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SULTANAEV YAUDAT TALGATOVICH

(to the 70th birthday)

On 19th July 2018 was 70th birthday of Yaudat Talgatovich Sultanaev, doctor of science (1990), professor (1991), honorary scientist of the Russian Federation, laureate of State award of the Republic of Bashkortostan in the field of science and technology, professor of the Bashkir State Pedagogical University, member of the Editorial Board of the Eurasian Mathematical Journal.

Ya.T. Sultanaev was born in the sity of Orsk. In 1971 he graduated from the Bashkir State University and then completed his postgraduate studies in the Moscow State University. Ya.T. Sultanaev's scientific supervisors were distinguished mathematicians A.G. Kostyuchenko and B.M. Levitan.

Ya.T. Sultanaev is a famous specialist in the spectral theory of differential operators and the qualitative theory of ordinary differential equations.

He obtained bilateral Tauberian theorems of Keldysh type, completely solved the problem on spectral assymptotics for semi-bounded ordinary differential operators, suggested a new method of investigation of assymptotic behaviour of solutions to singular differential equations which allowed him to essentially weaken the conditions on coefficients.

Jointly with V.A. Sadovnichii and A.M. Akhtyamov, he investigated inverse spectral problems with non-separated boundary conditions.

He published more than 70 papers in leading mathematical journals.

Among pupils of Ya.T. Sultanaev there are more than 20 candidates of science and one doctor of science.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yaudat Talgatovich on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

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ON THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION WITH THE BOUNDARY VALUE IN MORREY SPACE

N.R. Ahmedzade, Z.A. Kasumov

Communicated by D. Yang

Key words: Morrey-Poisson class, maximal function, Morrey-Lebesgue space.

AMS Mathematics Subject Classification: 30E25, 46E30.

Abstract. The class of Poisson-Morrey harmonic functions in the unit circle is introduced, some properties of functions of this class are studied. Nontangential maximal function is considered and it is estimated from above via maximum operator, and the proof is carried out for the Poisson-Stieltjes integral, when the density belongs to the corresponding Morrey-Lebesgue space. The obtained results are applied to solving of the Dirichlet problem for the Laplace equation with the boundary value in Morrey-Lebesgue space.

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1 Introduction

Let $\omega = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk on $\mathbb{C}$ and $\gamma = \partial\omega$ be its circumference.

Consider the following Dirichlet problem for the Laplace equation

$$\left.\begin{array}{l} \Delta u = 0, \ \text{ in } \ \omega\,, \\ u\,|_{\gamma} = f\,, \end{array}\right\} \tag{1.1}$$

where $f : \gamma \to R$ is a real-valued function. Let $u_r(t) = u\left(re^{it}\right)$ and

$$h_p = \left\{ u : \ \Delta u = 0 \ \text{ in } \ \omega\,, \ \text{ and } \ \|u\|_{h_p} < +\infty \right\},$$

where

$$\|u\|_{h_p} = \sup_{0<r<1} \|u_r\|_p\,,$$

$$\|g\|_p = \left(\int_{-\pi}^{\pi} |g(t)|^p\, dt \right)^{\frac{1}{p}}, 1 \le p < +\infty.$$

Denote by $P_z(\varphi)$ the Poisson kernel for the unit circle

$$P_z(\varphi) = Re\frac{e^{i\varphi} + re^{it}}{e^{i\varphi} - re^{it}} = \frac{1 - r^2}{1 - 2r\cos(t - \varphi) + r^2}, z = re^{it} \in \omega.$$

If $f \in L_p(\gamma) =: L_p$, then problem (1.1) is solvable in the class $h_p, 1 < p < +\infty$ (see e.g. [10]) and its solution can be represented as a Poisson-Lebesgue integral

$$u\left(re^{it}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_z(\varphi)\, f(\varphi)\, d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(t - \varphi) + r^2} f(\varphi)\, d\varphi,$$

wherein a boundary value $u\mid_{\gamma}=f$ in (1.1) is understood in the sense that the nontangential values on γ:

$$u\left(e^{it}\right)=\lim_{z\to e^{it}}u\left(z\right),$$

exist and almost everywhere on γ coincide with $f\left(e^{it}\right)$, i.e.

$$u\left(e^{it}\right)=f\left(e^{it}\right),\ \text{a.e. } t\in\left(-\pi,\pi\right), \tag{1.2}$$

and moreover

$$\lim_{r\to 1^{-}}\left\|u_r\left(\cdot\right)-f\left(\cdot\right)\right\|_p=0. \tag{1.3}$$

These results are well known and illuminated, e.g., in the monograph I.I. Danilyuk [10] .

