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HARDY SPACES, APPROXIMATION ISSUES AND BOUNDARY VALUE PROBLEMS

V.I. Vlasov

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Key words: Hardy spaces, analytic and harmonic functions, boundary value problem solvability, harmonic polynomials, approximation issues.

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Abstract. The weighted Hardy spaces $e_p(\mathscr{B};\, \rho)$ of harmonic functions are introduced on simply connected domains $\mathscr{B}$ with rectifiable boundaries. Boundary properties of functions in these spaces are investigated, the solvability of the Dirichlet problem is established, while its solution with its derivatives are estimated. Approximation properties of the system of harmonic polynomials in $e_p(\mathscr{B};\, \rho)$ are studied.

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1 Introduction

This paper deals with the weighted Hardy spaces $e_p(\mathscr{B};\rho)$, $p \in (1,\infty)$, of harmonic functions u on domains $\mathscr{B}$ of the special class (a) defined below. A domain of such type disposed on the complex z–plane is simply connected, locally one–sheeted and possesses a rectifiable boundary $\mathscr{C}$. A condition, ensuring that a function u belongs to the class $e_p(\mathscr{B};\rho)$, is the uniform boundedness of weighted L_p–norms over the family of parallel boundary contours $\mathscr{C}_r$ approaching the boundary $\mathscr{C}$ as $r \to 1$, i.e.

$$\sup_{r\in(0,1)} \int_{\mathscr{C}_r} |u(z)|^p\, |\rho(z)|\, |dz| \,<\, \infty. \tag{1.1}$$

Note that (1.1) is similar to the condition that defines the classical Hardy spaces H_p of analytic functions and h_p spaces of harmonic ones on the disk $\mathbb{U} := \{|\zeta| < 1\}$, as well as the Hardy–Smirnov spaces $E_p(\mathscr{B})$ of functions analytic on domains with rectifiable boundaries [6],[13],[20]. The weight is chosen to be the modulus of a function ρ analytic in $\mathscr{B}$ and subject to some requirements $(b)_p$ of compatibility with the geometry of the domain $\mathscr{B}$ expressed in terms of the so–called outer functions and the (A_p) Muckenhoupt condition [3]. The norm $\|u; e_p(\mathscr{B};\rho)\|$ is defined as the limit of the weighted $L_p(\mathscr{C}_r)$–norms as $r \to 1^-$.

We show that a function $u \in e_p(\mathscr{B};\rho)$ possesses a trace $u(z')$ in $L_p(\mathscr{C};\rho)$ on the boundary understood as the set of all non–tangential limits at $\mathscr{C}$. We prove that a function u of the class $e_p(\mathscr{B};\rho)$ possesses a harmonic conjugate v of the same class satisfying the M. Riesz–type inequality [21]

$$\|v; e_p\,(\mathscr{B};\, \rho)\| \,\le\, C\|u; e_p\,(\mathscr{B};\, \rho)\| \tag{1.2}$$

with the constant C independent of u, whenever $v(z_0) = 0$ at a fixed point $z_0 \in \mathscr{B}$.

For functions in $e_p(\mathscr{B};\,\rho)$, an analogue of the F. Riesz theorem [22] is proved, and it is established that the operator $\mathbb{S}: u(z) \to u(z')$ realizes an isometric isomorphism between the spaces $e_p(\mathscr{B};\,\rho)$ and $L_p(\mathscr{C};\,\rho)$. Hence, the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \;\; \mathscr{B}, \qquad u(z') = g(z') \in L_p(\mathscr{C};\,\rho), \tag{1.3}$$

has a unique solution in the class $e_p(\mathscr{B};\,\rho)$. We find estimates for this solution and all its derivatives that are uniform on each compact subset $E \subset D$ in terms of $L_p(\mathscr{C};\,\rho)-$ norm of the boundary function $g(z')$.

As a matter of fact a number of authors [4],[9],[10],[12],[14],[15] have introduced Hardy spaces of harmonic functions u on Lipschitz domains, but they used definitions that differed substantially from the one in our study. Instead of condition (1.1) mentioned above, a condition much more difficult to verify was adopted, namely that the nontangential maximal function u^* belongs to the class L_p. According to Burgholder, Gundy & Silverstein Theorem [1], the aforementioned condition is equivalent to the requirement that inequality (1.2) holds, i.e., that both functions u and v do belong to the class $e_p(\mathscr{B};\,\rho)$. Thus, the problem of proving inequality (1.2) was bypassed by virtue of the definition. We also note that Hardy–type classes or other similar constructions were employed to study significantly more general elliptic boundary value problems but only for domains with rather smooth (or piecewise smooth) boundary (see papers [2],[7],[8],[16],[18],[19],[23], [24],[25],[26] and references therein).

We prove that the system $\Xi := \{\xi_n\}$ defined by the formulas

$$\xi_{2n-1} := \operatorname{Re}(z - z_0)^{n-1}, \qquad \xi_{2n} := \operatorname{Im}(z - z_0)^n, \quad n = 1, 2, \ldots \tag{1.4}$$

is complete in the class $e_p(\mathscr{B};\rho)$, and under the condition $z_0 \in \mathscr{B}$ is minimal in this class. A direct corolarry of this is the convergence of the projection method of solving problem (1.3) based on system Ξ.

