ISSN (Print): 2077–9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2018, Volume 9, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

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The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 515
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 9, Number 3 (2018), 58 – 72

EFFICIENT NUMERICAL METHODS OF AITKEN TYPE AND THEIR DYNAMICS

P. Jain, P.B. Chand, K. Sethi

Communicated by T.V. Tararykova

Key words: Steffensen method, Aitken method, secant method, dynamics, fixed and critical points, basin of attraction.

AMS Mathematics Subject Classification: 65H05.

Abstract. In this paper, we derive some general Aitken type methods. These methods involve arbitrary methods of orders p and q which enable us to construct the method of any desired order. Further, it is shown that these methods can be combined with generalized secant method and as result, in the limiting case, the efficiency can be increased to 2. We also discuss the stability of the iterative method with the help of basins of attraction in the complex plane. Some numerical examples are provided in support of the theoretical results.

DOI: https://doi.org/10.32523/2077-9879-2018-9-3-58-72

1 Introduction

Non-linear equations are encountered in all branches of science and engineering. There hardly exist analytical methods for solving such equations and therefore it is desirable to obtain approximate solutions by methods which are based on iterative procedures. For a given non-linear equation

$$f(x) = 0,$$

a very well known method widely used is the Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The derivative free Steffensen method is given by

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)}. \tag{1.1}$$

Both Newton method as well as Steffensen method are quadratically convergent. Moreover, the efficiency index of both the methods is $2^{1/2} \approx 1.414$. There have been several ways by which the order of convergence can be increased [10]. Solemani et. al. in [11], presented a two point fourth order method based on Steffensen method given by

$$\left.\begin{aligned} w_n &= x_n + \beta f(x_n), \quad \beta \in \mathbb{R}\setminus\{0\} \\ y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ x_{n+1} &= x_n - \frac{f(x_n)^2}{f[x_n, w_n](f(x_n) - f(y_n) - (f(y_n)^2)/f(w_n)}, \end{aligned}\right\} \tag{1.2}$$

where, $f[x_n, w_n] = \dfrac{f(x_n) - f(w_n)}{x_n - w_n}$ is the first divided difference. Further, recently, in [9], Păvăloiu and Cătinaş obtained and studied the following Aitken type method:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f[y_n, z_n]}. \end{aligned}\right\} \tag{1.3}$$

Along with other considerations, it was proved in [9] that method (1.3) is of order 6 with efficiency index 1.4309 which is higher than for the Newton method or the standard Aitken method (having efficiency index 1.414).

It can be noted that method (1.3) involves the derivative of the function f. Sometimes, it is not possible to proceed if at some iterative step $f'(x_n) = 0$. So, in this paper, to begin with, we propose the following method in which the Newton iterates in (1.3) are replaced by Steffensen type method based on forward and backward differences which do not involve derivative of the function f:

$$\left.\begin{aligned} w_n &= x_n + \beta f(x_n), \\ y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ w_n^* &= y_n - \beta f(y_n), \\ z_n &= y_n - \frac{f(y_n)}{f[y_n, w_n^*]}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f[y_n, z_n]}, \end{aligned}\right\} \tag{1.4}$$

where, $\beta \in \mathbb{R} \setminus \{0\}$.

We prove that the order of convergence of method (1.4) is 6 and efficiency index is 1.4309 which is same as for method (1.3). This is done in Section 2.

Further, we propose a method more general than (1.3) or (1.4). We replace, in (1.3), the Newton iterates by the iterates of arbitrary methods. Let $\phi(x)$ and $\psi(x)$ be iterative functions such that the methods

$$x_{n+1} = \phi(x_n)$$

and

$$x_{n+1} = \psi(x_n)$$

are of order p and q, respectively. We propose the following generalized Aitken-type method:

$$\left.\begin{aligned} y_n &= \phi(x_n), \\ z_n &= \psi(y_n), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f[y_n, z_n]}. \end{aligned}\right\} \tag{1.5}$$

We prove, in Section 2, that method (1.5) is of order $pq + p$. This strategy would enable to produce an iterative method of any desired order by choosing appropriately the functions $\phi(x)$ and $\psi(x)$.

Next, note that the last iterate in method (1.3) is, in fact, the secant iterate which uses the previously calculated nodes y_n and z_n. In [6] and [7], the authors generalized the secant method which involves arbitrary number of previously calculated nodes. We exploit this generalized secant method in Section 3. In fact, we replace in (1.3), the secant iterate by generalized secant iterate. We show that as the number of iterate increases, not only the order but also the efficiency of the corresponding method increases. Moreover, in the limiting case as the number of iterates increases to infinity, the efficiency tends to 2. In Section 4, some numerical examples are provided based on the methods developed in this paper. Finally, in Section 5, we discuss the dynamics of the new method (1.4) and determine the basins of attraction of the method for the quadratic and cubic polynomials in the complex plane.