In this paper Morrey-Poisson class of harmonic functions in the unit circle ω is introduced, Dirichlet problem (1.1) with the boundary value from the Morrey-Lebesgue space is considered. The analogues of relations (1.2) and (1.3) in this case are proved.

It should be noted that the concept of Morrey space was introduced by C. Morrey [27] in 1938 in the study of qualitative properties of the solutions of elliptic type equations with BMO (Bounded Mean Oscillations) coefficients (see also [26, 8]). This space provides a large class of weak solutions to the Navier-Stokes system [23]. In the context of fluid dynamics, Morrey-type spaces have been used to model the fluid flow in case where the vorticity is a singular measure supported on some sets in R^n [14]. There appeared lately a large number of research works which considered many problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc. in Morrey-type spaces (for more details see [26, 8, 23, 14, 30, 44, 22, 19, 33, 34, 35, 17, 18, 2, 28, 40, 5, 3, 29, 9, 12, 11, 20, 39, 38, 37, 16, 25, 41, 42, 43, 1, 6, 7, 15, 24, 31, 32]). It should be noted that the matter of approximation in Morrey-type spaces has only started to be studied recently (see, e.g., [17, 18, 5, 3]), and many problems in this field are still unsolved.

In the present paper the class of Poisson-Morrey harmonic functions is introduced in the unit circle, some properties of the functions of this class are studied. Nontangential maximal function is considered and it is estimated from above a maximum operator, and the proof is carried out for the Poisson-Stieltjes integral, when the density belongs to the corresponding Morrey-Lebesgue space. The obtained results are applied to the solution of the Dirichlet problem for the Laplace equation with boundary value from Morrey-Lebesgue space.

It should be noted that similar problems with respect to the analytical functions from Hardy classes were considered in [5, 3, 4].

2 Preliminaries

We will need some facts about the theory of Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane $\mathbb{C}$. Denote by $|M|_{\Gamma}$ the linear Lebesgue measure of the set $M\subset\Gamma$. All the constants throughout this paper (can be different in different places) will be denoted by c.

By Morrey-Lebesgue space $L^{p,\alpha}\left(\Gamma\right)$, $0<\alpha\leq 1$, $p\geq 1$, we mean the normed space of all measurable functions $f\left(\cdot\right)$ on Γ with the finite norm

$$\|f\|_{L^{p,\alpha}(\Gamma)}=\sup_{B}\left(\left|B\bigcap\Gamma\right|_{\Gamma}^{\alpha-1}\int_{B\bigcap\Gamma}\left|f\left(\xi\right)\right|^p\left|d\xi\right|\right)^{\frac{1}{p}}<+\infty.$$

$L^{p,\alpha}(\Gamma)$ is a Banach space with $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$. Similarly we define the weighted Morrey-Lebesgue space $L^{p,\alpha}_\mu(\Gamma)$ with the weight function $\mu(\cdot)$ on Γ equipped with the norm

$$\|f\|_{L^{p,\alpha}_\mu(\Gamma)} = \|f\mu\|_{L^{p,\alpha}(\Gamma)}, \, f \in L^{p,\alpha}_\mu(\Gamma).$$

The inclusion $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 < \alpha_1 \le \alpha_2 \le 1$. Thus, $L^{p,\alpha}(\Gamma) \subset L_1(\Gamma)$, $\forall\alpha \in (0, 1]$, $\forall p \ge 1$. For $\Gamma = \gamma$ we will use the notation $L^{p,\alpha}(\gamma) = L^{p,\alpha}$ and the spaces $L^{p,\alpha}(\gamma)$ and $L^{p,\alpha}(-\pi, \pi)$ we will identify by usual method.

