2\pi(N - |\nu|)r + \frac{\pi}{2}\alpha]$$
$$+ \cos[2\pi(N + |\nu|)r - \frac{\pi}{2}(m - \alpha)] \Big\}.$$

It is clear that the integral with the cosine will be $O(|\nu|^{-1} M^{-1-\alpha})$, thus taking into account (2.21), we obtain

$$c_\nu = \frac{1}{|\nu|^{\frac{m-1}{2}}} \int_{\frac{1}{2}M}^{R} \psi(\frac{r}{M}) \theta(\pi r N) \sin[2\pi(N - |\nu|)r + \frac{\pi}{2}\alpha] \frac{dr}{r^{1+\alpha}}$$
$$+ O\left\{ M^{-\alpha} |\nu|^{-\frac{m+1}{2}} \left(M^{-1} + \frac{N}{T} \log \frac{R}{M} \right) \right\},$$

with a constant in O independent of α. It is easy to see that the last integral is

$$O\left\{ M^{-\alpha}[1 + M||\nu| - N||]^{-1} \right\},$$

with a constant in O independent of α. Hence,

$$|c_\nu| \le \frac{C}{|\nu|^{\frac{m-1}{2}} M^\alpha} \left\{ [1 + M||\nu| - N||]^{-1} + |\nu|^{-1} \left(M^{-1} + \frac{N}{T} \log \frac{R}{M} \right) \right\}.$$

It follows from this, by (2.20), that

$$j \le C_1 M^{-2\alpha} \Big\{ \sum_{\frac{1}{4}N^2 \le k \le 4N^2} \frac{r_m(k)}{[1 + \frac{M}{N}|N^2 - k|]^2 k^{\frac{m-1}{2}}}$$
$$+ \left(M^{-1} + \frac{N}{T} \log \frac{R}{M} \right)^2 \sum_{\frac{1}{2}N \le |\nu| \le 2N} \frac{1}{|\nu|^{m+1}} \Big\} + O(N^{-m} M^{-2\alpha}).$$

Since

$$\sum_{\frac{1}{2}N\le|\nu|\le 2N} |\nu|^{-m-1} \le \int\limits_{\frac{1}{4}N\le|x|\le 3N} |x|^{-m-1}\,dx \le \frac{C_2}{N},$$

it follows for

$$M \le N^{\frac{m-1}{m+1}-\delta}, \tag{2.22}$$

by Proposition 2.4, that

$$j \le C_2 M^{-2\alpha}\left\{M^{-1} + \left(M^{-1} + \frac{N}{T}\log\frac{R}{M}\right)^2 N^{-1}\right\},$$

and since $T \ge CN$,

$$\int\limits_I |\Phi(x)|^2 dx \le C_3 M^{-1-2\alpha}. \tag{2.23}$$

As above, by means of Hölder's inequality, we get

$$\|\chi\Phi\| \le C_4|\mathcal{E}|^{\frac{2-p}{2p}} M^{-\frac{1}{2}-\alpha}. \tag{2.24}$$

4. Let us find the lower bound for $\|S_N^\alpha(x)\|$. We will assume that $p \in [1, \frac{2m}{m+1}]$. For each p, define the critical order α_p by

$$\alpha_p = \frac{m}{p} - \frac{m+1}{2}.$$

In the sequel we shall assume that

$$\alpha \le \alpha_p.$$

Restricting ourselves to the case $\mathcal{E} \subseteq T \setminus \{x : |x| \le C_0 N^{-1}\}$ so far, we make use of the formula (1.60) in § 1, putting $T = N^2$ in it. Take the constant C_0 large enough to have, for $|x|N \ge C_0$,

$$\left|\sum_{\nu=1}^{q} \frac{a_\nu(|x|)}{(|x|N)^{\nu-1}}\right| \le 2\max|a_1| = C_m. \tag{2.25}$$

It is easy to see that (1.5) in § 1 yields

$$\left|\sum_{\nu=0}^{[\frac{\kappa+\varepsilon}{2}]} (-1)^\nu N^{-2\nu} \frac{\Gamma(\alpha+1)}{\nu!\,\Gamma(\alpha+1-\nu)} \zeta(\kappa-2\nu, x)\right| \le A_m|x|^{-m+\kappa}. \tag{2.26}$$

By estimates (2.9) and (2.24) and inequality (2.26), it follows from (1.60) by means of the triangular inequality that

$$\|\chi S_N^\alpha\| \le \frac{\Gamma(\alpha+1)}{\pi^{\alpha+1}} N^{\frac{m-1}{2}-\alpha-\kappa}\|\chi\Psi\| + J \tag{2.27}$$

and

$$\|\chi S_N^\alpha\| \ge \frac{\Gamma(\alpha+1)}{\pi^{\alpha+1}} N^{\frac{m-1}{2}-\alpha-\kappa}\|\chi\Psi\| - J, \tag{2.28}$$

where

$$J = C|\mathcal{E}|^{\frac{2-p}{2p}} N^{\frac{m-1}{2}-\alpha-\kappa}[N^{\delta-1-2\alpha} + M^{-\frac{1}{2}-\alpha}]$$
$$+ A_m \left(\int\limits_I \chi(x)|x|^{-p(m-\kappa)}dx \right)^{\frac{1}{p}} + CN^{-\kappa-\varepsilon} \left(\int\limits_I \chi(x)|x|^{-p(m+\varepsilon)}dx \right)^{\frac{1}{p}}$$
$$+ C_\varepsilon |\mathcal{E}|^{\frac{1}{p}} N^{-\frac{\varepsilon}{2}}. \tag{2.29}$$