We also study some properties of conformal mappings of domains of the considered classes. For a domain $\mathscr{B}$ with a piecewise smooth boundary and a power weight ρ, a criterion for satisfying conditions $(b)_p$ of compatibility $\mathscr{B}$ with ρ is stated. Note that several of the results presented in this paper were previously obtained in [27], [28].

Let us start with a reminder of the definition of concepts used in the sequel, such as the outer function, the Muckenhoupt condition, the *BMO* and *VMO* spaces [3]. A function $\mathscr{E}(\zeta)$ regular in the circle $\mathbb{U}$ is called outer (and we write $\mathscr{E} \in \mathrm{Out}$) if it can be represented as follows:

$$\mathscr{E}(\zeta) \;=\; e^{i\varkappa}\, \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + \zeta}{e^{it} - \zeta} \log \eta(t)\, dt \right\},$$

where $\varkappa \in \mathbb{R}$ and a function $\eta(\theta) \geq 0$ is such that $\log \eta(\theta) \in L_1(0, 2\pi)$.

Let I be an interval on the circle $\mathbb{T} := \partial\mathbb{U}$. We say that a non–negative function $q \in L_1(\mathbb{T})$ satisfies the Muckenhoupt condition (A_p), $p \in (1,\,\infty)$, if

$$\sup_{I \subset \mathbb{T}} \left(\frac{1}{|I|} \int_I q(t)\, dt \right) \left(\frac{1}{|I|} \int_I q(t)^{-1/(p-1)}\, dt \right)^{p-1} < \infty.$$

Note that the class of these functions is enlarging with increasing p.

For $f \in L_1(\mathbb{T})$ let us denote

$$M_\delta(f) := \sup_{|I| < \delta} \frac{1}{|I|} \int_I |f(\theta) - f_I|\, d\theta; \quad f_I := \frac{1}{|I|} \int_I f(t)\, dt.$$

If $\sup_{\delta > 0} M_\delta(f) < \infty$, then following [3], we say that f is of bounded mean oscillation and write $f \in BMO$. Let $f \in BMO$ and $\lim_{\delta \to 0} M_\delta(f) = 0$. Then we say that f has a vanishing mean oscillation and write $f \in VMO$ (see [3]).

2 Classes of domains and weight functions

Definition 1. A finite simply connected and locally one–sheeted domain $\mathscr{B}$ with rectifiable boundary $\mathscr{C}$ is said to belong to the class (a) if a conformal mapping $\omega:\ \mathbb{U} \overset{\text{conf}}{\longrightarrow} \mathscr{B}$ is continuous on the closure $\overline{\mathbb{U}}$ of the circle $\mathbb{U}$.

In particular, any Jordan domain with a rectifiable boundary belongs to (a). The following statements directly follow from theorems of I.I. Privalov, N.N. Luzin and C. Caratheodory (see [6], [17],[20], where the concept of prime ends was also given)):

Proposition 1. *If a domain $\mathscr{B}$ belongs to the class (a), then:*

1) its boundary $\mathscr{C}$ consists of prime ends of the first kind only;

2) the map $\omega:\mathbb{U} \overset{\text{conf}}{\longrightarrow} \mathscr{B}$ is conformal almost everywhere on $\mathbb{T}$ and the inverse mapping $\chi:\mathscr{B} \overset{\text{conf}}{\longrightarrow} \mathbb{U}$ is conformal almost everywhere on $\mathscr{C}$;

3) the function $z=\omega(\zeta)$ is absolutely continuous on $\mathbb{T}$ and the inverse function $\zeta=\chi(z)$ is absolutely continuous on $\mathscr{C}$;

4) $\omega' \in H_1$.

Note that a domain $\mathscr{B}\in(a)$ is not necessarily a Jordan one, as it may, for example, contain cuts that end at the boundary. Although it has no internal branch points (i.e. $\omega'(\zeta)\neq 0$ for all $\zeta\in\mathbb{U}$), it can be non one sheeted, containing some branch points on its boundary.

Definition 2. Let $\mathscr{B}\in(a)$ and $p\in(1,\infty)$. We say that the domain $\mathscr{B}$ belongs to the class $(a)_p$ if: 1) $\omega'(\zeta)$ is an outer function ($\omega'\in$ Out) and 2) $|\omega'(e^{i\theta})|$ considered as a function of the variable θ, satisfies the Muckenhoupt condition (A_p).

Note that the class $(a)_p$ is enlarging with increasing p. An arbitrary domain $\mathscr{B}\in(a)_p$ satisfies the Smirnov condition (S) [6],[13],[20]. This condition means that the harmonic function $\log|\omega'(\zeta)|$ is represented by the Poisson integral with summable density. Moreover, if $\mathscr{B}\in(a)_p$ then $\log\omega'\in H_\lambda$ for all $\lambda\in(0,\infty)$.