More details on Morrey-type spaces can be found in [26, 8, 23, 14, 30, 44, 22, 19, 33, 17, 18, 2, 28, 40, 5, 3, 29, 9, 12, 11, 20, 39, 38, 37, 16].

We will use the following concepts. Let $\Gamma \subset C$ be some bounded rectifiable curve, $t = t(\sigma)$, $0 \le \sigma \le 1$, be its parametric representation with respect to the arc length σ, and l be the length of Γ. Let $d\mu(t) = d\sigma$, i.e. let $\mu(\cdot)$ be a linear measure on Γ. Let

$$\Gamma_t(r) = \{\tau \in \Gamma : |\tau - t| < r\}, \Gamma_{t(s)}(r) = \{\tau(\sigma) \in \Gamma : |\sigma - s| < r\}.$$

It is absolutely clear that $\Gamma_{t(s)}(r) \subset \Gamma_t(r)$.

Definition 1. A curve Γ is said to satisfy the Carleson condition, if there exists $c > 0$ such that

$$\sup_{t\in\Gamma} \mu(\Gamma_t(r)) \le cr, \forall r > 0.$$

A curve Γ is said to satisfy the chord-arc condition at the point $t_0 = t(s_0) \in \Gamma$ if there exists a constant $m > 0$ independent of t such that $|s - s_0| \le m\,|t(s) - t(s_0)|$, $\forall t(s) \in \Gamma$. Γ satisfies the chord-arc condition uniformly on Γ if there exists $m > 0$ such that $|s - \sigma| \le m\,|t(s) - t(\sigma)|$, $\forall t(s), t(\sigma) \in \Gamma$.

Let us recall some facts about the homogeneous Morrey-type spaces from the work [33]. Let $(X; d; \nu)$ be a homogeneous space equipped with the quasi-distance $d(\cdot; \cdot)$ and the measure $\nu(\cdot)$. Recall that a quasi-distance $d : X^2 \to \mathbb{R}_+ = (0, +\infty)$ is a function which satisfies the following conditions:

i) $d(x; y) \ge 0$ & $d(x; y) = 0 \Leftrightarrow x = y$; $\forall x, y \in X$;

ii) $d(x; y) \le c(d(x; z) + d(z; y))$, $\forall x, y \in X$.

Let $B_r(x)$ be the open ball

$$B_r(x) = \{y \in X : d(x; y) < r\}.$$

Set

$$\nu(B_r(x)) = \int_{B_r(x)} 1\, d\nu.$$

Assume that X has a constant homogeneous dimension æ > 0, i.e. there exist $c_1, c_2 > 0$ such that

$$c_1 r^{æ} \le \nu(B_r(x)) \le c_2 r^{æ}, \ \forall x \in X, \forall r > 0. \tag{2.1}$$

In this case, the Morrey space $L^{p,\lambda}(X)$ is defined by means of the norm

$$\|f\|_{L^{p,\lambda}(X)} = \sup_{x\in X, r>0} \left\{ \frac{1}{r^\lambda} \int_{B_r(x)} |f(y)|^p \, d\nu(y) \right\}^{\frac{1}{p}}.$$

Theorem 2.1 ([33]). *Let $(X; d; \nu)$ be a homogeneous space equipped with the quasi-metrics d and the measure ν with $\nu(X) = +\infty$. If condition (2.1) is satisfied, then the maximal operator*

$$M_\nu f(x) = \sup_{r>0} \frac{1}{|B_r(x)|_\nu} \int_{B_r(x)} |f(y)|\, d\nu(y),$$

where $|B_r(x)|_\nu =: \nu(B_r(x))$, *is bounded in* $L^{p,\lambda}(X)$ *for* $1 < p < +\infty$, $0 \leq \lambda < æ$.

