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SOME BOUNDARY VALUE PROBLEMS FOR THE CAUCHY-RIEMANN EQUATION IN HALF LENS

N. Taghizadeh, V. Soltani Mohammadi

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Key words: Schwarz-type operator, Schwarz problem, Dirichlet problem, Neumann problem, half lens.

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Abstract. In this article, the boundary behaviour of the Schwarz-type operator on the half lens Ω will be discussed and the existence of boundary values at corner points is proved. Finally, two basic boundary value problems, namely, Dirichlet and Neumann problems for analytic functions and more generally the Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in Ω is investigated.

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1 Introduction

The solutions of boundary value problems on some special domains have been explicitly obtained. Those special domains include the unit disc [4], [3], the half-plane [5], [9], the quarter plane [1]–[2], the circular ring [12], [11], lenses and lunes [8] and so on. In particular, the Schwarz problem for the half lens domain is considered in [10]. Let $\Omega = \{z \in \Delta | \text{Im} z > 0\}$, where Δ is the lens defined in [8],

$$\Delta = \mathbb{D} \cap D_m(r),$$

where $\mathbb{D} = \{z : |z| < 1\}$, $D_m(r) = \{z : |z-m| < r\}$, $0 < r < 1 < m$, and $r^2 + 1 = m^2$.

In [10], by the reflection at the boundary of the half lens Ω and by using the modified Cauchy-Pompeiu formula on Ω, the following Schwarz-Poisson formula was obtained:

$$\begin{aligned}
\omega(z) &= \tfrac{1}{2\pi i}\int_{m-r}^{1} \text{Re}\,\omega(t)\big[u_t(z) + u_t(\tfrac{1}{z}) + u_t(\tfrac{m-z}{1-mz}) + u_t(\tfrac{1-mz}{m-z})\big]\mathrm{d}t \\
&+ \tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}} \text{Re}\,\omega(\zeta)\big[v_\zeta(z) - v_{\overline{\zeta}}(z) + v_\zeta\left(\tfrac{m-z}{1-mz}\right) - v_{\overline{\zeta}}\left(\tfrac{m-z}{1-mz}\right)\big]\tfrac{\mathrm{d}\zeta}{\zeta} \\
&+ \tfrac{1}{2\pi i}\int_{\partial\Omega_{D_m}} \text{Re}\,\omega(\zeta)\big[w_\zeta(z) - w_{\overline{\zeta}}(z) + w_\zeta\left(\tfrac{m-z}{1-mz}\right) - w_{\overline{\zeta}}\left(\tfrac{m-z}{1-mz}\right)\big]\tfrac{\mathrm{d}\zeta}{\zeta-m} \\
&+ \tfrac{2}{\pi}\int_{\partial\Omega_{\mathbb{D}}} \text{Im}\,\omega(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta} + \tfrac{2}{\pi}\int_{\partial\Omega_{D_m}} \text{Im}\,\omega(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta-m} \\
&- \tfrac{1}{2\pi}\int_{\Omega} \Big\{ \tfrac{\omega_{\overline{\zeta}}(\zeta)}{\zeta}\big[v_\zeta(z) + v_\zeta(\tfrac{1}{z}) + v_\zeta(\tfrac{m-z}{1-mz}) + v_\zeta(\tfrac{1-mz}{m-z})\big] \\
&\qquad - \tfrac{\overline{\omega_{\overline{\zeta}}(\zeta)}}{\overline{\zeta}}\big[v_{\overline{\zeta}}(z) + v_{\overline{\zeta}}(\tfrac{1}{z}) + v_{\overline{\zeta}}(\tfrac{m-z}{1-mz}) + v_{\overline{\zeta}}(\tfrac{1-mz}{m-z})\big]\Big\}\mathrm{d}\xi\mathrm{d}\eta, \qquad (1.1)
\end{aligned}$$

where $\zeta=\xi+i\eta, \partial\Omega_{\mathbb{D}}=\partial\Omega\cap\partial\mathbb{D}$ and $\partial\Omega_{D_m}=\partial\Omega\cap\partial D_m(r), u_t(z)=\frac{2}{t-z}, v_\zeta(z)=\frac{2\zeta}{\zeta-z}, w_\zeta(z)=\frac{2(\zeta-m)}{\zeta-z}$.

Formula (1.1) provides a solution to the Schwarz boundary value problem for the inhomogeneous Cauchy-Riemann equation in Ω (see [10]):

Theorem 1.1. *The Schwarz problem*

$$\begin{gathered}\omega_{\overline{z}}=f, \quad in\ \Omega, \quad \mathrm{Re}\omega=\gamma \quad on \quad \partial\Omega, \quad \gamma(m-r)=\gamma(1)=0,\\ \frac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\mathrm{Im}\omega(\zeta)\frac{d\zeta}{\zeta}+\frac{2}{\pi i}\int_{\partial\Omega_{D_m}}\mathrm{Im}\omega(\zeta)\frac{d\zeta}{\zeta-m}=c,\end{gathered} \tag{1.2}$$

with given $f\in L_p(\Omega,\mathbb{C})$ *,* $p>2$ *,* $\gamma\in C(\partial\Omega,\mathbb{R})$ *,* $c\in\mathbb{R}$ *is uniquely solvable by*

$$\begin{aligned}\omega(z) &= \frac{1}{2\pi i}\int_{m-r}^{1}\gamma(t)\big[u_t(z)+u_t\left(\tfrac{1}{z}\right)+u_t\left(\tfrac{m-z}{1-mz}\right)+u_t\left(\tfrac{1-mz}{m-z}\right)\big]\mathrm{d}t\\ &+\frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\big[v_\zeta(z)-v_{\overline{\zeta}}(z)+v_\zeta\left(\tfrac{m-z}{1-mz}\right)-v_{\overline{\zeta}}\left(\tfrac{m-z}{1-mz}\right)\big]\frac{\mathrm{d}\zeta}{\zeta}\\ &+\frac{1}{2\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\big[w_\zeta(z)-w_{\overline{\zeta}}(z)+w_\zeta\left(\tfrac{m-z}{1-mz}\right)-w_{\overline{\zeta}}\left(\tfrac{m-z}{1-mz}\right)\big]\frac{\mathrm{d}\zeta}{\zeta-m}+ic\\ &-\frac{1}{2\pi}\int_{\Omega}\Big\{\frac{f(\zeta)}{\zeta}\big[v_\zeta(z)+v_\zeta\left(\tfrac{1}{z}\right)+v_\zeta\left(\tfrac{m-z}{1-mz}\right)+v_\zeta\left(\tfrac{1-mz}{m-z}\right)\big]\\ &\qquad-\frac{\overline{f(\zeta)}}{\overline{\zeta}}\big[v_{\overline{\zeta}}(z)+v_{\overline{\zeta}}\left(\tfrac{1}{z}\right)+v_{\overline{\zeta}}\left(\tfrac{m-z}{1-mz}\right)+v_{\overline{\zeta}}\left(\tfrac{1-mz}{m-z}\right)\big]\Big\}\mathrm{d}\xi\mathrm{d}\eta.\end{aligned} \tag{1.3}$$