It is easy to check that the estimates hold

$$\int\limits_I \chi(x)|x|^{-p(m+\varepsilon)}dx \le \frac{v_{m-1}}{p(m+\varepsilon)-m} \left(\frac{N}{C_0}\right)^{p(m+\varepsilon)-m} \tag{2.30}$$

and

$$\int\limits_I \chi(x)|x|^{-p(m-\kappa)}dx \le \frac{v_{m-1}}{m-p(m-\kappa)}[1-(C_0N^{-1})^{m-p(m-\kappa)}], \tag{2.31}$$

where v_{m-1} is the volume of the $m-1$ dimensional unit sphere. Keeping in mind that $N \gg M$ and taking into account the last two inequalities, we obtain

$$\|\chi S_N^\alpha\| \ge \frac{\Gamma(\alpha+1)}{\pi^{\alpha+1}} N^{\frac{m-1}{2}-\alpha-\kappa} \Big\{ \|\chi\Psi\| - C_1|\mathcal{E}|^{\frac{2-p}{2p}} M^{-\frac{1}{2}-\alpha}$$
$$- C_1 C_0^{-(1-\frac{1}{p})m-\varepsilon} N^{\alpha-\alpha_p} \Big\} - C_1 \left[\frac{1-(C_0N^{-1})^{\gamma_m}}{\gamma_m} \right]^{\frac{1}{p}}$$
$$- C_\varepsilon |\mathcal{E}|^{\frac{1}{p}} N^{-\frac{\varepsilon}{2}}, \tag{2.32}$$

where $\gamma_m = m - p(m-\kappa)$.

Let us estimate $\|\chi\Psi\|$ from below. Since

$$\left| \mathrm{Re} \sum_{\substack{|n+x|\le M \\ n\ne 0}} \left[1-\psi\left(\frac{|n+x|}{M}\right)\right] \frac{\exp[2\pi i N|n+x| - i\omega_m]}{|n+x|^{\frac{m+1}{2}+\alpha}} \sum_{\nu=0}^q \frac{a_\nu(|n+x|)}{(|n+x|N)^\nu} \right|$$
$$\le B'_m \sum_{0<|n|\le M+\sqrt{m}} |n|^{-\frac{m+1}{2}-\alpha} \le B_m M^{\frac{m-1}{2}-\alpha}, \tag{2.33}$$

it follows from formula (2.12) and inequality (2.25) that

$$\|\chi\Psi\| \ge J_0 - J_1, \qquad \|\chi\Psi\| \le J_0 + J_1,$$

where

$$J_0 = \left(\int\limits_I \chi(x)\, |\cos[2\pi N|x| - \omega_m]|^p\, |x|^{-p(\frac{m+1}{2}+\alpha)}\, dx \right)^{\frac{1}{p}}$$

and

$$J_1 = \frac{C_m}{C_0} \left(\int\limits_I \chi(x)\, |x|^{-p(\frac{m+1}{2}+\alpha)}\, dx \right)^{\frac{1}{p}} + C|\mathcal{E}|^{\frac{1}{p}} M^{\frac{m-1}{2}-\alpha}.$$

Considering Δ to be a small constant, we take in the previous formulas the spherical layer

$$\mathcal{E} = \{x : C_0 N^{-1} \le |x| \le \Delta\}$$

as $\mathcal{E}$. It is easy to see that the following relations take place:

$$\begin{aligned} J_0^p &\ge v_{m-1} \int\limits_{C_0 N^{-1}}^{\Delta} \rho^{p(\alpha_p - \alpha)-1} \cos^2[2\pi N\rho - \omega_m]\, d\rho \\ &= \frac{v_{m-1}}{2} \frac{\Delta^{p(\alpha_p-\alpha)} - (C_0 N^{-1})^{p(\alpha_p-\alpha)}}{p(\alpha_p - \alpha)} \\ &+ \frac{v_{m-1}}{2} N^{-p(\alpha_p-\alpha)} \int\limits_{C_0}^{N\Delta} \rho^{p(\alpha_p-\alpha)-1} \cos[4\pi\rho - 2\omega_m]\, d\rho. \end{aligned}$$

From this, by means of simple calculations, we obtain for $N \ge N_0$,

$$J_0^p \ge \frac{v_{m-1}}{4} \frac{\Delta^{p(\alpha_p-\alpha)} - (C_0 N^{-1})^{p(\alpha_p-\alpha)}}{p(\alpha_p - \alpha)}.$$

Set

$$\Psi_{N,\alpha} = \left[\frac{1 - (C_0 \Delta^{-1} N^{-1})^{p(\alpha_p-\alpha)}}{p(\alpha_p - \alpha)}\right]^{\frac{1}{p}} v_{m-1}^{\frac{1}{p}}.$$

Thus,

$$\|\chi\Psi\| \ge \Delta^{\alpha_p-\alpha}(4^{-\frac{1}{p}} - C_m C_0^{-1})\Psi_{N,\alpha} - C_1 \Delta^{\frac{m}{p}} M^{\frac{m-1}{2}-\alpha}.$$