3 $h_\rho^{p,\alpha}$ classes and Hardy-Littlewood operator

Let $\rho : [-\pi, \pi] \to \mathbb{R}_+$ be a weight function. Consider the weighted Morrey-type space $h_\rho^{p,\alpha}$ of harmonic functions in ω equipped with the norm

$$\|u\|_{h_\rho^{p,\alpha}} = \sup_{0<r<1} \|u_r(\cdot)\,\rho(\cdot)\|_{p,\alpha},$$

where

$$u_r(t) = u\left(re^{it}\right) = u(r\cos t;\, r\sin t).$$

Assume that the weight $\rho(\cdot)$ satisfies the following condition

$$\rho^{-1} \in L_q,\ \frac{1}{p} + \frac{1}{q} = 1. \tag{3.1}$$

Applying Hölder's inequality we obtain

$$\int_{-\pi}^{\pi} |u_r(\cdot)|\, dt \leq \left(\int_{-\pi}^{\pi} |u_r(\cdot)\,\rho(\cdot)|^p\, dt\right)^{\frac{1}{p}} \left(\int_{-\pi}^{\pi} \rho^{-q}(t)\, dt\right)^{\frac{1}{q}} \leq$$

$$\leq (2\pi)^{\frac{1-\alpha}{p}} \sup_{I\in[-\pi,\pi]} \left(\frac{1}{|I|^{1-\alpha}} \int_I |u_r\rho|^p\, dt\right)^{\frac{1}{p}} \left\|\rho^{-1}\right\|_{L_q} =$$

$$= (2\pi)^{\frac{1-\alpha}{p}} \left\|\rho^{-1}\right\|_{L_q} \|u_r\|_{h_\rho^{p,\alpha}}.$$

It follows immediately that if condition (3.1) is true, then $u \in h_1$. Consequently, every function $u \in h_\rho^{p,\alpha}$ has the nontangential boundary values $u^+(e^{it})$ on γ. Then, by Fatou's lemma (see e.g. [21, 36, 13]) we have $u_r(e^{it}) \to u^+(e^{it})$ as $r \to 1^-$ a.e. in $[-\pi, \pi]$. Applying Fatou's lemma on passage to the limit, we obtain

$$\int_I \left|u^+\left(e^{it}\right)\rho(t)\right|^p dt \leq \varliminf_{r\to 1^-} \int_I \left|u_r\left(e^{it}\right)\rho(t)\right|^p dt \leq$$

$$\leq \|u\|_{h_\rho^{p,\alpha}}^p\, |I|^{1-\alpha},$$

because

$$\left|u_r\left(e^{it}\right)\rho(t)\right| \to \left|u^+\left(e^{it}\right)\rho(t)\right|, r \to 1^-, \quad \text{for a.e.}\, t \in [-\pi, \pi].$$

It follows immediately that $u^+ \in L_\rho^{p,\alpha}$ and

$$\left\|u^+\right\|_{p,\alpha;\rho} \leq \|u\|_{h_\rho^{p,\alpha}}.$$

If the relation

$$\rho^{-1} \in L_{q+0}(-\pi, \pi),\ \text{ i.e. there exists } \varepsilon > 0 \text{ such that } \rho^{-1} \in L_{q+\varepsilon}(-\pi, \pi), \tag{3.2}$$

true, then we have

$$\int_{\pi}^{\pi} |u_r(\cdot)|^{1+\delta} dt \le \left(\int_{-\pi}^{\pi} |u_r(\cdot)\,\rho(\cdot)|^p dt \right)^{\frac{1+\delta}{p}} \left(\int_{-\pi}^{\pi} |\rho(\cdot)|^{-\frac{pq}{p-q\delta}} dt \right)^{\frac{1}{q}-\frac{\delta}{p}} \le c_\delta \|u\|_{h_\rho^{p,\alpha}}^{1+\delta},$$

where $\delta > 0$ is a sufficiently small number, and c_δ is a constant depending only on δ. Then, in view of the classical results, the representation

$$u\left(re^{it}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u^+(s)\, P(r;\, s-t)\, ds, \tag{3.3}$$

is true, where $u^+(s) =: u^+(e^{is})$, $s \in [-\pi,\, \pi]$, and $P_z(\varphi) =: P(r;\, \theta - \varphi)$ is a Poisson kernel for the unit disk

$$P_z(\varphi) = P_r(\theta-\varphi) = P(r;\, \theta-\varphi) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2},\ z = re^{i\theta}.$$

Thus, if $u \in h_\rho^{p,\alpha}$ and $\rho(\cdot)$ satisfies condition (3.2), then $u^+ \in L_\rho^{p,\alpha}$ and relation (3.3) holds.