Next we show that side condition (1.2) is satisfied by the given solution of the Schwarz problem . Let $\omega_3(z)$ denote the area integral

$$\begin{aligned}\omega_3(z) &= -\frac{1}{2\pi}\int_{\Omega}\Big\{\frac{f(\zeta)}{\zeta}\big[v_\zeta(z)+v_\zeta\left(\tfrac{1}{z}\right)+v_\zeta\left(\tfrac{m-z}{1-mz}\right)+v_\zeta\left(\tfrac{1-mz}{m-z}\right)\big]\\ &\qquad-\frac{\overline{f(\zeta)}}{\overline{\zeta}}\big[v_{\overline{\zeta}}(z)+v_{\overline{\zeta}}\left(\tfrac{1}{z}\right)+v_{\overline{\zeta}}\left(\tfrac{m-z}{1-mz}\right)+v_{\overline{\zeta}}\left(\tfrac{1-mz}{m-z}\right)\big]\Big\}\mathrm{d}\xi\mathrm{d}\eta,\end{aligned}$$

and

$$\begin{aligned}\omega_0(z) &= \frac{1}{2\pi i}\int_{m-r}^{1}\gamma(t)\Big[\frac{2}{t-z}+\frac{2z}{zt-1}+\frac{2(1-mz)}{t(1-mz)-(m-z)}+\frac{2(m-z)}{t(m-z)-(1-mz)}\Big]\mathrm{d}t,\\ \omega_1(z) &= \frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\Big[\frac{2\zeta}{\zeta-z}-\frac{2\overline{\zeta}}{\overline{\zeta}-z}+\frac{2\zeta(1-mz)}{\zeta(1-mz)-(m-z)}-\frac{2\overline{\zeta}(1-mz)}{\overline{\zeta}(1-mz)-(m-z)}\Big]\frac{\mathrm{d}\zeta}{\zeta},\\ \omega_2(z) &= \frac{1}{2\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\Big[\frac{2(\zeta-m)}{\zeta-z}-\frac{2(\overline{\zeta}-m)}{\overline{\zeta}-z}+\frac{2(\zeta-m)(1-mz)}{\zeta(1-mz)-(m-z)}-\frac{2(\overline{\zeta}-m)(1-mz)}{\overline{\zeta}(1-mz)-(m-z)}\Big]\frac{\mathrm{d}\zeta}{\zeta-m}.\end{aligned}$$

In order to check side condition (1.2) the relations

$$
\begin{aligned}
\frac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\omega_0(z)\frac{\mathrm{d}z}{z} &= \frac{2}{\pi i}\int_{m-r}^{1}\gamma(t)\Bigg\{\frac{1}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\Big[\frac{1}{t-z}+\frac{(1-mz)}{t(1-mz)-(m-z)}\Big]\frac{\mathrm{d}z}{z} \\
&\quad -\frac{1}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\Big[\frac{1}{t-\overline{z}}+\frac{(1-m\overline{z})}{t(1-m\overline{z})-(m-\overline{z})}\Big]\frac{\mathrm{d}\overline{z}}{\overline{z}}\Bigg\}\mathrm{d}t, \\
\frac{2}{\pi i}\int_{\partial\Omega_{D_m}}\omega_0(z)\frac{\mathrm{d}z}{z-m} &= \frac{2}{\pi i}\int_{m-r}^{1}\gamma(t)\Bigg\{\frac{1}{\pi i}\int_{\partial\Omega_{D_m}}\Big[\frac{1}{t-z}+\frac{(1-mz)}{t(1-mz)-(m-z)}\Big]\frac{\mathrm{d}z}{z-m} \\
&\quad -\frac{1}{\pi i}\int_{\partial\Omega_{D_m}}\Big[\frac{1}{t-\overline{z}}+\frac{(1-m\overline{z})}{t(1-m\overline{z})-(m-\overline{z})}\Big]\frac{\mathrm{d}\overline{z}}{\overline{z}-m}\Bigg\}\mathrm{d}t, \\
\frac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\omega_1(z)\frac{\mathrm{d}z}{z} &= \frac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\Bigg\{\frac{1}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\Big[\frac{\zeta}{\zeta-z}+\frac{\zeta(1-mz)}{\zeta(1-mz)-(m-z)}\Big]\frac{\mathrm{d}z}{z} \\
&\quad +\frac{1}{\pi i}\int_{\overline{\partial\Omega_{\mathbb{D}}}}\Big[\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}+\frac{\overline{\zeta}(1-m\overline{z})}{\overline{\zeta}(1-m\overline{z})-(m-\overline{z})}\Big]\frac{\mathrm{d}\overline{z}}{\overline{z}}\Bigg\}\frac{\mathrm{d}\zeta}{\zeta}, \\
\frac{2}{\pi i}\int_{\partial\Omega_{D_m}}\omega_1(z)\frac{\mathrm{d}z}{z-m} &= \frac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\Bigg\{\frac{1}{\pi i}\int_{\partial\Omega_{D_m}}\Big[\frac{\zeta}{\zeta-z}+\frac{\zeta(1-mz)}{\zeta(1-mz)-(m-z)}\Big]\frac{\mathrm{d}z}{z-m} \\
&\quad +\frac{1}{\pi i}\int_{\overline{\partial\Omega_{D_m}}}\Big[\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}+\frac{\overline{\zeta}(1-m\overline{z})}{\overline{\zeta}(1-m\overline{z})-(m-\overline{z})}\Big]\frac{\mathrm{d}\overline{z}}{\overline{z}-m}\Bigg\}\frac{\mathrm{d}\zeta}{\zeta}, \\
\frac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\omega_2(z)\frac{\mathrm{d}z}{z} &= \frac{2}{\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\Bigg\{\frac{1}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\Big[\frac{\zeta-m}{\zeta-z}+\frac{(\zeta-m)(1-mz)}{\zeta(1-mz)-(m-z)}\Big]\frac{\mathrm{d}z}{z} \\
&\quad +\frac{1}{\pi i}\int_{\overline{\partial\Omega_{\mathbb{D}}}}\Big[\frac{\overline{\zeta}-m}{\overline{\zeta}-\overline{z}}+\frac{(\overline{\zeta}-m)(1-m\overline{z})}{\overline{\zeta}(1-m\overline{z})-(m-\overline{z})}\Big]\frac{\mathrm{d}\overline{z}}{\overline{z}}\Bigg\}\frac{\mathrm{d}\zeta}{\zeta-m}, \\
\frac{2}{\pi i}\int_{\partial\Omega_{D_m}}\omega_2(z)\frac{\mathrm{d}z}{z-m} &= \frac{2}{\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\Bigg\{\frac{1}{\pi i}\int_{\partial\Omega_{D_m}}\Big[\frac{\zeta-m}{\zeta-z}+\frac{(\zeta-m)(1-mz)}{\zeta(1-mz)-(m-z)}\Big]\frac{\mathrm{d}z}{z-m} \\
&\quad +\frac{1}{\pi i}\int_{\overline{\partial\Omega_{D_m}}}\Big[\frac{\overline{\zeta}-m}{\overline{\zeta}-\overline{z}}+\frac{(\overline{\zeta}-m)(1-m\overline{z})}{\overline{\zeta}(1-m\overline{z})-(m-\overline{z})}\Big]\frac{\mathrm{d}\overline{z}}{\overline{z}-m}\Bigg\}\frac{\mathrm{d}\zeta}{\zeta-m}, \\
\frac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\omega_3(z)\frac{\mathrm{d}z}{z} &= -\frac{2}{\pi i}\int_{\Omega}\frac{f(\zeta)}{\zeta}\Bigg\{\frac{1}{\pi}\int_{\partial\Delta\cap\partial\mathbb{D}}\Big[\frac{\zeta}{\zeta-z}+\frac{\zeta(1-mz)}{\zeta(1-mz)-(m-z)}\Big]\frac{\mathrm{d}z}{z}\Bigg\}\mathrm{d}\xi\mathrm{d}\eta \\
&\quad -\frac{2}{\pi i}\int_{\Omega}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Bigg\{\frac{1}{\pi}\int_{\partial\Delta\cap\partial\mathbb{D}}\Big[\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}+\frac{\overline{\zeta}(1-m\overline{z})}{\overline{\zeta}(1-m\overline{z})-(m-\overline{z})}\Big]\frac{\mathrm{d}\overline{z}}{\overline{z}}\Bigg\}\mathrm{d}\xi\mathrm{d}\eta, \\
\frac{2}{\pi i}\int_{\partial\Omega_{D_m}}\omega_3(z)\frac{\mathrm{d}z}{z-m} &= -\frac{2}{\pi i}\int_{\Omega}\frac{f(\zeta)}{\zeta}\Bigg\{\frac{1}{\pi}\int_{\partial\Delta\cap\partial D_m}\Big[\frac{\zeta}{\zeta-z}+\frac{\zeta(1-mz)}{\zeta(1-mz)-(m-z)}\Big]\frac{\mathrm{d}z}{z-m}\Bigg\}\mathrm{d}\xi\mathrm{d}\eta \\
&\quad -\frac{2}{\pi i}\int_{\Omega}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Bigg\{\frac{1}{\pi}\int_{\partial\Delta\cap\partial D_m}\Big[\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}+\frac{\overline{\zeta}(1-m\overline{z})}{\overline{\zeta}(1-m\overline{z})-(m-\overline{z})}\Big]\frac{\mathrm{d}\overline{z}}{\overline{z}-m}\Bigg\}\mathrm{d}\xi\mathrm{d}\eta, \\
&\quad \frac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}} ic\frac{\mathrm{d}z}{z}+\frac{2}{\pi i}\int_{\partial\Omega_{D_m}} ic\frac{\mathrm{d}z}{z-m}=ci,
\end{aligned}
$$