Take relation (2.32), with $\varepsilon = 1$, and make use of the last inequality. Then

$$\begin{aligned} \|\chi S_N^\alpha\| \ge A N^{\frac{m-1}{2}-\alpha-\kappa} \Big[& \Delta^{\alpha_p-\alpha}\Psi_{N,\alpha} - B\Delta^{m\frac{2-p}{2p}} M^{-\frac{1}{2}-\alpha} \\ & - B\Delta^{\frac{m}{p}} M^{\frac{m-1}{2}-\alpha} - BC_0^{-1} N^{\alpha-\alpha_p}\Big] - C_1 \left[\frac{1-(C_0 N^{-1})^{\gamma_m}}{\gamma_m}\right]^{\frac{1}{p}} \\ & - C_2 \Delta^{\frac{m}{p}} N^{-\frac{1}{2}}. \end{aligned}$$

Consider the expression in the brackets. After elementary transformations, it can be written as

$$\Delta^{\alpha_p-\alpha}\left[\Psi_{N,\alpha} - B\Delta^{-\alpha-\frac{1}{2}} M^{-\frac{1}{2}-\alpha}(1 + M^{\frac{m}{2}}\Delta^{\frac{m}{2}})\right] - BC_0^{-1} N^{\alpha-\alpha_p}.$$

Taking $\Delta = M^{-1}$ and choosing M large enough, we conclude that, for $N \ge N_0$, the last expression will be not less than

$$\frac{1}{2}\Delta^{\alpha_p-\alpha}\Psi_{N,\alpha}.$$

Assuming that

$$\frac{m-1}{2} - \alpha - \kappa \ge 0, \tag{2.34}$$

we obtain, for $N \ge N_0$,

$$\|S_N^\alpha\| \ge A_0 \left[\frac{1 - N^{-p(\alpha_p-\alpha)}}{p(\alpha_p - \alpha)}\right]^{\frac{1}{p}} N^{\frac{m-1}{2}-\alpha-\kappa} - C_1 \left[\frac{1-(C_0 N^{-1})^{\gamma_m}}{\gamma_m}\right]^{\frac{1}{p}}. \tag{2.35}$$

5. Let us introduce the Lebesgue constants of spherical means of periodic functions in the L^p class. If $f \in L^p$, we will denote by $S_N^\alpha(x, f)$ the Riesz mean of order α of a function f. We will call the quantity

$$\Lambda_p(N, \alpha) = \sup_{\|f\| \le 1} \|S_N^\alpha(x, f)\|$$

the Lebesgue constant of these means. It is well known that if $p \ge 1$, $\frac{1}{p'} + \frac{1}{p} = 1$, then

$$\Lambda_{p'}(N, \alpha) = \Lambda_p(N, \alpha). \tag{2.36}$$

By this, it suffices to consider the case $1 \le p \le 2$. The critical order α_p has been defined above for $p \in [1, \frac{2m}{m+1}]$. For $p' \in [\frac{2m}{m-1}, \infty)$, set

$$\alpha_{p'} = \frac{m-1}{2} - \frac{m}{p'} = \alpha_p.$$

Theorem 2.1. *If $p \in [1, \frac{2m}{m+1}]$, $0 \le \alpha \le \alpha_p$, then*

$$\Lambda_p(N, \alpha) = \Lambda_{p'}(N, \alpha) \ge A \left[\frac{N^{p(\alpha_p - \alpha)} - 1}{p(\alpha_p - \alpha)} \right]^{\frac{1}{p}}. \tag{2.37}$$

Proof. Let $p \in [1, \frac{2m}{m+1}]$. Let us specify some quantity κ so that

$$m - p(m - \kappa) < 0.$$

We define $S_{N,\kappa}^\alpha(x)$ and $V_{2N}(x)$ according to this κ. Setting $f(x) = \frac{V_{2N}(x)}{\|V_{2N}\|}$, we derive that

$$S_N^\alpha(x, f) = \frac{1}{\|V_{2N}\|} S_{N,\kappa}^\alpha(x).$$

Therefore,

$$\Lambda_p(N, \alpha) \ge \frac{\|S_{N,\kappa}^\alpha\|}{\|V_{2N}\|}. \tag{2.38}$$

Let us estimate $\|V_{2N}\|$ from above by making use of relations (1.61) and (1.62) in § 1. It follows from (1.61) in § 1 that for $N|x| \le C_0$,

$$|V_{2N}(x)| \le C N^{m-\kappa},$$

and from (1.62) in § 1 and (1.3) in § 1 it follows that for $N|x| \ge C_0$,

$$|V_{2N}(x)| \le C \left[|x|^{-m+\kappa} + |x|^{\kappa-\mu} N^{m-\kappa} \right].$$

Hence,

$$\begin{aligned}\|V_{2N}\|^p &\le C_1 N^{p(m-\kappa)-m} + C_1 \int\limits_{CN^{-1} \le |x| \le 1} \left[|x|^{-(m-\kappa)p} + |x|^{p(\kappa-\mu)} N^{p(m-\kappa)} \right] dx \\ &\le C_2 \frac{N^{(m-\kappa)p-m}}{(m-\kappa)p - m},\end{aligned}$$

where the constant C_2 depends only on m and p. The last estimate and inequalities (2.35) and (2.38) give (2.37). $\square$

Remark 1. It is quite probable that inequality (2.37) is sharp in power degree and

$$\Lambda_p(N,\alpha) \leq B\left[\frac{N^{p(\alpha_p-\alpha)}-1}{p(\alpha_p-\alpha)}\right]^{\frac{1}{p}}.$$

However, we have failed to prove this in the whole range. Mention that it is easy to do this for $p=\infty$. It is clear that

$$\Lambda_\infty(N,\alpha) = \int_I |D_N^\alpha(x)|\,dx,$$

where $D_N^\alpha(x)$ is defined from formula (1.16) in § 1 by letting $\kappa = 0$.