Now let us prove the converse. In other words, let us prove that if $u^+ \in L_\rho^{p,\alpha}$ and representation (3.3) holds, then $u \in h_\rho^{p,\alpha}$. To do so, we need some auxiliary facts. First, in the following section we will consider a more general case.

4 Nontangential maximal function

Consider an arbitrary nontangential internal angle θ_0 with the vertex at a point $z = e^{it} \in \gamma$, $t \in [-\pi,\, \pi]$. Denote by $M_\mu f(t)$ the Hardy-Littlewood type maximal function (or the Hardy-Littlewood operator) of a function $f(\cdot)$:

$$M_\mu f(x) = \sup_{I \ni x} \frac{1}{\mu(I)} \int_I |f(t)|\, d\mu(t),$$

where sup is taken over all intervals $I \subset [-\pi,\, \pi]$ which contain x, and $\mu(\cdot)$ is a Borel measure on $[-\pi,\, \pi]$, which satisfies the condition

$$\mu(I) > 0, \text{ for } \forall I : |I| > 0.$$

Let us show that there exists a positive constant C_{θ_0}, depending only on θ_0, such that

$$\sup_{z \in \theta_0} |u_\mu(z)| \le C_{\theta_0} M_\mu f(t), \forall t \in [-\pi,\, \pi],$$

where

$$u_\mu(z) = u\left(re^{it}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r;\, s-t)\, u^+(s)\, d\mu(s).$$

For the usual maximal operator, this fact was established in [36, p. 237] and [13, p. 30].

Let us first prove this property for the Poisson kernel $P_z(t)$ in the upper half-plane

$$P_z(t) =: P_y(x-t) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2}, z = x + iy, y > 0.$$

Namely, let $f \in L_1\left(\frac{d\mu(t)}{1+t^2}\right)$ and consider the Poisson integral

$$u_\mu(x;\, y) = \int_{\mathbb{R}} P_y(x-s)\, f(s)\, d\mu(s).$$

Let us show that there exists $A_{\alpha_0} > 0$ such that

$$\sup_{\Gamma_{\mu;\,\alpha_0}(t)} |u_\mu(x;\,y)| \le A_{\alpha_0} M_\mu f(t)\,, \forall t \in \mathbb{R}, \tag{4.1}$$

where

$$\Gamma_{\mu;\,\alpha_0}(t) = \{(x;\,y):\ \mu((-|x-t|\,, |x-t|)) < \alpha_0 y,\ \ 0 < y < \infty\}\,,$$

and A_{α_0} is a constant, depending only on $\alpha_0 > 0$.

The method of proving Theorem 4.2 in [13] is also applicable to our case. Without loss of generality, it suffices consider the case $t = 0$. Following that method, we first consider the case $x = 0$. We have

$$u_\mu(0;\,y) = \int_{\mathbb{R}} P_y(s)\, f(s)\, d\mu(s)\,.$$

Let $I_h = (-h,\, h)$, $\forall h > 0$. Consider the sequence of step functions $h_n(s)$, which are nonnegative, even, nondecreasing for $s > 0$, and tend to $P_y(s)$ as $n \to \infty$ (for every fixed $y > 0$). Then it is clear that $h_n(\cdot)$ has the following form

$$h_n(s) = \sum_{k=1}^{m} a_k \chi_{I_{h_k}}(s)\,, \forall s \in \mathbb{R},$$

where $a_k \ge 0$. Assume that the measure $\mu(\cdot)$ satisfies the condition

$$m_\mu = \sup_{y>0;\, x\in\mathbb{R}} \int_{\mathbb{R}} P_y(s - |x|)\, d\mu(s) < +\infty. \tag{4.2}$$

Then we have

$$\int_{\mathbb{R}} h_n(s)\, d\mu(s) = \sum_{k=1}^{m} a_k \mu(I_{h_k}) \le \int_{\mathbb{R}} P_y(s)\, d\mu(s) \le m_\mu < +\infty.$$