Definition 3. Let a domain $\mathscr{B}\in(a)$ be bounded by a piecewise smooth contour $\mathscr{C}$; let its smooth arcs join each other at points z'_k, $1\le k\le N$, called corner ones, and the corresponding internal angles $\pi\alpha_k$ be positive. We then say that the domain $\mathscr{B}$ belongs to the class $(PS)_\alpha$ if $\max(1,\max\alpha_k)=\alpha$.

Note that the angles $\pi\alpha_k$ may take any positive values, including those greater than 2π since a domain $\mathscr{B}\in(a)$ may contain branch points on its boundary.

Lemma 1. *Let $\mathscr{B}\in(PS)_\alpha$ and let ζ'_k be the inverse images of z'_k under the mapping $\omega:\ \mathbb{U}\overset{conf}{\to}\mathscr{B}$. Then the following representation*

$$\omega(\zeta)=\omega({\zeta'}_k)+(\zeta-{\zeta'}_k)^{\alpha_k}\omega_k(\zeta),\quad k=\overline{1,N}, \tag{2.1}$$

for the function ω holds, where

$$\forall\lambda>0:\quad \omega_k\in H_\lambda,\ \ 1/\omega_k\in H_\lambda;\qquad \log|\omega_k(e^{i\theta})|\in VMO, \tag{2.2}$$

and its derivative admits the following representation:

$$\omega'(\zeta)=\tau(\zeta)\prod_{k=1}^{N}(\zeta-{\zeta'}_k)^{\alpha_k-1},\qquad \log\,|\omega'(e^{i\theta})|\in BMO, \tag{2.3}$$

where

$$\forall\lambda>0:\quad \tau\in H_\lambda,\quad 1/\tau\in H_\lambda,\qquad \log|\tau(e^{i\theta})|\in VMO. \tag{2.4}$$

A similar statement holds for the inverse mapping $\chi:\mathscr{B}\overset{\text{conf}}{\longrightarrow}\mathbb{U}$ as well. Representations analogous to (2.1), (2.3) were obtained in [5],[29] for domains with piecewise Lyapunov boundary.

In this case instead of (2.2), (2.4), we can state that the functions $|\omega_k|$ and $|\tau|$ are bounded above and below by positive constants.

Using Lemma 1, we obtain the following

Proposition 2. *A domain* $\mathscr{B} \in (PS)_\alpha$ *belongs to the class* $(a)_p$ *if and only if* $p > \alpha$ *.*

In order to illustrate this proposition, let us consider the following example where in the context of Definition 2 the condition $\omega' \in \mathrm{Out}$ is replaced by the following equivalent one

$$\frac{1}{\omega'} \in H_{\frac{1}{p-1}}. \tag{2.5}$$

Example 1. Let the function $z = \omega(\alpha, \gamma; \zeta)$ of the variable $\zeta \in \mathbb{U}$ with parameters α, γ be defined by the formula

$$\omega(\alpha, \gamma; \zeta) := \left(\frac{\zeta + 1}{\mathrm{e}^{\gamma+1}}\right)^\alpha \ln^\gamma\left(\frac{\mathrm{e}^{\gamma+1}}{\zeta + 1}\right), \qquad \alpha \in (1,\, 2), \qquad \gamma \in (-\infty,\, \infty)\,. \tag{2.6}$$

This function is regular on the circle $\mathbb{U}$ and maps it onto a domain denoted by $\mathscr{B}(\alpha,\, \gamma)$, and the point $\zeta = -1$ of its boundary $\partial\mathbb{U}$ into the point denoted by z_1', i.e.

$$\mathscr{B}(\alpha,\, \gamma) := \omega(\alpha, \gamma;\, \mathbb{U}), \qquad z_1' := \omega(\alpha, \gamma;\, -1). \tag{2.7}$$

The domain $\mathscr{B}(\alpha,\, \gamma)$ is a Jordan one, with a piecewise smooth contour. This contour contains the unique angular point z_1', whose angle at this point is equal to $\pi\alpha$. So the domain $\mathscr{B}(\alpha,\, \gamma)$ belongs to the class $(PS)_\alpha$.

According to Proposition 2, a domain $\mathscr{B}(\alpha,\, \gamma)$ belongs to $(a)_p$ for all $p > \alpha$. Let us show that it does not belong to the $(a)_p$ class with $p = \alpha$. Using the Pravitz lemma [6] and Fejer — Riesz inequality [11], we can establish the following properties of the function $1/\omega'(\alpha, \gamma; \zeta)$, entering condition (2.5) of the definition of the class $(a)_p$.