Theorem 2.2. *If* $0 \leq \alpha \leq \alpha_1 = \frac{m-1}{2}$, *then*

$$A\frac{N^{\alpha_1-\alpha}-1}{\alpha_1-\alpha} \leq \Lambda_\infty(N,\alpha) \leq B\frac{N^{\alpha_1-\alpha}-1}{\alpha_1-\alpha}. \tag{2.39}$$

Proof. It suffices for us to establish the right inequality in (2.39), since the left one is contained in (2.37) as a particular case. Note that

$$|D_N^\alpha(x)| \leq CN^m,$$

therefore

$$\int_{|x|\leq C_0N^{-1}} |D_N^\alpha(x)|\,dx \leq C_1. \tag{2.40}$$

To estimate the integral over the domain $\mathcal{E} = I \setminus \{x : |x| \leq C_0N^{-1}\}$, let us apply inequality (2.27), with $\kappa = 0$ and $M = 1$ in it. It follows from (2.29) that

$$J \leq C_2(N^{\alpha_1-\alpha} + \log N). \tag{2.41}$$

The upper estimate for $\|\chi\Psi\|$ is trivial, and we get

$$\|\chi\Psi\| \leq C_3\frac{1-N^{-(\alpha_1-\alpha)}}{\alpha_1-\alpha}. \tag{2.42}$$

The right inequality in (2.39) follows now from (2.27) by (2.40), (2.41) and (2.42). $\square$

Remark 2. It could be tempting to find an asymptotics of $\Lambda_\infty(N,\alpha)$, but this question seems to be extremely difficult.

§ 3

1. Let us apply previous results to the problem of divergence almost everywhere of multiple Fourier series. Consider the means of a function

$$\zeta(\kappa,x) = \sum_{0\neq n\in\mathbb{Z}^m} |n|^{-\kappa}\exp(2\pi i n\cdot x) = \sum_{k=1}^{\infty} k^{-\frac{\kappa}{2}} \sum_{|n|^2=k} e^{2\pi i n\cdot x}. \tag{3.1}$$

Theorem 3.1. *If $p \in [1, \frac{2m}{m+1})$, $\alpha < \alpha_p$ and κ satisfies the inequalities*

$$m - \frac{m}{p} < \kappa < \frac{m-1}{2} - \alpha, \tag{3.2}$$

then $\zeta(\kappa, x) \in L^p$, and its spherical means of order α diverge almost everywhere.

Proof. If the left inequality in (3.2) holds, then $\zeta(\kappa, x) \in L^p$ trivially follows from (1.4) and (1.5) in § 1. If we show that for an arbitrary measurable set $\mathcal{E} \subset I$, $|\mathcal{E}| > 0$,

$$\lim_{N\to\infty} \int\limits_{\mathcal{E}} |S_N^\alpha(x)|^p \, dx = \infty, \tag{3.3}$$

then the divergence almost everywhere of the $S_N^\alpha(x)$ means will be proved. Indeed, if the sequence $\{S_N^\alpha(x)\}$ converges on a set $\mathcal{E}_0$, $|\mathcal{E}_0| > 0$, then the Egorov theorem yields the uniform convergence on a set $\mathcal{E} \subset \mathcal{E}_0$, such that $|\mathcal{E}| > |\mathcal{E}_0| - \varepsilon > 0$, and thus (3.3) is invalid for this $\mathcal{E}$. This contradiction proves the divergence almost everywhere of the sequence $\{S_N^\alpha(x)\}$. Note that it suffices to prove (3.3) only for sets $\mathcal{E}$ which do not intersect with a small enough neighborhood of the point $x = 0$. So, the following will be devoted to the proof of (3.3). Make use of the relation (2.32) in § 2, in which χ is the characteristic function of $\mathcal{E}$. It is clear that

$$\lim_{N\to\infty} N^{-\frac{m-1}{2}+\alpha+\kappa} \|\chi S_N^\alpha\| \geq \frac{\Gamma(\alpha+1)}{\pi^{\alpha+1}} \left\{ \lim_{N\to\infty} \|\chi\Psi\| - C_1 |\mathcal{E}|^{\frac{2-p}{2p}} M^{-\frac{1}{2}-\alpha} \right\}, \tag{3.4}$$

and it remains to study $\lim\limits_{N\to\infty} \|\chi\Psi\|$. Since the quantity M is fixed, we have, by (1.49) in § 1 and (2.12) in § 2,

$$\Psi(x) = \sum_{|n+x|\leq M} \left[1 - \psi\left(\frac{|n+x|}{M} \right) \right] \frac{\cos(2\pi N|n+x| - \omega_m)}{|n+x|^{\frac{m+1}{2}+\alpha}} + O(N^{-1}),$$