Thus

$$\left|\int_{\mathbb{R}} h_n(s)\, f(s)\, d\mu(s)\right| \le \int_{\mathbb{R}} h_n(s)\, |f(s)|\, d\mu(s) =$$

$$= \sum_{k=1}^{m} a_k \mu(I_{h_k}) \frac{1}{\mu(I_{h_k})} \int_{I_{h_k}} |f(s)|\, d\mu(s) \le$$

$$\le \sum_{k=1}^{m} a_k \mu(I_{h_k})\, M_\mu f(0) \le m_\mu M_\mu f(0)\,.$$

Hence, applying Fatou's lemma, we obtain

$$\int_{\mathbb{R}} P_y(s)\, |f(s)|\, d\mu(s) \le m_\mu M_\mu f(0)\,,$$

and, consequently

$$|u_\mu(0;\,y)| \le \int_{\mathbb{R}} P_y(s)\, |f(s)|\, d\mu(s) \le m_\mu M_\mu f(0)\,.$$

In the general case, the existence of a number $A_1 > 0$ such that

$$|u_\mu(x;\, y)| \le A_1 M_\mu f(0)\,, \forall (x;\, y) \in \Gamma_{\mu;\,\alpha_0}(0)\,,$$

can be proved in a very similar way to [13] with the use of the above method.

Indeed, following the work [13], let us consider the function $\psi(s)$, which is even on $\mathbb{R}$ and majorizing $P_y(x-s)$, defined by the expression

$$\psi(s) = \sup_{|t|>s} P_y(x-t)\,, s \ge 0; \text{ and } \psi(-s) = \psi(s)\,, \forall s \in \mathbb{R},$$

where $(x,\, y) \in \Gamma_\alpha(0)$ is an arbitrary fixed point. Paying attention to the parity of the function $P_y(\cdot) : P_y(x-t) = P_y(t-x)$, for $\psi(s)$ we obtain the expression

$$\psi(s) = \begin{cases} P_y(s-|x|)\,, & s \ge |x|\,, \\ P_y(0)\,, \; 0 \le s < |x|\,, \end{cases}$$

for $\forall s \ge 0$. Let us approximate the function $\psi(\cdot)$ by step functions $h_n(\cdot)$, as in the case $P_y(\cdot)$, that is even, non-negative, not increasing, and, as $n \to \infty$, increasingly tends to $\psi(\cdot)$. So, let

$$h_n(s) = \sum_{k=1}^{m} a_k \chi_{(-x_k, x_k)}(s)\,, \quad s \in \mathbb{R}, \quad a_k \ge 0.$$

We have

$$\int_{\mathbb{R}} h_n(s)\, d\mu(s) = \sum_k a_k \mu((-x_k,\, x_k)) \le \int_{\mathbb{R}} \psi(s)\, d\mu(s) =$$

$$= \int_{-|x|}^{|x|} P_y(0)\, d\mu(s) + \int_{-\infty}^{-|x|} P_y(s-|x|)\, d\mu(s) + \int_{|x|}^{+\infty} P_y(s-|x|)\, d\mu(s) \le$$

$$\le \frac{1}{\pi} \frac{\mu((-|x|\,,\, |x|))}{y} + \int_{-\infty}^{+\infty} P_y(s-|x|)\, d\mu(s)\,.$$

Let $\alpha > 0$ be a fixed number. Suppose that condition (4.2) holds. Then we have

$$\int_{\mathbb{R}} h_n(s)\, d\mu(s) \le \frac{\alpha}{\pi} + \gamma, \forall (x;\, y) \in \Gamma_{\mu;\,\alpha}(0)\,.$$

Consequently,

$$\left| \int_{\mathbb{R}} h_n(s)\, f(s)\, d\mu(s) \right| \le \int_{\mathbb{R}} h_n(s)\, |f(s)|\, d\mu(s) = \sum_k a_k \int_{-x_k}^{x_k} |f(s)|\, d\mu(s) =$$

$$= \sum_k a_k \mu((-x_k,\, x_k)) \frac{1}{\mu((-x_k,\, x_k))} \int_{-x_k}^{x_k} |f(s)|\, d\mu(s) \le$$

$$\le \sum_k a_k \mu((-x_k,\, x_k))\, M_\mu f(0) \le A_\alpha M_\mu f(0)\,, \forall (x,\, y) \in \Gamma_{\mu;\,\alpha}(0)\,,$$

where $A_\alpha = \frac{\alpha}{\pi} + \gamma$.

So the following main lemma is true.