are to be used. Hence,

$$\frac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}} Im\omega(z)\frac{\mathrm{d}z}{z}+\frac{2}{\pi i}\int_{\partial\Omega_{D_m}} Im\omega(z)\frac{\mathrm{d}z}{z-m}=c.$$

In Theorem 1.1, the given boundary data γ is required to satisfy the condition at the corner points

$$\gamma(m-r)=\gamma(1)=0. \tag{1.4}$$

In this article, with the aid of the boundary behaviour of the Schwarz operator obtained in [10] under condition (1.4), the boundary behaviour of the Schwarz operator on the half lens Ω is investigated and the existence of boundary values at the corner points $m-r,1$ have been verified and hereby Theorem 1.1 remains true if condition (1.4) at the corner points is not satisfied. Further the Dirichlet and Neumann boundary value problems are investigated.

2 Boundary behaviour of the Schwarz operator

Similarly to [13], we introduce the Schwarz integral on the half lens Ω as follows

$$\begin{aligned} S[\gamma](z) &= \frac{1}{2\pi i}\int_{m-r}^{1}\gamma(t)[u_t(z)+u_t(\tfrac{1}{z})+u_t(\tfrac{m-z}{1-mz})+u_t(\tfrac{1-mz}{m-z})]\mathrm{d}t \\ &+\frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)[v_\zeta(z)-v_{\overline{\zeta}}(z)+v_\zeta(\tfrac{m-z}{1-mz})-v_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\frac{\mathrm{d}\zeta}{\zeta} \\ &+\frac{1}{2\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)[w_\zeta(z)-w_{\overline{\zeta}}(z)+w_\zeta(\tfrac{m-z}{1-mz})-w_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\frac{\mathrm{d}\zeta}{\zeta-m}, \end{aligned}$$

where $\gamma\in C(\partial\Omega;\mathbb{R})$. By a direct computation,

$$\frac{\partial S[\gamma](z)}{\partial\overline{z}}=0,\quad z\in\Omega,$$

so $S[\gamma](z)$ is analytic in Ω and as in the proof of Theorem 2.2 in [10], we show that under condition $\gamma(1)=\gamma(m-r)=0$,

$$ReS[\gamma](\zeta)=\gamma(\zeta),\quad \zeta\in\partial\Omega. \tag{2.1}$$

Lemma 2.1. *If* $\gamma(\zeta)\equiv 1$ *for* $\zeta\in\partial\Omega$, *then* $S[\gamma](z)\equiv 1, z\in\Omega$.

Proof. A simple computation gives

$$\begin{aligned} S[1](z) &= \frac{1}{2\pi i}\int_{m-r}^{1}[u_t(z)+u_t(\tfrac{1}{z})+u_t(\tfrac{m-z}{1-mz})+u_t(\tfrac{1-mz}{m-z})]\mathrm{d}t \\ &+\frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}[v_\zeta(z)-v_{\overline{\zeta}}(z)+v_\zeta(\tfrac{m-z}{1-mz})-v_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\frac{\mathrm{d}\zeta}{\zeta} \\ &+\frac{1}{2\pi i}\int_{\partial\Omega_{D_m}}[w_\zeta(z)-w_{\overline{\zeta}}(z)+w_\zeta(\tfrac{m-z}{1-mz})-w_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\frac{\mathrm{d}\zeta}{\zeta-m} \\ &= \frac{1}{2\pi i}\int_{m-r}^{1}[u_t(z)+u_t(\tfrac{1}{z})+u_t(\tfrac{m-z}{1-mz})+u_t(\tfrac{1-mz}{m-z})]\mathrm{d}t \\ &+\frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})]\mathrm{d}\zeta \\ &+\frac{1}{2\pi i}\int_{\partial\Omega_{D_m}}[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})]\mathrm{d}\zeta \\ &-\frac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\frac{\mathrm{d}\zeta}{\zeta}-\frac{2}{\pi i}\int_{\partial\Omega_{D_m}}\frac{\mathrm{d}\zeta}{\zeta-m}. \end{aligned}$$