and, consequently,

$$\begin{aligned} \|\chi\Psi\| &\geq \int\limits_I \chi(x)\Psi(x)\cos(2\pi N|x| - \omega_m)\, dx \\ &= \int\limits_I \chi(x) \frac{\cos^2(2\pi N|x| - \omega_m)}{|x|^{\frac{m+1}{2}+\alpha}}\, dx + j + O(N^{-1}), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} j = \int\limits_I \chi(x)\cos(2\pi N|x| - \omega_m) &\sum_{|n|\neq 0} \left[1 - \psi\left(\frac{|n+x|}{M} \right) \right] \\ \times \frac{\cos(2\pi N|n+x| - \omega_m)}{|n+x|^{\frac{m+1}{2}+\alpha}}\, dx. \end{aligned}$$

In order to estimate the obtained integrals, we shall prove a few simple statements.

2.

Proposition 3.1. *If $f(x)$ is integrable on $\mathbb{R}^m$, then*

$$\lim_{\lambda\to\infty} \int\limits_{\mathbb{R}^m} f(x)\exp(i\lambda|x|)\, dx = 0. \tag{3.6}$$

Proof. It is clear that

$$\int_{\mathbb{R}^m} f(x)\exp(i\lambda|x|)\,dx = \int_0^\infty g(\rho)\exp(i\lambda\rho)\,d\rho,$$

with

$$g(\rho) = \rho^{m-1}\int_{|x|=\rho} f(x)\,dv_{m-1},$$

where dv_{m-1} is the element of the volume of the $m-1$ dimensional unit sphere. Therefore, (3.6) follows from the classical Riemann-Lebesgue theorem. □

Proposition 3.2. *If $n \neq 0$, then*

$$\lim_{N\to\infty}\int_I \left[1-\psi\left(\frac{|n+x|}{M}\right)\right]\frac{\cos(2\pi N|x|-\omega_m)\cos(2\pi N|n+x|-\omega_m)}{|n+x|^{\frac{m+1}{2}+\alpha}} \times \chi(x)\,dx = 0. \tag{3.7}$$

Proof. Since the system $\{\exp[2\pi i\nu\cdot x]\}$ is dense in $L(I)$, we will have (3.7) if for any $\nu\in\mathbb{Z}^m$,

$$\lim_{N\to\infty}\int_I \left[1-\psi\left(\frac{|n+x|}{M}\right)\right] e^{2\pi i\nu\cdot x} \times \frac{\cos(2\pi N|x|-\omega_m)\cos(2\pi N|n+x|-\omega_m)}{|n+x|^{\frac{m+1}{2}+\alpha}}\,dx = 0.$$

Let

$$J_N = \int_I F(x)\exp\{2\pi i f_\pm(x)\}\,dx,$$

where

$$f_\pm(x) = |x| \pm |n+x| + \frac{\nu x}{N}, \qquad F(x) = \frac{1-\psi(\frac{|n+x|}{M})}{|n+x|^{\frac{m+1}{2}+\alpha}}.$$

First of all, note that if $g_\pm(x) = |x|\pm|n+x|$, then $\operatorname{grad} g_\pm(x) = 0$ if and only if $x = tn$, $-\infty < t < \infty$. Hence, if we construct a cylinder with axis $x = tn$ based on the ball $|x|\le h$ of a sufficiently small radius, and then remove from I the corresponding part of the indicated cylinder, then in the remaining set I_h

$$\min_{x\in I_h}|\operatorname{grad} f_\pm(x)| = \delta > 0,$$

provided $N \ge N_0(\nu)$. Mention also that if $x\in I_h$, then

$$\left|\frac{\partial^2 f_\pm(x)}{\partial x_l\partial x_k}\right| \le C, \qquad k,l = 1,\dots,m.$$

Make use of the obvious identity

$$\exp[2\pi i N f_\pm(x)] = -\frac{1}{4\pi^2N^2}\frac{\Delta[\exp(2\pi i N f_\pm(x))]}{|\operatorname{grad} f_\pm(x)|^2} - \frac{i}{2\pi N}\frac{\Delta f_\pm(x)}{|\operatorname{grad} f_\pm(x)|^2}\,e^{2\pi i N f_\pm(x)},$$

where Δ is the Laplace operator. This yields

$$J_{N,h} = \int\limits_{I_h} F(x)\exp[2\pi i N f_\pm(x)]\,dx$$
$$= -\frac{1}{4\pi^2N^2}\int\limits_{I_h} F(x)\frac{\Delta[\exp(2\pi i N f_\pm(x))]}{|\mathrm{grad} f_\pm(x)|^2}\,dx + O(N^{-1}).$$

Making obvious transforms, we obtain

$$\int\limits_{I_h} F(x)\frac{\Delta[\exp(2\pi i N f_\pm(x))]}{|\mathrm{grad} f_\pm(x)|^2}\,dx$$
$$= -\int\limits_{I_h} F(x)\sum_{k,l=1}^{m}\frac{\partial}{\partial x_l}\big(e^{2\pi i N f_\pm(x)}\big)\frac{\partial}{\partial x_k}\left(\frac{1}{|\mathrm{grad} f_\pm(x)|^2}\right)dx + O(N)$$
$$= O(N)\int\limits_{I_h}\left(\sum_{k,l=1}^{m}\left|\frac{\partial^2 f_\pm(x)}{\partial x_k\partial x_l}\right|^2\right)^{\frac{1}{2}}\frac{dx}{|\mathrm{grad} f_\pm(x)|^2} + O(N) = O(N).$$