Lemma 4.1. *Let $\mu(\cdot)$ be a Borel measure on $\mathbb{R}$ with*

$$\mu(I) > 0, \forall I : |I| > 0; \; \sup_{y>0;\, x\in\mathbb{R}} \int_{\mathbb{R}} P_y(s-|x|)\, d\mu < +\infty.$$

Then, for $f \in L_1\left(\frac{d\mu(t)}{1+t^2}\right)$*, the function*

$$u_\mu(x;\, y) = \int_{\mathbb{R}} P_y(x-s)\, f(s)\, d\mu(s)\,,$$

which is harmonic on the upper half-plane, satisfies the relation

$$\sup_{z\in\Gamma_{\mu;\,\alpha_0}(t)} |u_\mu(z)| \le A_{\alpha_0} M_\mu f(t)\,, t\in\mathbb{R},$$

where M_μ *is the Hardy-Littlewood type maximal function*

$$M_\mu f(x) = \sup_{I\ni x} \frac{1}{\mu(I)} \int_I |f(t)|\, d\mu(t)\,,$$

$$\Gamma_{\mu;\,\alpha_0}(t) = \{(x;\, y)\in C : \mu((-|x-t|\,,\, |x-t|)) < \alpha_0 y;\; y>0\}\,, \alpha_0>0,$$

and A_{α_0} *is a constant depending only on* α_0*.*

By M we denote the usual Hardy-Littlewood operator, i.e.

$$Mf(x) = \sup_{I\ni x} \frac{1}{|I|} \int_I |f(t)|\, dt,$$

where $|I|$ is a Lebesgue measure of the interval $I\subset[-\pi,\pi]$.

It is not difficult to see that the Lebesgue measure on $\mathbb{R}$ satisfies all the conditions of Lemma 4.1. Therefore we get, in particular, the following corollary.

Corollary 4.1. *Let* $f\in L_1\left(\frac{dt}{1+t^2}\right)$*. Then there exists* $A_{\alpha_0}>0$ *such that*

$$\sup_{z\in\Gamma_{\alpha_0}(t)} |u(z)| \le A_{\alpha_0} Mf(t)\,, \forall t\in\mathbb{R},$$

where $u(\cdot)$ *is the corresponding harmonic function on* $Imz>0$*, and* M *is the usual maximal operator.*

Let us go back to Theorem 1 [33]. Let condition (2.1) be satisfied. Note that Theorem 1 [33] is true in the case $\mu(X)<+\infty$, too. Since its proof is based on the Fefferman-Stein inequality which is also true in the case $\mu(X)<+\infty$. Let us apply this theorem to our case. In our case we have $X=\mathbb{R}$, $d(x;\, y)=|x-y|$ and æ $=1$. So, if the measure $\mu(\cdot)$ satisfies the conditions of Theorem 1 [33] in our case, then we have

$$\int_I |M_\mu f|^p\, d\mu \le c\,|I|^{1-\alpha}\,,$$

where $|I|$ is the Lebesgue measure of a set $I\subset\mathbb{R}$. Then from (4.1) it directly follows that $u_\mu\in h^{p,\alpha}(d\mu)$, where $h^{p,\alpha}(d\mu)$ is the class of all harmonic functions on the upper half-plane equipped, for which the norm

$$\|u_\mu\|_{h^{p,\alpha}(d\mu)} = \sup_{y>0}\sup_{I\subset\mathbb{R}} \left(\frac{1}{|I|^{1-\alpha}} \int_I |u_\mu(x;\, y)|^p\, d\mu(x)\right)^{1/p} < \infty.$$

So we get the validity of the following theorem.