2 General Aitken type methods and their convergence

We first establish the order of convergence of method (1.4).

Theorem 2.1. *Let f be a real or complex-valued sufficiently differentiable function defined on some interval I and $\beta \in \mathbb{R} \setminus \{0\}$. Let α be a simple root of the nonlinear equation $f(x) = 0$. Then method (1.4) is of the sixth order of convergence and satisfies the error equation given by*

$$e_{n+1} = C_2^5(1 - \beta f'(\alpha))(1 + \beta f'(\alpha))^3 e_n^6 + O(e_n^7),$$

where $C_n = \dfrac{f^{(n)}(\alpha)}{n!\ f'(\alpha)}$ *and* $e_n = x_n - \alpha$ *is the error in* x_n.

Proof. Here, Taylor's series expansion of $f(x_n)$ around α is given by:

$$f(x_n) = f'(\alpha)\left(e_n + C_2 e_n^2 + C_3 e_n^3 + C_4 e_n^4 + C_5 e_n^5 + C_6 e_n^6 + O(e_n^7)\right)$$

and as $w_n = x_n + \beta f(x_n)$, Taylor's series expansion of $f(w_n)$ around α is given by:

$$\begin{aligned} f(w_n) = f'(\alpha)\Big[&(1+\beta f'(\alpha))e_n + C_2\left(1 + 3\beta f'(\alpha) + \beta^2 f'(\alpha)^2\right)e_n^2 \\ &+ \left(2C_2^2\beta f'(\alpha)(1+\beta f'(\alpha)) + C_3\left(1 + 4\beta f'(\alpha) + 3\beta^2 f'(\alpha)^2 + \beta^3 f'(\alpha)^3\right)\right)e_n^3 \\ &+ \Big(C_4\left(1 + 5\beta f'(\alpha) + 6\beta^2 f'(\alpha)^2 + 4\beta^3 f'(\alpha)^3 + \beta^4 f'(\alpha)^4\right) \\ &+ C_2\beta f'(\alpha)\left(C_2^2\beta f'(\alpha) + C_3(5 + 8\beta f'(\alpha) + 3\beta^2 f'(\alpha)^2)\right)\Big)e_n^4 + O(e_n^5)\Big]. \end{aligned}$$

If $y_n - \alpha = d_n$, then from the second equation of (1.4), we get

$$\begin{aligned} d_n = {} & C_2(1+\beta f'(\alpha))e_n^2 + \left(-C_2^2(2 + 2\beta f'(\alpha) + \beta^2 f'(\alpha)^2) + C_3(2 + 3\beta f'(\alpha) + \beta^2 f'(\alpha)^2)\right)e_n^3 \\ & + \Big(C_2^3\left(4 + 5\beta f'(\alpha) + 3\beta^2 f'(\alpha)^2 + \beta^3 f'(\alpha)^3\right) + C_4\left(3 + 6\beta f'(\alpha) + 4\beta^2 f'(\alpha)^2 + \beta^3 f'(\alpha)^3\right) \\ & - C_2C_3\left(7 + 10\beta f'(\alpha) + 7\beta^2 f'(\alpha)^2 + 2\beta^3 f'(\alpha)^3\right)\Big)e_n^4 + O(e_n^5). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} f(y_n) &= f(d_n + \alpha) \\ &= f'(\alpha)\left(d_n + C_2 d_n^2 + C_3 d_n^3 + C_4 d_n^4 + \cdots\right) \\ &= f'(\alpha)\left(A_1 e_n^2 + A_2 e_n^3 + A_3 e_n^4 + O(e_n^5)\right), \end{aligned}$$

where

$$\begin{aligned} A_1 &= C_2(1 + \beta f'(\alpha)), \\ A_2 &= -C_2^2(2 + 2\beta f'(\alpha) + \beta^2 f'(\alpha)^2) + C_3(2 + 3\beta f'(\alpha) + \beta^2 f'(\alpha)^2), \\ A_3 &= C_2^3\left(5 + 7\beta f'(\alpha) + 4\beta^2 f'(\alpha)^2 + \beta^3 f'(\alpha)^3\right) + C_4\left(3 + 6\beta f'(\alpha) + 4\beta^2 f'(\alpha)^2 + \beta^3 f'(\alpha)^3\right) \\ &\quad - C_2C_3\left(7 + 10\beta f'(\alpha) + 7\beta^2 f'(\alpha)^2 + 2\beta^3 f'(\alpha)^3\right). \end{aligned}$$