So by the Residue Theorem, $S[1](z)=1$. □

Lemma 2.2. *If $\gamma(\zeta)\equiv\zeta$ for $\zeta\in\partial\Omega$, then $S[\gamma](z)\equiv 2z+\frac{2}{\pi i}(1-m+r)-m+\frac{2m}{\pi}\alpha, z\in\Omega$, where α is the argument of $\frac{1}{m}+i\frac{r}{m}$.*

Proof. Similarly to the proof of Lemma 2.1

$$\begin{aligned} S[\zeta](z) &= \tfrac{1}{2\pi i}\int_{m-r}^{1}[v_t(z)+v_t(\tfrac{1}{z})+v_t(\tfrac{m-z}{1-mz})+v_t(\tfrac{1-mz}{m-z})]\mathrm{d}t\\ &\quad+\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}[v_\zeta(z)-v_{\overline{\zeta}}(z)+v_\zeta(\tfrac{m-z}{1-mz})-v_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\mathrm{d}\zeta\\ &\quad+\tfrac{1}{2\pi i}\int_{\partial\Omega_{D_m}}[v_\zeta(z)-w_{\overline{\zeta}}(z)\tfrac{\zeta}{\zeta-m}+v_\zeta(\tfrac{m-z}{1-mz})-w_{\overline{\zeta}}(\tfrac{m-z}{1-mz})\tfrac{\zeta}{\zeta-m}]\mathrm{d}\zeta\\ &= \tfrac{1}{2\pi i}\int_{m-r}^{1}[v_t(z)+\tfrac{2}{zt-1}+v_t(\tfrac{m-z}{1-mz})+\tfrac{2(1-mz)}{t(m-z)-(1-mz)}]\mathrm{d}t\\ &\quad+\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}[v_\zeta(z)+\tfrac{2}{z\zeta-1}+v_\zeta(\tfrac{m-z}{1-mz})+\tfrac{2(1-mz)}{\zeta(m-z)-(1-mz)}]\mathrm{d}\zeta\\ &\quad+\tfrac{1}{2\pi i}\int_{\partial\Omega_{D_m}}[v_\zeta(z)+\tfrac{2}{z\zeta-1}+v_\zeta(\tfrac{m-z}{1-mz})+\tfrac{2(1-mz)}{\zeta(m-z)-(1-mz)}]\mathrm{d}\zeta\\ &\quad+\tfrac{2}{\pi i}\int_{m-r}^{1}\mathrm{d}t-\tfrac{2m}{\pi i}\int_{\partial\Omega_{D_m}}\tfrac{\mathrm{d}\zeta}{\zeta-m}=2z+\tfrac{2}{\pi i}(1-m+r)+\tfrac{2m}{\pi}\alpha-m, \end{aligned}$$

by the Residue Theorem. □

Lemma 2.3. *If $\gamma(\zeta)\equiv\overline{\zeta}$ for $\zeta\in\partial\Omega$, then $S[\gamma](z)\equiv\frac{2}{\pi i}(1-m+r)+m-\frac{2m}{\pi}\alpha, z\in\Omega$.*

Proof. First,

$$\begin{aligned} S[\overline{\zeta}](z) &= \tfrac{1}{2\pi i}\int_{m-r}^{1}[v_t(z)+v_t(\tfrac{1}{z})+v_t(\tfrac{m-z}{1-mz})+v_t(\tfrac{1-mz}{m-z})]\mathrm{d}t\\ &\quad+\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\overline{\zeta}[v_\zeta(z)-v_{\overline{\zeta}}(z)+v_\zeta(\tfrac{m-z}{1-mz})-v_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\tfrac{\mathrm{d}\zeta}{\zeta}\\ &\quad+\tfrac{1}{2\pi i}\int_{\partial\Omega_{D_m}}\overline{\zeta}[w_\zeta(z)-w_{\overline{\zeta}}(z)+w_\zeta(\tfrac{m-z}{1-mz})-w_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\tfrac{\mathrm{d}\zeta}{\zeta-m}. \end{aligned} \tag{2.2}$$

By a straightforward computation one has

$$\begin{aligned} &\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\overline{\zeta}[v_\zeta(z)-v_{\overline{\zeta}}(z)+v_\zeta(\tfrac{m-z}{1-mz})-v_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\tfrac{\mathrm{d}\zeta}{\zeta}\\ &= -\tfrac{1}{2\pi i}\int_{\overline{\partial\Omega_{\mathbb{D}}}}\zeta[v_{\overline{\zeta}}(z)-v_\zeta(z)+v_{\overline{\zeta}}(\tfrac{m-z}{1-mz})-v_\zeta(\tfrac{m-z}{1-mz})]\tfrac{\mathrm{d}\overline{\zeta}}{\overline{\zeta}}\\ &= -\tfrac{1}{2\pi i}\int_{\overline{\partial\Omega_{\mathbb{D}}}}\zeta[v_\zeta(z)-v_{\overline{\zeta}}(z)+v_\zeta(\tfrac{m-z}{1-mz})-v_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\tfrac{\mathrm{d}\zeta}{\zeta}, \end{aligned} \tag{2.3}$$

and similarly

$$\begin{aligned} &\tfrac{1}{2\pi i}\int_{\partial\Omega_{D_m}}\overline{\zeta}[w_\zeta(z)-w_{\overline{\zeta}}(z)+w_\zeta(\tfrac{m-z}{1-mz})-w_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\tfrac{\mathrm{d}\zeta}{\zeta-m}\\ &= -\tfrac{1}{2\pi i}\int_{\overline{\partial\Omega_{D_m}}}\zeta[w_\zeta(z)-w_{\overline{\zeta}}(z)+w_\zeta(\tfrac{m-z}{1-mz})-w_{\overline{\zeta}}(\tfrac{m-z}{1-mz})]\tfrac{\mathrm{d}\zeta}{\zeta-m}. \end{aligned} \tag{2.4}$$