For $\alpha \in (1, 2)$ and $\gamma \in (0,\, \alpha - 1]$, we have

$$\frac{1}{\omega'(\alpha, \gamma)} \notin H_{\frac{1}{\alpha-1}}, \qquad \text{but} \qquad \forall p \in (\alpha,\, \infty): \quad \frac{1}{\omega'(\alpha, \gamma)} \in H_{\frac{1}{p-1}}\,. \tag{2.8}$$

If $\alpha \in (1,\, 2)$ and $\gamma \in (\alpha - 1,\, \infty)$, then

$$\frac{1}{\omega'(\alpha, \gamma)} \in H_{\frac{1}{\alpha-1}}, \qquad \text{and} \qquad \forall p \in (1,\, \alpha) \quad \frac{1}{\omega'(\alpha, \gamma)} \notin H_{\frac{1}{p-1}}\,. \tag{2.9}$$

Relations (2.8) and (2.9) show that for the domain $\mathscr{B}(\alpha,\, \gamma)$ the first condition $1/\omega'(\alpha, \gamma) \in H_{1/(\alpha-1)}$ its membership in the class $(a)_\alpha$ in Definition 2 can be satisfied, or can be violated. As for the second condition $|\omega'(\alpha, \gamma; e^{i\theta})| \in (A_\alpha)$ in this definition, it can be shown that this condition is not satisfied for any $\gamma \in (-\infty, \infty)$. Thus, the domain $B(\alpha,\, \gamma)$ belongs to $(a)_p$ for all $p > \alpha$ and does not belong to $(a)_\alpha$ for any choice of γ. This conclusion fully aligns with Proposition 2.

Definition 4. Let a domain $\mathscr{B} \in (a)$ and a function $\rho(z)$ be analytic in $\mathscr{B}$. We say that the pair $(\mathscr{B}; \rho)$ satisfies the compatibility conditions $(b)_p$, $p \in (1,\, \infty)$, if

1) the function $\rho \circ \omega$ is outer and belongs to H_δ for some $\delta > 0$,

2) the function

$$\lambda(\zeta) := \big[\rho \circ \omega(\zeta)\big]\omega'(\zeta) \tag{2.10}$$

is outer,

3) $|\lambda(e^{i\theta})|$, as a function of θ, satisfies the Muckenhoupt condition (A_p).

The meaning of this definition for piecewise smooth domains is clarified by the following theorem.

Theorem 1. *Let* $\mathscr{B} \in (a)$ *be a domain with piecewise smooth boundary* $\mathscr{C}$. *Let* $\pi\alpha_k$ *be angles at the corner points* $z'_k \in \mathscr{C}$, $k = \overline{1, N}$. *Let the function* $\rho(z)$ *have the following form:*

$$\rho(z) = \prod\nolimits_{k=1}^{N} (z - z'_k)^{\beta_k - 1}.$$

Then the pair $(\mathscr{B}; \rho)$ *satisfies condition* $(b)_p$ *if and only if*

$$0 < \alpha_k \beta_k < p$$

for all k.

Proposition 2 follows for $\rho \equiv 1$. We note also that the inclusion $\mathscr{B} \in (S)$ follows from the inclusion $(\mathscr{B}; \rho) \in (b)_p$.

3 Hardy spaces $e_p(\mathscr{B}; \rho)$ of harmonic functions

First, let us consider the Hardy spaces without weight.

Definition 5. Let $\mathscr{B} \in (a)$ and $\mathscr{C}_r$ be the image of the circle $\{|\zeta| = r\}$, $r \in (0, 1)$, under a mapping $\omega : \mathbb{U} \overset{\text{conf}}{\longrightarrow} \mathscr{B}$. Let us denote by $e_p(\mathscr{B})$, $p \in (1, \infty)$, the class of all harmonic functions $u(z)$ in $\mathscr{B}$ satisfying the following condition

$$\sup_{0<r<1} \int_{\mathscr{C}_r} |u(z)|^p ds < \infty. \tag{3.1}$$

Note that whether $u(z)$ belongs to $e_p(\mathscr{B})$ or not, it does not depend on the choice of a mapping ω. Moreover, this class can be equivalently defined without using any mapping. This was done in [13] for the $E_p(\mathscr{B})$ class: a function $u(z)$ harmonic in $\mathscr{B}$ belongs to the class $e_p(\mathscr{B})$ if there exists a family of rectifiable contours $\mathscr{C}_r \subset \mathscr{B}$ approaching $\mathscr{C}$ as $r \to 1^-$, such that condition (3.1) be satisfied.

For a function $u(z)$ defined on the domain $\mathscr{B}$ its trace $u(z')$ is understood as the set of all non–tangential limits at the boundary $\mathscr{C}$.

Theorem 2. *If a function* $u(z)$ *belongs to* $e_p(\mathscr{B})$ *with* $\mathscr{B} \in (a)_p$, $p \in (1, \infty)$ *then:*

1) it does possess a trace $u(z')$ *at the boundary* $\mathscr{C}$ *that belongs to* $L_p(\mathscr{C}; \rho)$ *and the following equalities*

$$\lim_{r\to 1} \int_{\mathscr{C}_r} |u(z)|^p \, |dz| = \int_{\mathscr{C}} |u(z')|^p \, |dz'|, \tag{3.2}$$

$$\lim_{r\to 1} \int_{\mathscr{C}} \left| u(z_r) - u(z') \right|^p |dz'| = 0,$$

hold with $z_r = \omega(r\,\chi(z'))$, $\chi = \omega^{-1}$;

2) the function $v(z)$, *harmonically conjugate to* u, *belongs to* $e_p(\mathscr{B})$ *as well, while under the condition* $v(z_0) = 0$ *with* z_0 *being a point of the domain* $\mathscr{B}$, *holds the inequality*

$$\|v; e_p(\mathscr{B})\| \le C \|u; e_p(\mathscr{B})\|$$

with the constant C *independent of* u.