Thus,

$$J_{N,h} = O(N^{-1}),$$

and since

$$J_N = J_{N,h} + O(h^m),$$

we have $\lim\limits_{N\to\infty} J_N = 0$, by the arbitrariness of h, and hence, in turn, relation (3.7) holds true. □

3. It follows from Proposition 3.2 that

$$\lim_{N\to\infty} j = 0,$$

and by Proposition 3.1,

$$\lim_{N\to\infty}\int\limits_{I}\chi(x)\frac{\cos^2(2\pi N|x| - \omega_m)}{|x|^{\frac{m+1}{2}+\alpha}}\,dx = \frac{1}{2}\int\limits_{I}\chi(x)\,\frac{dx}{|x|^{\frac{m+1}{2}+\alpha}}.$$

Therefore, it follows from (3.5) that

$$\varliminf_{N\to\infty}\|\chi\Psi\| \geq \frac{1}{2}\int\limits_{I}\chi(x)\,\frac{dx}{|x|^{\frac{m+1}{2}+\alpha}},$$

and from (3.4) we derive, by the arbitrariness of M,

$$\varliminf_{N\to\infty} N^{-\frac{m-1}{2}+\alpha+\kappa}\|\chi S_N^\alpha\| \geq \frac{\Gamma(\alpha+1)}{2\pi^{\alpha+1}}\int\limits_{I}\frac{\chi(x)}{|x|^{\frac{m+1}{2}+\alpha}}\,dx,$$

which proves (3.3) and hence the theorem. □

Remark. In the plane of the variables α and p^* consider the set R_m by relating to it the points (α, p), $p \geq 1$, for which $f(x) \in L^p$ exists such that the Riesz means of order α diverge almost everywhere.

If $m = 1$, then the known result due to A.N. Kolmogorov asserts that $(0,1) \subseteq R_1$. As is now became known, the set R_1 consists of one point. For $m \geq 2$, the picture is essentially different. In virtue of the proven above,

$$R_m \supset \left\{\alpha, p : 0 \leq \alpha < \alpha_p, 1 \leq p < \frac{2m}{m+1}\right\}.$$

In our next work we shall study the set of the points

$$\left\{\alpha, p : \alpha = \alpha_p, 1 \leq p \leq \frac{2m}{m+1}\right\}.$$

It is quite possible that the conjecture

$$R_m = \left\{\alpha, p : 0 \leq \alpha \leq \alpha_p, 1 \leq p \leq \frac{2m}{m+1}\right\}$$

can be proved in affirmative.

§ 4

1. The results in § 2 shed light on the problem of summability by the Riesz means of order $\alpha < \frac{m-1}{2}$. It is well known that if $\alpha > \frac{m-1}{2}$, then the Fourier series of a continuous function is uniformly summable by the means of order α. This is not the case for $\alpha < \frac{m-1}{2}$, and additional constraints should be posed on the function for the summability to take place.

We will consider one type of such constraints connected with the extension of the fractional differentiation to the case of many variables. More precisely, we will consider the classes of periodic function which admit the representation

$$f(x) = a_0 + \frac{1}{(2\pi)^\kappa} \int_I g(\xi)\zeta(\kappa, x-\xi)\, d\xi, \tag{4.1}$$

where $g(\xi) \in L^p(I)$, $p \geq 1$.

Theorem 4.1. *If $f(x)$ is representable in form (4.1), where $g(\xi) \in L^p(I)$, $p > \frac{m}{\kappa}$, $\kappa \leq \frac{m-1}{2}$, then the Fourier series of the function $f(x)$ is uniformly summable by the Riesz means of order $\alpha = \frac{m-1}{2} - \kappa$.*

Proof. It is clear that

$$f(x) - S_N^\alpha(x,f) = (2\pi)^{-\kappa} \int_I g(x-\xi)\left[\zeta(x,\xi) - S_{N,\kappa}^\alpha(\xi)\right] d\xi.$$

Alter this integral, assuming $g \in L^2(I)$ so far. Make use of formulas (1.59) and (1.60) in § 1 by taking $T = N^2$ in them. In virtue of (2.9) in § 2 the convolution of $R_N^\alpha(x)$ and $g(x)$ will be $O(N^{\kappa - \frac{m}{2} + \delta}) = O(1)$, and in virtue of (2.23) in § 2 the convolution of Φ and g will be $O(M^{-\frac{1}{2}-\alpha})$. It follows from (1.5) in § 1 that for $N|x| \geq C_0$,

$$\sum_{\nu=1}^{[\frac{\kappa+\varepsilon}{2}]} (-1)^\nu N^{-2\nu} \frac{\Gamma(\alpha+1)}{\nu!\Gamma(\alpha+1-\nu)} \zeta(\kappa - 2\nu, x) = O(N^{-\varepsilon}|x|^{-m+\kappa-\varepsilon}).$$