By substituting (2.3) and (2.4) in (2.2) one gets

$$
\begin{aligned}
S[\overline{\zeta}](z) &= \tfrac{1}{2\pi i}\int_{m-r}^{1}[v_t(z)+\tfrac{2}{zt-1}+v_t(\tfrac{m-z}{1-mz})+\tfrac{2(1-mz)}{t(m-z)-(1-mz)}]\mathrm{d}t \\
&\quad -\tfrac{1}{2\pi i}\int_{\overline{\partial\Omega_{\mathbb{D}}}}[v_\zeta(z)+\tfrac{2}{z\zeta-1}+v_\zeta(\tfrac{m-z}{1-mz})+\tfrac{2(1-mz)}{\zeta(m-z)-(1-mz)}]\mathrm{d}\zeta \\
&\quad -\tfrac{1}{2\pi i}\int_{\overline{\partial\Omega_{D_m}}}[v_\zeta(z)+\tfrac{2}{z\zeta-1}+v_\zeta(\tfrac{m-z}{1-mz})+\tfrac{2(1-mz)}{\zeta(m-z)-(1-mz)}]\mathrm{d}\zeta \\
&\quad +\tfrac{2}{\pi i}\int_{m-r}^{1}\mathrm{d}t+\tfrac{2m}{\pi i}\int_{\overline{\partial\Omega_{D_m}}}\tfrac{\mathrm{d}\zeta}{\zeta-m},
\end{aligned}
$$

so by the Residue Theorem,

$$
S[\overline{\zeta}](z)=\tfrac{2}{\pi i}(1-m+r)+m-\tfrac{2m}{\pi}\alpha.
$$

□

Now, we construct the following function:

$$
\begin{aligned}
L_\gamma(z) &= \tfrac{\gamma(1)\left[\mathrm{Re}(z)-(m-r)\right]-\gamma(m-r)\left[\mathrm{Re}(z)-1\right]}{1-(m-r)} \\
&= \tfrac{\gamma(1)-\gamma(m-r)}{1-(m-r)}\mathrm{Re}(z)+\tfrac{\gamma(m-r)-(m-r)\gamma(1)}{1-(m-r)}.
\end{aligned} \tag{2.5}
$$

Obviously, $L_\gamma(1)=\gamma(1), L_\gamma(m-r)=\gamma(m-r)$.

Lemma 2.4. *If* $\gamma\in C(\partial\Omega;\mathbb{C})$, *then*

$$
S[L_\gamma](z)=\tfrac{\gamma(1)-\gamma(m-r)}{1-(m-r)}z+\tfrac{\gamma(m-r)-(m-r)\gamma(1)}{1-(m-r)}-\tfrac{2i}{\pi}\left(\gamma(1)-\gamma(m-r)\right). \tag{2.6}
$$

Proof. The lemma follows by Lemmas 2.1–2.3 and by the obvious equality

$$
\mathrm{Re}(z)=\tfrac{z+\overline{z}}{2}.
$$

□

Now, in the next theorem we show that Theorem 1.1 remains true if condition (1.4) at the corner points is not satisfied.

Theorem 2.1. *If* $\gamma\in C(\partial\Omega;\mathbb{R})$ *then*

$$
\mathrm{Re}\{S[\gamma]\}(\zeta)=\gamma(\zeta),\quad \zeta\in\partial\Omega, \tag{2.7}
$$

in particular,

$$
\mathrm{Re}\{S[\gamma]\}(1)=\gamma(1),\quad \mathrm{Re}\{S[\gamma]\}(m-r)=\gamma(m-r).
$$

Proof. Since γ is a real-valued function, by (2.5) and (2.6) we obtain that

$$
\begin{aligned}
\mathrm{Re}\{S[\gamma](z)\} &= \mathrm{Re}\{S[\gamma-L_\gamma](z)+S[L_\gamma](z)\} \\
&= \mathrm{Re}\{S[\gamma-L_\gamma](z)\}+L_\gamma(z),\quad z\in\Omega,
\end{aligned} \tag{2.8}
$$

and $L_\gamma(1)=\gamma(1), L_\gamma(m-r)=\gamma(m-r)$, so $\gamma-L_\gamma$ satisfies the conditions on γ defined in Theorem 2.2 in [10]. Hence by (2.1) we have

$$
Re\{S[\gamma-L_\gamma](\zeta)\}=\gamma(\zeta)-L_\gamma(\zeta),\quad \zeta\in\partial\Omega. \tag{2.9}
$$

Thus, (2.8) and (2.9) lead to the desired conclusion (2.7). □

3 The Dirichlet boundary value problem for the Cauchy-Riemann equation

In this section, we first consider the Dirichlet boundary value problem for the homogeneous Cauchy-Riemann equation. To solve boundary value problems for analytic functions in Ω the following representation formula is important.

Theorem 3.1. *Any* $\omega \in C^1(\Omega;\mathbb{C}) \cap C(\Omega;\mathbb{C})$ *can be represented as*

$$\begin{aligned}\omega(z) &= \tfrac{1}{4\pi i}\int_{\partial\Omega}\omega(\zeta)\big[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})\big]\mathrm{d}\zeta \\ &\quad -\tfrac{1}{2\pi}\int_{\Omega}\omega_{\overline{\zeta}}(\zeta)\big[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})\big]\mathrm{d}\xi\mathrm{d}\eta. \qquad (3.1)\end{aligned}$$

Proof. By the Cauchy-Pompeiu representations for $z \in \Omega$ and $\frac{1}{z}$, $\frac{m-z}{1-mz}$, $\frac{1-mz}{m-z} \notin \Omega$ one has,

$$\omega(z) = \tfrac{1}{2\pi i}\int_{\partial\Omega}\tfrac{\omega(\zeta)}{\zeta-z}\mathrm{d}\zeta - \tfrac{1}{\pi}\int_{\Omega}\omega_{\overline{\zeta}}(\zeta)\tfrac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta-z},$$

$$0 = \tfrac{1}{2\pi i}\int_{\partial\Omega}\tfrac{\omega(\zeta)z}{z\zeta-1}\mathrm{d}\zeta - \tfrac{1}{\pi}\int_{\Omega}\omega_{\overline{\zeta}}(\zeta)\tfrac{z\mathrm{d}\xi\mathrm{d}\eta}{z\zeta-1},$$

$$0 = \tfrac{1}{2\pi i}\int_{\partial\Omega}\omega(\zeta)\tfrac{1-mz}{\zeta(1-mz)-(m-z)}\mathrm{d}\zeta - \tfrac{1}{\pi}\int_{\Omega}\omega_{\overline{\zeta}}(\zeta)\tfrac{1-mz}{\zeta(1-mz)-(m-z)}\mathrm{d}\xi\mathrm{d}\eta,$$

$$0 = \tfrac{1}{2\pi i}\int_{\partial\Omega}\omega(\zeta)\tfrac{m-z}{\zeta(m-z)-(1-mz)}\mathrm{d}\zeta - \tfrac{1}{\pi}\int_{\Omega}\omega_{\overline{\zeta}}(\zeta)\tfrac{m-z}{\zeta(m-z)-(1-mz)}\mathrm{d}\xi\mathrm{d}\eta.$$

Adding the resulting above four relations, leads to the claimed representation formula. □

Formula (3.1) provides a solution to the Dirichlet boundary value problem for the homogeneous Cauchy-Riemann equation in Ω.