The first assertion is a counterpart of the F. Riesz theorem [22] while the second one is a counterpart of M. Riesz theorem [21].

Theorem 3. *For an arbitrary boundary function* g *in* $L_p(\mathscr{C})$, $p \in (1, \infty)$, *a solution to the Dirichlet problem*

$$\Delta u = 0 \ \text{ in } \mathscr{B}, \qquad u(z') = g(z'), \tag{3.3}$$

exists and is unique in the class $e_p(\mathscr{B})$ *; here, the second equality in (3.3) holds almost everywhere on* $\mathscr{C}$ *, and is necessarily satisfied at the points of continuity of the function* $g(z')$ *.*

By Theorems 2 and 3, the class $e_p(\mathscr{B})$ is a Banach space with the norm defined as $\|u; e_p(\mathscr{B})\| := \lim_{r\to 1}\|u;\ L_p(\mathscr{C}_r)\|$ which according to (3.2) equals to $\|u;\ L_p(\mathscr{C})\|$. This brings us to the following statement.

Proposition 3. *An operator* $\mathbb{S}$ *which assigns to each function* $u \in e_p(\mathscr{B})$ *its trace* $u(z')$, $z' \in \mathscr{C}$, *is an isometric isomorphism* $e_p(\mathscr{B}) \to L_p(\mathscr{C})$ *with* $p \in (1, \infty)$.

Remark 1. Proposition 3 can be reformulated as follows: let $\mathscr{B} \in (a)$, for an operator $\mathbb{S}$ to map isomorphically $e_p(\mathscr{B})$ onto $L_p(\mathscr{C})$, is suffices to assume that the domain $\mathscr{B}$ satisfies the conditions in Definition 2, namely: 1) $1/\omega' \in H_{1/(p-1)}$; 2) $|\omega'(e^{i\theta})| \in (A_p)$. (Theorems 2 and 3 also allow similar reformulations.)

The question arises: is it possible to relax the conditions in Remark 1? To this end, we turn to the following example.

Example 2. Consider again the domain $\mathscr{B}(\alpha,\gamma)$ defined by equalities (2.6), (2.7). In Example 1, we have shown that this domain does not belong to the class $(a)_\alpha$ for any $\gamma \in (-\infty, \infty)$. Now we show that for this domain there is no isomorphism between the spaces $e_\alpha(\mathscr{B})$ and $L_\alpha(\mathscr{C})$ when $\gamma \in (-\infty, 0]$.

Let the function $u_0(\alpha,\gamma; z)$, also denoted for brevity by $u_0(z)$, be given by the formula

$$u_0(z) := U_0 \circ \chi(\alpha,\gamma;\ z), \qquad U_0(\zeta) := \operatorname{Re}\Big[(1-\zeta)/(1+\zeta)\Big],$$

where $\zeta = \chi(\alpha,\gamma; z)$ is the inverse mapping to $z = \omega(\alpha,\gamma;\zeta)$ defined by (2.6).

The function u_0 is obviously harmonic in $\mathscr{B}(\alpha,\gamma)$. It is the real part of the mapping of $\mathscr{B}(\alpha,\gamma)$ onto the right half–plane $\{\operatorname{Re} w > 0\}$, which maps the point z_1' into $\zeta = \infty$. The function u_0 is positive in $\mathscr{B}(\alpha,\gamma)$, continuous in $\overline{\mathscr{B}}(\alpha,\gamma) \setminus z_1'$, and the relation $u_0(z) \to \infty$ holds if $\mathscr{B} \ni z \to -1$. Its trace on $\mathscr{C}$ is equal to identical zero, thus $\|u_0; L_\alpha(\mathscr{C})\| = 0$. At the same time, one can verify that the limit as $r \to 1^-$ of norms $\|u_0; L_\alpha\mathscr{C}_r)\|$ takes different values for different values of γ:

$$\lim_{r\to 1}\|u_0; L_\alpha(\mathscr{C}_r)\| = \begin{cases} 0, & \gamma \in (-\infty,\ 0), \\ \text{const} \neq 0, & \gamma = 0, \\ \infty, & \gamma \in (0,\ \infty). \end{cases} \tag{3.4}$$

As follows from the first two lines of (3.4), the norms $\|u_0; L_\alpha(\mathscr{C}_r)\|$ as $r \in (0,1)$ are uniformly bounded for $\gamma \in (-\infty, 0]$. Hence, the function u_0 belongs to the class $e_\alpha(\mathscr{B})$ with $\mathscr{B} = \mathscr{B}(\alpha,\gamma)$ for these γ. Nonetheless, $u_0 \not\equiv 0$ has a null trace on $\mathscr{C}$. The absence of isomorphism of $e_\alpha(\mathscr{B}) \to L_\alpha(\mathscr{C})$ hence immediately follows.

Let us turn now to the weighted Hardy spaces.