Theorem 4.1. *Assume that the measure $\mu(\cdot)$ satisfies the conditions (I is an interval)*

$$\mu(I) \sim |I|, \forall I \subset \mathbb{R}; \sup_{y>0;\, x\in\mathbb{R}} \int_{\mathbb{R}} P_y(s-|x|)\, d\mu(s) < +\infty.$$

Let

$$u_\mu(z) = u_\mu(x;\, y) = \int_{\mathbb{R}} P_y(x-t) f(t)\, d\mu(t),\, f \in L^{p,\alpha}(d\mu), 0 \le 1-\alpha < 1,$$

where $L^{p,\alpha}(d\mu)$ is the Morrey space equipped with the norm

$$\|f\|_{p,\alpha;\, d\mu} = \sup_{I\subset\mathbb{R}} \left\{ \frac{1}{|I|^{1-\alpha}} \int_I |f(y)|^p\, d\mu(y) \right\}^{\frac{1}{p}}.$$

Then for $\forall\alpha_0 > 0$, there exists $A_{\alpha_0} > 0$ such that

$$\sup_{(x;\, y)\in\Gamma_{\alpha_0}(t)} |u_\mu(x;y)| \le A_{\alpha_0} M_\mu f(t), \forall t \in \mathbb{R}, \tag{4.3}$$

and $u_\mu^* \in h^{p,\alpha}(d\mu)$:

$$\left\|u_\mu^*\right\|_{h^{p,\alpha}(d\mu)} \le A_{\alpha_0} \|f\|_{p,\alpha;\, d\mu}, \tag{4.4}$$

where $u_\mu^*(\cdot)$ is the nontangential maximal function for u:

$$u_\mu^*(t) = \sup_{z\in\Gamma_{\alpha_0}(t)} |u_\mu(z)|, t \in \mathbb{R}.$$

In fact, the validity of inequality (4.3) is already proved, and inequality (4.4) follows directly from (4.3) and from the boundedness of the maximal operator M_μ in $M^{p,\alpha}(d\mu)$.

In the sequel we will need some properties of the Poisson kernel for the unit disk, namely

i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\, dt = 1, \forall r \in [0;\, 1)$;

ii) $\sup_{|t|>\delta} P_r(t) \to 0$, as $r \to 1^-$, $\forall\delta > 0$;

iii) $\int_{|t|>\delta} P_r(t)\, dt \to 0$, as $r \to 1^-$, $\forall\delta > 0$.

Using these properties of the kernel $P_r(\cdot)$ we obtain

$$\|(P_r * f)(\cdot) - f(\cdot)\|_{p,\alpha;\,\rho} = \left\| \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t) f(t-s)\, dt - \right.$$

$$\left. -\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t) f(s)\, dt \right\|_{p,\alpha;\,\rho} \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t) \|f(t-\cdot) -$$

$$-f(\cdot)\|_{p,\alpha;\,\rho}\, dt = \frac{1}{2\pi}\left[\int_{|t|>\delta} P_r(t) \|f(t-\cdot) - \right.$$

$$\left. -f(\cdot)\|_{p,\alpha;\,\rho}\, dt + \int_{|t|\le\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\,\rho}\, dt \right].$$

We have

$$\frac{1}{2\pi}\int_{|t|\le\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\,\delta}\, dt \le$$

$$\le \sup_{|t|\le\delta} \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\,\delta} \to 0, \quad \text{as} \quad \delta \to 0,$$

if $f \in M_\rho^{p,\alpha}$. Assume

$$BM_\rho^{p,\alpha} = \left\{ f \in M_\rho^{p,\alpha} : \sup_{s \in \mathbb{R}} \|f(\cdot - s)\|_{p,\alpha;\rho} < +\infty \right\}.$$

Let $f \in BM_\rho^{p,\alpha} \bigcap M_\rho^{p,\alpha}$. We have

$$\|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho} \le \|f\|_{p,\alpha;\rho} + \|f(t-\cdot)\|_{p,\alpha;\rho} \le$$

$$2 \sup_{s \in \mathbb{R}} \|f(\cdot - s)\|_{p,\alpha;\rho} \int_{|t|>\delta} P_r(t)\, dt \to 0, \quad \text{as } r \to 1^-.$$

Thus, the following theorem is proved.

Theorem 4.2. *Let $f \in BM_\rho^{p,\alpha} \bigcap M_\rho^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \le 1$. Then $\|P_r * f - f\|_{p,\alpha;\rho} \to 0$, as $r \to 1^-$.*

This theorem has the following corollary.

Corollary 4.2. *Let $f \in BM_\rho^{p,\alpha} \bigcap M_\rho^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \le 1$. Then Dirichlet problem* (1.1) *is solvable in the classes $h_\rho^{p,\alpha}$.*

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