Finally, mention that $S^\alpha_{N,\kappa}(x) = O(N^{m-\kappa})$. Taking into account all these remarks, we get

$$\begin{aligned} f(x) - S^\alpha_N(x,f) = &-\frac{\Gamma(\alpha+1)}{2^\kappa \pi^{\alpha+\kappa+1}} \int\limits_{I_N} g(x-\xi)\Psi(\xi)\, d\xi \\ &+ \int\limits_{I_N} g(x-\xi)[\zeta(\kappa,\xi) + O(N^{m-\kappa})]\, d\xi \\ &+ O(N^{-\varepsilon}) \int\limits_{I_N} |g(x-\xi)|\,|\xi|^{-m+\kappa-\varepsilon}\, d\xi + O(M^{-\frac{1}{2}-\alpha}) + o(1), \end{aligned}$$

where $I_N = I \setminus \{x : |x| \le C_0 N^{-1}\}$, and Ψ is given by formula (2.12) in § 2.

Applying (1.49) in § 1 and (2.33) in § 2, we obtain after obvious transformations

$$\begin{aligned} \int\limits_{I_N} g(x-\xi)\Psi(\xi)\, d\xi = &\int\limits_{I_N} g(x-\xi) \frac{\cos(2\pi N|\xi| - \omega_m)}{|\xi|^{m-\kappa}}\, d\xi \\ &+ \int\limits_{I} \sum_{|n|\ne 0} \left[1 - \psi\left(\frac{|n+\xi|}{M}\right)\right] \frac{\cos(2\pi N|n+\xi| - \omega_m)}{|n+\xi|^{\frac{m+1}{2}+\alpha}} g(x-\xi)\, d\xi \\ &+ O(M^{\frac{m-1}{2}-\alpha}) \int\limits_{|\xi|\le C_0N^{-1}} |g(x-\xi)|\, d\xi + O(N^{-1}M^{\frac{m-1}{2}-\alpha}). \end{aligned}$$

Set

$$K_\varepsilon(\xi,N) = \begin{cases} |\xi|^{-m+\kappa}, & |\xi| \le C_0 N^{-1} \\ N^{-\varepsilon}|\xi|^{-m+\kappa-\varepsilon}, & |\xi| > C_0 N^{-1}, \end{cases}$$

and

$$\Psi_0(\xi) = \sum_{|n|\ne 0} \left[1 - \psi\left(\frac{|n+\xi|}{M}\right)\right] \frac{\cos(2\pi N|n+\xi| - \omega_m)}{|n+\xi|^{\frac{m+1}{2}+\alpha}}.$$

Estimating $\zeta(\kappa,x)$ by means of (1.5) in § 1, we obtain, for an arbitrary $g \in L^2(I)$,

$$\begin{aligned} f(x) - S^\alpha_N(x,f) = &-\frac{\Gamma(\alpha+1)}{2^\kappa \pi^{\alpha+\kappa+1}} \int\limits_{I} g(x-\xi) \left[\frac{\cos(2\pi N|\xi| - \omega_m)}{|\xi|^{m-\kappa}} + \Psi_0(\xi)\right] d\xi \\ &+ O(1) \int\limits_{I} |g(x-\xi)| K_\varepsilon(\xi,N)\, d\xi \\ &+ O(M^{-\frac{1}{2}-\alpha} + M^{\frac{m-1}{2}} N^{-1}) + o(1). \end{aligned} \tag{4.2}$$

Now, taking $g \in L^p(I)$, $p > \frac{m}{\kappa}$, we easily estimate the integrals in (4.2). Indeed, it follows from Hölder's inequality that the second integral is $o(1)$ uniformly in x and M, and Proposition 3.1 easily yields that, for any fixed M, the first integral tends to zero uniformly in x. By the arbitrariness of M, this implies

$$|f(x) - S^\alpha_N(x,f)| \le \delta$$

for $N \ge N_\delta$. □

Remark. Changing slightly the final part of the proof of Theorem 4.1, it is easy to get formally more general

Theorem 4.1′. *Let $f(x)$ be representable in the form (4.1), where $g(\xi) \in L^2(I)$, $\kappa \le \frac{m-1}{2}$. Let $\mathcal{I} \subset I$ be an open set, and $g \in L^p(\mathcal{I})$, where $p > \frac{m}{\kappa}$. Then, uniformly in any inner subdomain $\mathcal{I}_1 \subset \mathcal{I}$,*

$$\lim_{N\to\infty} S_N^\alpha(x,f) = f(x), \qquad \alpha = \frac{m-1}{2} - \kappa.$$

It is absolutely clear that Theorem 4.1 is sharp in the sense that the order of the means $\alpha = \frac{m-1}{2} - \kappa$ cannot be taken smaller with the assertion of the theorem still to be valid.