Definition 6. Let a pair $(\mathscr{B};\rho)$ satisfy the conditions $(b)_p$, $p \in (1,\infty)$, introduced in Definition 4. Let us denote by $e_p(\mathscr{B};\rho)$ the class of all harmonic in $\mathscr{B}$ functions $u(z)$ such that

$$\sup_{0<r<1}\int_{\mathscr{C}_r}|u(z)|^p|\rho(z)|ds < \infty,$$

where the contours $\mathscr{C}_r$ are the same as in Definition 5.

Theorem 4. *Let* $u \in e_p(\mathscr{B};\rho)$ *with* $(\mathscr{B};\rho) \in (b)_p$, $p \in (1,\infty)$, *then:*

1) it does possess a trace $u(z')$ *on the boundary* $\mathscr{C}$ *that belongs to* $L_p(\mathscr{C};\rho)$, *and the following equality holds*

$$\lim_{r\to 1}\|u;\ L_p(\mathscr{C}_r;\rho)\| = \|u;\ L_p(\mathscr{C};\rho)\|; \tag{3.5}$$

2) *the function* v, *harmonically conjugate to* u, *belongs to* $e_p(\mathscr{B};\rho)$ *as well, while under the condition* $v(z_0)=0$ *with* z_0 *being a point of the domain* $\mathscr{B}$, *holds inequality (1.2) with a constant* C *independent on* u;

3) *the following equalities*

$$\lim_{r\to 1}\int_{\mathscr{C}}\left|u\left(z'_r\right)-u(z')\right|^p|\rho(z')|\,|dz'|=0,$$

$$\lim_{r\to 1}\int_{\mathscr{C}}\left|u\left(z'_r\right)\right|^p|\rho(z')|\,|dz'|=\int_{\mathscr{C}}\left|u\left(z'\right)\right|^p|\rho(z')|\,|dz'|,$$

hold with $z'_r=\omega\left(r\,\chi(z')\right)$, $\chi=\omega^{-1}$.

Assertions 1) and 3) are weighted counterparts of the F. Riesz theorem [22] while Assertions 2) is a weighted counterpart of the M. Riesz theorem [21].

Theorem 5. *Dirichlet problem (1.3) for the Laplace equation with a boundary function* $g\in L_p(\mathscr{C};\rho)$, $p\in(1,\infty)$, *has a unique solution in the class* $e_p(\mathscr{B};\rho)$. *Moreover, the second equality in (1.3) is satisfied almost everywhere on* $\mathscr{C}$ *and necessarily holds at the points of continuity of the boundary function* $g(z')$.

It follows from Theorems 4 and 5 that $e_p(\mathscr{B};\rho)$ is a Banach space with the norm defined as $\|u;e_p\,(\mathscr{B};\,\rho)\|:=\lim_{r\to 1}\|u;\;L_p(\mathscr{C}_r;\rho)\|$ which according to (3.5), is equal to $\|u;\;L_p(\mathscr{C};\rho)\|$. Moreover, the following statement holds.

Proposition 4. *The operator* $\mathbb{S}$ *which assigns to each function* $u\in e_p(\mathscr{B};\rho)$ *its trace* $u(z')\in L_p(\mathscr{C};\rho)$ *is an isometric isomorphism* $e_p(\mathscr{B};\rho)\to L_p(\mathscr{C};\rho)$ *for* $p\in(1,\infty)$.

4 Some estimates for the solution to the Dirichlet problem

Let, as before, $\zeta=\chi(z)$ be the inverse mapping to $\omega:\mathbb{U}\overset{\text{conf}}{\longrightarrow}\mathscr{B}$, and λ be defined by (2.10). We introduce the notation $D^{(l,n)}:=\partial^{l+n}/\partial^l x\,\partial^n y$, where $z=x+iy$.

Theorem 6. *Let* $p\in(1,\,\infty)$, *and the pair* $(\mathscr{B};\rho)$ *satisfies condition* $(b)_t$ *for some* $t\in(1,p]$. *Let* $u(z)$ *be a solution to problem (1.3) with* $g(z')\in L_p(\mathscr{C};\rho)$. *Then for any compact* $E\subset\mathscr{B}$ *and non–negative integers* l,n, *the following estimate holds:*

$$\max_{z\in E}\left|D^{(l,\,n)}\,u(z)\right|\;\leq\;\frac{A_{l+n}(\delta)}{\delta^{\,l+n+t/p}}\;\;\|g;\,L_p(\mathscr{C};\rho)\|,\tag{4.1}$$

where δ *is the distance of* $\chi(E)$ *to* $\mathbb{T}$. *The factor* $A_k(\delta)$ *is a polynomial in* δ *of degree* $(k-1)$ *whose coefficients depend on* k,t,p, *as well as on the function* λ *and on the values of* $\max\limits_{z\in E}|\chi^{(m)}(z)|$, $m=\overline{1,k}$. *In particular,*

$$A_0(\delta)=\frac{1}{\pi^{t/p}}\left\|\frac{1}{\lambda};\;L_{1/(t-1)}(\mathbb{T})\right\|^{1/p},\qquad A_1(\delta)=A_0(\delta)\,\max_{z\in E}|\,\chi'(z)|.\tag{4.2}$$

Note that $A_k(\delta)$ is bounded for domains with boundaries of the class $C^{k,\,\mu}$, $\mu\in(0,1)$.