Theorem 3.2. *The Dirichlet problem*

$$\begin{aligned}&\omega_{\overline{z}} = 0, \quad \text{in} \quad \Omega,\\ &\omega = \gamma, \quad \text{on} \quad \partial\Omega, \gamma \in C(\partial\Omega;\mathbb{C}),\end{aligned} \qquad (3.2)$$

is solvable, if and only if,

$$\tfrac{1}{4\pi i}\int_{\partial\Omega}\gamma(\zeta)\big[u_\zeta(\overline{z})+u_\zeta(\tfrac{1}{\overline{z}})+u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\mathrm{d}\zeta = 0, \qquad (3.3)$$

and the unique solution can be represented as

$$\omega(z) = \tfrac{1}{4\pi i}\int_{\partial\Omega}\gamma(\zeta)\big[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})\big]\mathrm{d}\zeta, z \in \Omega. \qquad (3.4)$$

Proof. Let ω defined by (3.4) be a solution to the Dirichlet problem. This formula can be decomposed into the sum of Cauchy type integrals, which implies that

$$\omega = \gamma, \quad \text{on } \partial\Omega. \qquad (3.5)$$

We consider the following function

$$h(z) = \tfrac{1}{4\pi i}\int_{\partial\Omega}\gamma(\zeta)\big[u_\zeta(\overline{z})+u_\zeta(\tfrac{1}{\overline{z}})+u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\mathrm{d}\zeta. \qquad (3.6)$$

Obviously,

$$\begin{aligned}
\omega(z)-h(z) &= \tfrac{1}{4\pi i}\int_{m-r}^{1}\gamma(t)\big[u_t(z)-u_t(\overline{z})+u_t(\tfrac{1-mz}{m-z})-u_t(\tfrac{1}{\overline{z}})+u_t(\tfrac{m-z}{1-mz})\\
&\quad -u_t(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_t(\tfrac{1}{z})-u_t(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\mathrm{d}t\\
&\quad +\tfrac{1}{4\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\big[v_\zeta(z)+v_{\overline{\zeta}}(\overline{z})-v_{\overline{\zeta}}(z)-v_\zeta(\overline{z})+v_\zeta(\tfrac{m-z}{1-mz})\\
&\quad +v_{\overline{\zeta}}(\tfrac{m-\overline{z}}{1-m\overline{z}})-v_{\overline{\zeta}}(\tfrac{m-z}{1-mz})-v_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})\big]\tfrac{\mathrm{d}\zeta}{\zeta}\\
&\quad +\tfrac{1}{4\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\big[w_\zeta(z)+w_{\overline{\zeta}}(\overline{z})-w_\zeta(\overline{z})-w_{\overline{\zeta}}(z)+w_\zeta(\tfrac{m-z}{1-mz})\\
&\quad +w_{\overline{\zeta}}(\tfrac{m-\overline{z}}{1-m\overline{z}})-w_{\overline{\zeta}}(\tfrac{m-z}{1-mz})-w_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})\big]\tfrac{\mathrm{d}\zeta}{\zeta-m}\\
&= \tfrac{1}{2\pi i}\int_{m-r}^{1}\gamma(t)\big[\tfrac{z-\overline{z}}{|t-z|^2}-\tfrac{z-\overline{z}}{|zt-1|^2}+\tfrac{r^2(z-\overline{z})}{|t(1-mz)-(m-z)|^2}\\
&\quad -\tfrac{r^2(z-\overline{z})}{|t(m-z)-(1-mz)|^2}\big]dt+\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\big[\tfrac{1-|z|^2}{|\zeta-z|^2}\\
&\quad -\tfrac{1-|z|^2}{|\overline{\zeta}-z|^2}-\tfrac{r^2(1-|z|^2)}{|\zeta(1-mz)-(m-z)|^2}\\
&\quad +\tfrac{r^2(1-|z|^2)}{|\overline{\zeta}(1-mz)-(m-z)|^2}\big]\tfrac{d\zeta}{\zeta}+\tfrac{1}{2\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\big[\tfrac{r^2-|z-m|^2}{|\zeta-z|^2}\\
&\quad -\tfrac{r^2-|z-m|^2}{|\overline{\zeta}-z|^2}-\tfrac{r^2-|z-m|^2}{|1-z\overline{\zeta}|^2}+\tfrac{r^2-|z-m|^2}{|1-z\zeta|^2}\big]\tfrac{d\zeta}{\zeta-m},
\end{aligned}\tag{3.7}$$

so by Theorem 2.2 in [10]

$$\lim_{z\to\zeta}\big[\omega(z)-h(z)\big]=\gamma(\zeta),\qquad \zeta\in\partial\Omega.\tag{3.8}$$

By (3.5) and (3.8), one has $\lim_{z\to\zeta,\zeta\in\partial\Omega}h(z)=0$. Since $\overline{h(z)}$ is analytic for $z\in\Omega$, by the maximum principle for analytic functions, $h(z)\equiv 0$ for $z\in\Omega$, which is just condition (3.3). Now, we verify that condition (3.3) is sufficient. If condition (3.3) is satisfied, then,

$$\begin{aligned}
\omega(z) &= \tfrac{1}{4\pi i}\int_{\partial\Omega}\gamma(\zeta)\big[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})\\
&\quad +u_\zeta(\overline{z})+u_\zeta(\tfrac{1}{\overline{z}})+u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\mathrm{d}\zeta,
\end{aligned}$$

is equal to (3.4). Hence $\omega_{\overline{z}}=0,\ z\in\Omega$ and by Theorem 2.2 in [10], $\lim_{z\to\zeta}\omega(z)=\gamma(\zeta),\ \zeta\in\partial\Omega$. □

In the next step, the inhomogeneous Dirichlet problem for the inhomogeneous Cauchy-Riemann equation is solved in Ω. By using the definition and properties of the Pompeiu operator, the inhomogeneous problem is reduced to the homogeneous case.

Theorem 3.3. *The Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in* Ω,

$$\begin{aligned}
&\omega_{\overline{z}}=f(z),\quad z\in\Omega, f\in L_p(\Omega;\mathbb{C}), p>2,\\
&\omega=\gamma,\ \ on\ \ \partial\Omega,\gamma\in C(\partial\Omega;\mathbb{C}),
\end{aligned}\tag{3.9}$$

is solvable, if and only if, for $z\in\Omega$,

$$\begin{aligned}
&\tfrac{1}{4\pi i}\int_{\partial\Omega}\gamma(\zeta)\big[u_\zeta(\overline{z})+u_\zeta(\tfrac{1}{\overline{z}})+u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\mathrm{d}\zeta\\
&=\tfrac{1}{2\pi}\int_{\Omega}f(\zeta)\big[u_\zeta(\overline{z})+u_\zeta(\tfrac{1}{\overline{z}})+u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]d\xi d\eta,
\end{aligned}\tag{3.10}$$

and its solution can be uniquely expressed as

$$\begin{aligned}\omega(z) &= \frac{1}{4\pi i}\int_{\partial\Omega}\omega(\zeta)\big[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})\big]\mathrm{d}\zeta\\ &-\frac{1}{2\pi}\int_{\Omega}f(\zeta)\big[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})\big]d\xi d\eta,\ z\in\Omega.\end{aligned}\tag{3.11}$$

Proof. By Theorem 3.1 if the Dirichlet problem (3.9) is solvable, then its solution can be represented as (3.11).