Theorem 4.2. *Let $0 < \kappa < \frac{m-1}{2}$ and $p > \frac{m}{\kappa}$ be constants such as in Theorem 4.1 and let $0 \le \alpha < \frac{m-1}{2} - \kappa$. Then there exists a function $g \in L^p(I)$, and such a function $f(x)$, defined by (4.1), that*

$$\overline{\lim_{N\to\infty}} |S_N^\alpha(0,f)| = \infty. \tag{4.3}$$

Proof. Since $p > \frac{2m}{m-1}$, we have that $p' = \frac{p}{p-1}$ satisfies the inequality $p' < \frac{2m}{m+1}$. Since $\alpha < \frac{m-1}{2} - \kappa$, it is easy to check that $\alpha < \alpha_{p'}$ and $\gamma_m = m - p'(m-\kappa) > 0$. Hence it follows from inequality (2.35) in § 2, taken for the $L^{p'}$-norm, that

$$\lim_{N\to\infty} \int_I |S_N^\alpha(x)|^{p'}\,dx = \infty,$$

and thus there exists $g(\xi) \in L^p$ such that

$$\overline{\lim_{N\to\infty}} \left| \int_I g(-\xi) S_N^\alpha(\xi)\,d\xi \right| = \infty.$$

By this, (4.3) holds for the function $f(x)$ defined by relation (4.1). □

2. Condition $p > \frac{m}{\kappa}$ is essential in the formulation of Theorem 4.1, since for $p \le \frac{m}{\kappa}$, generally speaking, $f(x) \notin C(I)$, where from its Fourier series cannot be uniformly summable. It is clear that if $2 \le p \le \frac{m}{\kappa}$, the Fourier series of $f(x)$ will converge almost everywhere, and hence will be summable almost everywhere by the Riesz means. It is not difficult to characterize the set of divergence points for the means of order $\alpha = \frac{m-1}{2} - \kappa$. Indeed, by Proposition 3.1, it follows from (4.2) that

$$\lim_{N\to\infty} S_N^\alpha(x,f) = f(x), \qquad \alpha = \frac{m-1}{2} - \kappa, \tag{4.4}$$

provided

$$\int_I \frac{|g(x-\xi)|}{|\xi|^{m-\kappa+\varepsilon}}\,d\xi < \infty.$$

Taking $\varepsilon > 0$ sufficiently small and applying the Sobolev-Kondrashov theorem [8], we conclude that (4.4) holds almost everywhere in every k-dimensional hyperplane, provided

$$k > m - p\kappa. \tag{4.5}$$

Theorem 4.3. *Let $f(x)$ be representable in the form (4.1), where $g \in L^p(I)$, $2 \le p \le \frac{m}{\kappa}$. If $\alpha = \frac{m-1}{2} - \kappa$ and a natural k satisfies (4.5), then in any k-dimensional hyperplane (4.4) holds almost everywhere.*

In the theory of Fourier series for functions of one variable a number of results is known [1], [7] in which for those or another classes of functions with almost everywhere convergent Fourier series additional information is pointed out on the exceptional set, for instance, in terms of its capacity. Though for functions of one variable such a setting may look somewhat precious, for functions of several variables it seems to be necessary. Say, for functions which satisfy Hölder's condition of order γ, $0 < \gamma \leq \frac{m-1}{2}$, it seems natural to give a subtle characteristic of the exceptional set of measure zero, where the Fourier series may be not summable by means of order $\alpha \leq \frac{m-1}{2} - \gamma$. Though Theorem 4.3 gives an example of such a characteristic, it might be more fruitful to characterize an exceptional set by means of either its Hausdorff measures or capacity.

References

[1] A. Beurling, *Ensembles exceptionnels*, Acta Math. 72 (1940), 1–13.

[2] G. Hardy, E. Landau, *The lattice points of a circle*, Proc. Royal Soc. (A) (1925), 224–258.

[3] V.A. Ilyin, *Problems of localization and convergence for Fourier series in fundamental systems of the Laplace operator*, Uspekhi Mat.Nauk 23 (1968), 61–120 (in Russian). - English transl. in Russian Math. Surveys 23 (1968), 59–116.

[4] A.N. Kolmogorov, *Une série de Fourier-Lebesgue divergente partout*, C.R. Acad. Sci. Paris, Ser.I 183 (1926), 1327–1328.

[5] B.M. Levitan, *On summability of multiple series and Fourier integrals*, Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 1073–1076 (in Russian).

[6] A. Oppenheim, *Some identities in the theory of numbers*, Proc. London Math. Soc. (2) 26 (1927), 295–350.

[7] R. Salem, A. Zygmund, *Capacity of sets and Fourier series*, Trans. Amer. Math. Soc. 59 (1946), 23–41.

[8] S.L. Sobolev, *Applications of functional analysis in mathematical physics*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950 (in Russian). - English transl. in Translations of Mathematical Monographs, Vol. 7 American Mathematical Society, Providence, R.I. 1963.

[9] E. Stein, *Localization and summability of multiple Fourier series*, Acta Math. 100, No. 1-2 (1958), 93–147.

[10] E.S. Titchmarsh, *The theory of the Riemann Zeta-function*, Oxford, Clarendon Press, 1951. - Russian transl. in IL, Moscow, 1953.

[11] B.A. Venkov, *Elementary number theory*, ML, 1937 (in Russian). - English transl. in Wolters-Noordhoff Publishing, Groningen, 1970.

[12] I.M. Vinogradov, *Elements of number theory*, ML, 1952 (in Russian). - English transl. in Dover Publications, Inc., New York, 1954.

[13] A. Walfisz, *Über zwei Gitterpunktprobleme*, Math. Annalen 95 (1926), 69–83.

[14] A. Walfisz (Val'fish), *On the representation of numbers by sums of squares. Asymptotic formulas*, Uspekhi Matem. Nauk 7 (1952), 97–178 (in Russian).

Received: 27.02.2017