Corollary 1. *Under the assumptions that the conditions of Theorem 4 are satisfied, for the solution* $u(z)$ *to Dirichlet problem (1.3) and for its gradient, the following estimates hold:*

$$\max_{z\in E}|u(z)|\;\leq\;\frac{1}{(\pi\delta)^{t/p}}\;\;\|g;L_p(\mathscr{C};\rho)\|\;\left\|\frac{1}{\lambda};\;L_{1/(t-1)}(\mathbb{T})\right\|^{1/p}\tag{4.3}$$

$$\max_{z\in E}|\mathrm{grad}\,u(z)|\;\leq\;\frac{2\,(2\pi)^{-\,t/p}}{\delta^{1+t/p}}\;\;\max_{z\in E}\;|\chi'(z)|\;\|g;L_p(\mathscr{C};\rho)\|\;\left\|\frac{1}{\lambda};\,L_{1/(t-1)}(\mathbb{T})\right\|^{1/p}.\tag{4.4}$$

Note that λ should be replaced by ω' in (4.2) – (4.4) in the non–weighted case ($\rho\equiv 1$).

5 Approximation properties of harmonic polynomials in $e_p(\mathscr{B};\rho)$ and the method of solving the Dirichlet problem

We consider the approximation properties of the system Ξ given by formulas (1.4), and also the projection method for solving Dirichlet problem (1.3). The functions $\xi_n(z) := \xi_n(z_0;\, z)$, $n \in \mathbb{N}$, $z = x+iy$, forming this system, are harmonic polynomials of x, y. To emphasize the parametric dependence of this system on z_0, we denote it by Ξ_{z_0}. The approximation properties under consideration are given in the following theorem.

Theorem 7. *Let $\mathscr{B}$ be a Jordan domain and $(\mathscr{B};\rho) \in (b)_p$, $p \in (1, \infty)$. Then the system Ξ_{z_0} is complete in $e_p(\mathscr{B};\rho)$ for any $z_0 \in \mathbb{C}$, whereas for its minimality in $e_p(\mathscr{B};\rho)$ it is necessary and sufficient that $z_0 \in \mathscr{B}$.*

Based on this theorem, we construct the solution to problem (1.3) using the following projection method:

$$u(z) \;=\; \lim_{K\to\infty} u^K(z)\,, \qquad u^K(z) := \sum\nolimits_{n=1}^{K} a_n^K\, \xi_n(z_0;\, z). \tag{5.1}$$

We determine the coefficients a_n^K of the approximate solutions u^K by means of the conditions $(u^K - g,\, \xi_n) = 0,\, n = \overline{1,\, K}$, where (a, b) is defined by the equality

$$(a, b) = \int_{\mathscr{C}} a\, b\, |\rho|\, |dz|$$

for $a \in L_p(\mathscr{C}, \rho)$, $b \in L'_p(\mathscr{C}, \rho)$ with $1/p + 1/p' = 1$.

The specified conditions lead to the following system of linear equations:

$$\sum\nolimits_{n=1}^{K} c_{m\,n}\, a_n^K \;=\; g_m, \qquad m = \overline{1,\, K}, \tag{5.2}$$

where $c_{m\,n} = (\xi_m,\, \xi_n)$, $g_m = (g,\, \xi_m)$. The convergence of the above–described method is established by the following theorem.

Theorem 8. *Let $\mathscr{B}$ be a Jordan domain and $(\mathscr{B};\rho) \in (b)_p$ with $p \in (1, \infty)$. Let $u \in e_p(\mathscr{B};\rho)$ be a solution to problem (1.3), and approximate solutions $u^K(z)$ be determined by (5.1), where the coefficients a_n^K are defined by system (5.2). Then:*

1)

$$\lim_{K\to\infty} \|u^K - g;\, L_2(\mathscr{C};\rho)\| = 0;$$

2) for any non–negative integers l, m, the sequence $\{D^{(l,\,m)}\, u^K\}_K$ converges uniformly to $D^{(l,\,m)}\, u$ on any compact $E \subset \mathscr{B}$ as $K \to \infty$;

3) if $z_0 \in \mathscr{B}$, then the limit $\lim\limits_{K\to\infty} a_n^K =: a_n$ exists for all n, and the series

$$u(z) \;=\; \sum\nolimits_{n=1}^{\infty} a_n\, \xi_n(z_0;\, z), \tag{5.3}$$

converges inside the circle $\{z : |z - z_0| < \min\limits_{z' \in \mathscr{C}} |z' - z_0|\}$, and this equality can be differentiated term by term arbitrarily many times.

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References

[1] D.L. Burgholder, R.F Gundy, M.L. Silverstein, *A maximal function characterisation of the class* H_p. Trans. Amer. Math. Soc. 157 (1971), 137–153.

[2] G. Cimmino, *Nuovo tipo di condizione al contorno e nuovo metodo di trattazione per il problema generalizzato di Dirichlet.* Rend. Circolo Math. Palermo. 61 (1937), no. 2, 117–221.