Introducing the new unknown function $\varphi=\omega-Tf$, then we arrive at the following boundary value problem

$$\varphi_{\overline{z}}=0,\quad in\ \ \Omega,\quad \varphi=\gamma-Tf,\quad on\ \ \partial\Omega,\tag{3.12}$$

equivalent to equation (3.9). By Theorem 3.2 the solvability condition for equation (3.12) is

$$\frac{1}{4\pi i}\int_{\partial\Omega}(\gamma(\zeta)-T[f](\zeta))\left[u_\zeta(\overline{z})+u_\zeta(\tfrac{1}{\overline{z}})+u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\right]\mathrm{d}\zeta=0,\tag{3.13}$$

and

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\partial\Omega}T[f](\zeta)\big[u_\zeta(\overline{z})+u_\zeta(\tfrac{1}{\overline{z}})+u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\mathrm{d}\zeta\\ &=\frac{1}{\pi}\int_{\Omega}f(\tilde\zeta)\frac{1}{2\pi i}\int_{\partial\Omega}\big[u_\zeta(\overline{z})+u_\zeta(\tfrac{1}{\overline{z}})+u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\frac{\mathrm{d}\zeta}{\zeta-\tilde\zeta}\mathrm{d}\tilde\xi\mathrm{d}\tilde\eta,\end{aligned}$$

which is just condition (3.10) by direct computation.

Conversely, if condition of solvability (3.10) is satisfied, (3.11) can be rewritten as,

$$\begin{aligned}\omega(z) &= \frac{1}{4\pi i}\int_{\partial\Omega}\gamma(\zeta)\big[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})\\ &\quad u_\zeta(\overline{z})+u_\zeta(\tfrac{1}{\overline{z}})+u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\mathrm{d}\zeta\\ &\quad-\frac{1}{2\pi}\int_{\Omega}f(\zeta)\big[u_\zeta(z)-u_\zeta(\overline{z})+u_\zeta(\tfrac{1}{z})-u_\zeta(\tfrac{1}{\overline{z}})+u_\zeta(\tfrac{m-z}{1-mz})\\ &\quad-u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})+u_\zeta(\tfrac{1-mz}{m-z})-u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\mathrm{d}\xi\mathrm{d}\eta.\end{aligned}\tag{3.14}$$

Since the area integral tends to 0 as $z\to\zeta\in\partial\Omega$, by Theorem 2.2 in [10], (3.14) implies that $\lim_{z\to\zeta}\omega(z)=\gamma(\zeta),\zeta\in\Omega$ and obviously $\omega_{\overline{z}}=f(z),z\in\Omega$. The uniqueness of the solution follows from the fact that the corresponding homogeneous problem

$$\begin{aligned}&\omega_{\overline{z}}=0,\quad z\in\Omega,\\ &\omega=0,\ \ on\ \ \partial\Omega,\end{aligned}$$

has only the trivial solution. □

4 The Neumann boundary value problem for the Cauchy-Riemann equation

To formulate the Neumann boundary value problem, we need to define the outward normal derivative at the boundary of Ω. The normal derivative on the boundary of Ω is given by the formulas

$$\partial_{\nu_z}=\begin{cases}-i(\partial z-\partial\overline{z}), & z\in(m-r,1),\\ z\partial_z+\overline{z}\partial_{\overline{z}}, & z\in\partial\Omega_{\mathbb{D}}\setminus\{1,\frac{1}{m}+i\frac{r}{m}\},\\ (\frac{z-m}{r})\partial z+(\frac{\overline{z}-m}{r})\partial\overline{z}, & z\in\partial\Omega_{D_m}\setminus\{m-r,\frac{1}{m}+i\frac{r}{m}\},\end{cases}$$

Theorem 4.1. *The Neumann problem*

$$\begin{aligned} &\omega_{\overline{z}} = 0, \quad z \in \Omega, \\ &\partial_{\nu_z}\omega = \gamma \ \textit{on} \ \partial\Omega \setminus \{m-r, 1, \tfrac{1}{m} + i\tfrac{r}{m}\}, \gamma \in C(\partial\Omega; \mathbb{C}), \omega(1) = 0, \end{aligned} \tag{4.1}$$

where

$$\partial_{\nu_z} = \begin{cases} -i\omega_z & z \in (m-r, 1), \\ z\omega_z & z \in \partial\Omega_{\mathbb{D}} \setminus \{1, \frac{1}{m} + i\frac{r}{m}\}, \\ (\frac{z-m}{r})\omega_z & z \in \partial\Omega_{D_m} \setminus \{m-r, \frac{1}{m} + i\frac{r}{m}\}, \end{cases}$$

is solvable, if and only if, for $z \in \Omega$ *,*

$$\begin{aligned} &\tfrac{1}{4\pi} \int_{m-r}^{1} \gamma(t)\big[u_t(\overline{z}) + u_t(\tfrac{1}{\overline{z}}) + u_t(\tfrac{m-\overline{z}}{1-m\overline{z}}) + u_t(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\mathrm{d}t \\ &+\tfrac{1}{4\pi i} \int_{\partial\Omega_{\mathbb{D}}} \gamma(\zeta)\big[u_\zeta(\overline{z}) + u_\zeta(\tfrac{1}{\overline{z}}) + u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}}) + u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\tfrac{\mathrm{d}\zeta}{\zeta} \\ &+\tfrac{r}{4\pi i} \int_{\partial\Omega_{D_m}} \gamma(\zeta)\big[u_\zeta(\overline{z}) + u_\zeta(\tfrac{1}{\overline{z}}) + u_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}}) + u_\zeta(\tfrac{1-m\overline{z}}{m-\overline{z}})\big]\tfrac{\mathrm{d}\zeta}{\zeta-m} = 0, \end{aligned}$$

and its solution is

$$\begin{aligned} \omega(z) \quad = \quad & \tfrac{1}{2\pi} \int_{m-r}^{1} \gamma(t)\Big[(z-1) + \tfrac{(z-1)t}{t-m} + \tfrac{m(z-1)t}{mt-1} - t\log(\tfrac{t-z}{t-1}) \\ & +\tfrac{1}{t}\log(\tfrac{zt-1}{t-1}) - \tfrac{r^2 t}{(1-mt)^2}\log\left(\tfrac{(mt-1)z-(t-m)}{(mt-1)-(t-m)}\right) + \tfrac{r^2 t}{(t-m)^2} \\ & \times\log\left(\tfrac{z(t-m)-(mt-1)}{(t-m)-(mt-1)}\right)\Big]\tfrac{dt}{t} + \tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}} \gamma(\zeta)\Big[\tfrac{(z-1)}{\zeta} + \tfrac{(z-1)}{\zeta-m} \\ & -\tfrac{m(z-1)}{1-m\zeta} - \log(\tfrac{\zeta-z}{\zeta-1}) + \tfrac{1}{\zeta^2}\log(\tfrac{z\zeta-1}{\zeta-1}) \\ & -\tfrac{r^2}{(1-m\zeta)^2}\log\left(\tfrac{z(1-m\zeta)-(m-\zeta)}{(1-m\zeta)-(m-\zeta)}\right) - \tfrac{r^2}{(\zeta-m)^2}\log\left(\tfrac{z(\zeta-m)-(m\zeta-1)}{(\zeta-m)-(m\zeta-1)}\right)\Big]\tfrac{\mathrm{d}\zeta}{\zeta} \\ & +\tfrac{r}{2\pi i}\int_{\partial\Omega_{D_m}} \gamma(\zeta)\Big[\tfrac{(z-1)}{\zeta} + \tfrac{(z-1)}{\zeta-m} - \tfrac{m(z-1)}{1-m\zeta} \\ & -\log(\tfrac{\zeta-z}{\zeta-1}) + \tfrac{1}{\zeta^2}\log(\tfrac{z\zeta-1}{\zeta-1}) - \tfrac{r^2}{(1-m\zeta)^2}\log\left(\tfrac{z(1-m\zeta)-(m-\zeta)}{(1-m\zeta)-(m-\zeta)}\right) \\ & -\tfrac{r^2}{(\zeta-m)^2}\log\left(\tfrac{z(\zeta-m)-(m\zeta-1)}{(\zeta-m)-(m\zeta-1)}\right)\Big]\tfrac{\mathrm{d}\zeta}{\zeta-m}. \end{aligned} \tag{4.2}$$

Proof. If ω is a solution to the Neumann problem, then $\varphi = \omega_z$ is a solution to the Dirichlet problem

$$\varphi_{\overline{z}} = 0, \quad \textit{in} \ \Omega, \quad \varphi = \omega_z \quad \textit{on} \ \partial\Omega, \tag{4.3}$$

where ω_z on $\partial\Omega$ is represented by,

$$\omega_z(z) = \begin{cases} i\gamma & z \in (m-r, 1), \\ \overline{z}\gamma & z \in \partial\Omega_{\mathbb{D}} \setminus \{1, \frac{1}{m} + i\frac{r}{m}\}, \\ (\frac{\overline{z}-m}{r})\gamma & z \in \partial\Omega_{D_m} \setminus \{m-r, \frac{1}{m} + i\frac{r}{m}\}. \end{cases}$$

The solution of the Dirichlet problem is

$$\begin{aligned}\omega_z(z) &= \tfrac{1}{4\pi}\int_{m-r}^{1}\gamma(t)\Big[u_t(z)+u_t(\tfrac{1}{z})+u_t(\tfrac{m-z}{1-mz})+u_t(\tfrac{1-mz}{m-z})\Big]\mathrm{d}t\\ &+\tfrac{1}{4\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\Big[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})\Big]\tfrac{\mathrm{d}\zeta}{\zeta}\\ &+\tfrac{r}{4\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\Big[u_\zeta(z)+u_\zeta(\tfrac{1}{z})+u_\zeta(\tfrac{m-z}{1-mz})+u_\zeta(\tfrac{1-mz}{m-z})\Big]\tfrac{\mathrm{d}\zeta}{\zeta-m}.\end{aligned}\tag{4.4}$$

The primitive of the function in (4.4) is

$$\begin{aligned}\omega(z) &= \tfrac{1}{2\pi}\int_{m-r}^{1}\gamma(t)\Big[z+\tfrac{zt}{t-m}+\tfrac{mzt}{mt-1}-t\log(t-z)+\tfrac{1}{t}\log(zt-1)\\ &-\tfrac{r^2t}{(1-mt)^2}\log\Big((mt-1)z-(t-m)\Big)+\tfrac{r^2t}{(t-m)^2}\\ &\times\log\Big(z(t-m)-(mt-1)\Big)\Big]\tfrac{dt}{t}+\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\Big[\tfrac{z}{\zeta}+\tfrac{z}{\zeta-m}-\tfrac{mz}{1-m\zeta}\\ &-\log(\zeta-z)+\tfrac{1}{\zeta^2}\log(z\zeta-1)-\tfrac{r^2}{(1-m\zeta)^2}\log\Big(z(1-m\zeta)-(m-\zeta)\Big)\\ &-\tfrac{r^2}{(\zeta-m)^2}\log\Big(z(\zeta-m)-(m\zeta-1)\Big)\Big]\tfrac{d\zeta}{\zeta}+\tfrac{r}{2\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\Big[\tfrac{z}{\zeta}+\tfrac{z}{\zeta-m}\\ &-\tfrac{mz}{1-m\zeta}-\log(\zeta-z)+\tfrac{1}{\zeta^2}\log(z\zeta-1)-\tfrac{r^2}{(1-m\zeta)^2}\\ &\times\log\Big(z(1-m\zeta)-(m-\zeta)\Big)-\tfrac{r^2}{(\zeta-m)^2}\log\Big(z(\zeta-m)-(m\zeta-1)\Big)\Big]\tfrac{d\zeta}{\zeta-m}\\ &+c_0,\end{aligned}$$

where $c_0\in\mathbb{C}$. The solution to Neumann problem (4.1) has form (4.2), if one defines c_0 by,

$$\begin{aligned}c_0 &= -\tfrac{1}{2\pi}\int_{m-r}^{1}\gamma(t)\Big[1+\tfrac{t}{t-m}+\tfrac{mt}{mt-1}-t\log(t-1)+\tfrac{1}{t}\log(t-1)\\ &-\tfrac{r^2t}{(1-mt)^2}\log\Big((mt-1)-(t-m)\Big)+\tfrac{r^2t}{(t-m)^2}\\ &\times\log\Big((t-m)-(mt-1)\Big)\Big]\tfrac{dt}{t}-\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\big[\tfrac{1}{\zeta}+\tfrac{1}{\zeta-m}-\tfrac{m}{1-m\zeta}\\ &-\log(\zeta-1)+\tfrac{1}{\zeta^2}\log(1\zeta-1)-\tfrac{r^2}{(1-m\zeta)^2}\log\Big((1-m\zeta)-(m-\zeta)\Big)\\ &-\tfrac{r^2}{(\zeta-m)^2}\log\Big(1(\zeta-m)-(m\zeta-1)\Big)\Big]\tfrac{d\zeta}{\zeta}-\tfrac{r}{2\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\Big[\tfrac{1}{\zeta}+\tfrac{1}{\zeta-m}\\ &-\tfrac{m}{1-m\zeta}-\log(\zeta-1)+\tfrac{1}{\zeta^2}\log(\zeta-1)-\tfrac{r^2}{(1-m\zeta)^2}\\ &\times\log\Big((1-m\zeta)-(m-\zeta)\Big)-\tfrac{r^2}{(\zeta-m)^2}\log\Big((\zeta-m)-(m\zeta-1)\Big)\Big]\tfrac{d\zeta}{\zeta-m}.\end{aligned}$$

□

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