[3] J.B. Garnett, *Bounded analytic functions.* Academic Press, New York, 1981.

[4] J. Garsia–Cuerva, *Weighted* H^p *–spaces.* Rospr. Mat. (1985), no. 162. 62 pp.

[5] C. Gattegno, A. Ostrowski, *Représentation conforme a' la frontiere; domaines particuliers.* Mem. or Sc. Math. Fasc. CX. Paris, 1949.

[6] G.M. Goluzin, *Geometric theory of functions of a complex variable.* Amer. Math. Soc., Providence, 1969.

[7] A.K. Gushchin, *The Dirichlet problem for a second–order elliptic equation with an* L_p *boundary function.* Matem. Sb. 203 (2012), no. 1, 1–27 (in Russion), English translation in.

[8] A.K. Gushchin, V.P, Mikhailov, *On the existence of boundary values of solutions of en elliptic equation.* Matem. Sb. 73 (1991), no. 1. 171–194 (in Russion), English translation in.

[9] B.E.J. Dahlberg, *Harmonic functions in Lipshitz domains.* Harm. Anal. Euclidean Spaces. Proc. Symp. Pure Math. Amer. Math. Soc. Williamstone, Mass. 1978. Part 1, Providence R.I., 1979, 313–332.

[10] B.E. Dahlberg, C.E. Kenig, *Hardy spaces and the Neumann problem in* L_p *for Laplace's equation in Lipshitz domains.* Ann. of Math. 125 (1987), no. 3, 437–455.

[11] P.L. Duren, *Theory of* H^p *spaces.* Academic Press: N.Y. and London, 1970.

[12] D.S. Jerison, C.E. Kenig, *The Neuman problem om Lipschitz domains.* Bull. (New Series) American Math. Soc. 4 (1981), no 2, 201–207.

[13] M.V. Keldysh, M.A. Lavrentieff, *Sur la representation conforme des domaines limites par les curves rectifiables.* Ann. sci. Ecole norm. supér. 54 (1937), no. 1, 1–38.

[14] C.E. Kenig, *Weighted* H^p *spaces on Lipshitz domains.* Amer. J. Math. 102 (1980), no. 1, 129–163.

[15] C.E. Kenig, *Recente progress on boundary–value problems on Lipshitz domains.* Proc. Symp. Pure Math. Amer. Soc. 43 (1985), 185–205.

[16] E. Magenes, G. Stampacchia, *Il problemi al contorno per le equazioni differeziali di tipo ellittico.* Ann. Scuola normale super. Pisa. 12 (1958), no. 3, 247–357.

[17] A.I. Markushevich, *Theory of functions of a complex variable. Vol. III.* Translated and edited by Richard A. Silverman. Second English edition. Chelsea Publishing Co., New York, 1977.

[18] V.G. Maz'ya, *The degenerate problem with an oblique derivative.* Matem. Sb. (N.S.) 87(129) (1972), no. 3, 417–454 (in Russion), English translation in.

[19] C. Miranda, *Partial differential equations of elliptic type.* Springer, Berlin, Heidelberg, New York, 1970.

[20] I.I. Privalov, *The boundary properties of analitic functions.* Gostehizdat, Moscow, Leningrad, 1950, in Russion.

[21] M. Riesz, *Sur les functions conjugees.* Math. Zeitshr. 27 (1927), no. 2., 218–244.

[22] F. Riesz, *Über die Randwerte einer analytischen Funcktion.* Math. Zeitshr. 18 (1923), 87–96.

[23] A.P. Soldatov, *Weighted Hardy classes of analytic functions.* Differenth. Uravneniya. 38 (2002), no. 6, 855–864 (in Russian). English translation: Differ. Equat., 38:6 (2002), 855–864

[24] A.P. Soldatov, *The Hardy spaces of solutions to first–order elliptic systems.* Dokl. Akad. nauk, 76 (2007), no. 2, 660–664 (in Russian). English translation: Russian Acad. Sci. Dokl. Math, 2007, vol. 78, No2.

[25] A.P. Soldatov, *Hardy spaces of solutions to second–order elliptic systems.* Dokl. Akad. nauk, 77 (2008), no. 1, 38–41 (in Russian). English translation: Russian Acad. Sci. Dokl. Math. 418 (2008), no. 2, 162-167

[26] E.M. Stein, G. Weiss, *On the theory of harmonic functions of several variables. The theory of H^p –spaces.* Acta Math. 103 (1960), 25–62.

[27] V.I. Vlasov, *Boundary value problems in domains with curvilinear boundary.* D. Sc. thesis, Moscow, Computing Center of the Russ. Acad. Sci, 1990 (in Russian).

[28] V.I. Vlasov, A.V. Rachkov , *On weighted spaces of Hardy type.* Dokl. Akad. Nauk, 328 (1993), no. 3, 281–284 (in Russian). English translation: Russian Acad. Sci. Dokl. Math, 47 (1993), no. 1, 57–61

[29] S. Warschawski, *Über das Ranwerhalten der Ableitung der Abbildungsfunction bei conformer Abbildung.* Math. Zeitshr. 35 (1932), no. 3–4, 321–456.

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