Since, $w_n^* = y_n - \beta f(y_n)$, after some tedious calculations and using Taylor's expansion, we can find the value of $f(w_n^*)$ as

$$f(w_n^*) = f'(\alpha)\left(B_1 e_n^2 + B_2 e_n^3 + B_3 e_n^4 + O(e_n^5)\right),$$

where

$$\begin{aligned}
B_1 &= C_2(1-\beta^2 f'(\alpha)^2,\\
B_2 &= (-1+\beta f'(\alpha))\left(C_2^2(2+2\beta f'(\alpha)+\beta^2 f'(\alpha)^2) + C_3(2+3\beta f'(\alpha)+\beta^2 f'(\alpha)^2)\right),\\
B_3 &= C_2^3\left(5-6\beta^2 f'(\alpha)^2 - 3\beta^3 f'(\alpha)^3\right) + C_4\left(3+3\beta f'(\alpha) - 2\beta^2 f'(\alpha)^2 - 3\beta^3 f'(\alpha)^3 - \beta^4 f'(\alpha)^4\right)\\
&\quad + C_2C_3\left(-7-3\beta f'(\alpha)+3\beta^2 f'(\alpha)^2 + 5\beta^3 f'(\alpha)^3 + 2\beta^4 f'(\alpha)^4\right).
\end{aligned}$$

Now, if θ_n is the error in z_n, then using the above considerations in the fourth equation of (1.4), we obtain

$$\begin{aligned}
\theta_n &= C_2^3(1-\beta f'(\alpha))(1+\beta f'(\alpha))^2 e_n^4\\
&\quad + 2C_2^2(-1+\beta^2 f'(\alpha)^2)\left(C_2^2(2+2\beta f'(\alpha)+\beta^2 f'(\alpha)^2) - C_3(2+3\beta f'(\alpha)+\beta^2 f'(\alpha)^2)\right) e_n^5\\
&\quad + O(e_n^6),
\end{aligned}$$

so that

$$\begin{aligned}
f(z_n) &= f(\theta_n+\alpha),\\
&= f'(\alpha)\left(\theta_n + C_2\theta_n^2 + \cdots\right)\\
&= f'(\alpha)\left(D_1 e_n^4 + D_2 e_n^5 + O(e_n^6)\right),
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= C_2^3(1-\beta f'(\alpha))(1+\beta f'(\alpha))^2,\\
D_2 &= 2C_2^2(-1+\beta^2 f'(\alpha)^2)\left(C_2^2\left(2+2\beta f'(\alpha)+\beta^2 f'(\alpha)^2\right) - C_3\left(2+3\beta f'(\alpha)+\beta^2 f'(\alpha)^2\right)\right).
\end{aligned}$$

Now, the last equation of (1.4) gives

$$\begin{aligned}
e_{n+1} &= \theta_n - \frac{f(z_n)(d_n-\theta_n)}{f(y_n)-f(z_n)}\\
&= C_2^5(1-\beta f'(\alpha))(1+\beta f'(\alpha))^3 e_n^6 + O(e_n^7),
\end{aligned}$$

which is the required error equation and the assertion is proved. □

Remark 1. *It follows from Theorem 2.1 that if* $\beta = \pm\dfrac{1}{f'(\alpha)}$, *then the order of method (1.4) is at least* 7.

Next, we study the convergence of general Aitken method (1.5).

Theorem 2.2. *Let* f *be a sufficiently differentiable function in a neighbourhood of* α *which is a simple root of* $f(x)=0$. *If* $\phi(x)$ *and* $\psi(x)$ *are iterative functions such that the methods*

$$y_n = \phi(x_n) \tag{2.1}$$

and

$$z_n = \psi(x_n) \tag{2.2}$$

have order of convergence p *and* q, *respectively, then method (1.5) has order of convergence* $pq+p$.

Proof. Let e_n, d_n, θ_n denote the errors involved in the iterates x_n, y_n, z_n, respectively. Since methods (2.1) and (2.2) are of order p and q, respectively, the error equations for the iterates y_n and z_n in (1.5) are given, respectively, by

$$d_n = Ae_n^p + O(e_n^{p+1}), \tag{2.3}$$

$$\theta_n = Bd_n^q + O(d_n^{q+1}), \tag{2.4}$$

where A and B are certain constants. Now, using Taylor's expansion, we have

$$\begin{aligned} f(z_n) &= f(\theta_n + \alpha), \\ &= f'(\alpha)\left[\theta_n + C_2\theta_n^2 + C_3\theta_n^3 + O(\theta_n^4)\right], \end{aligned}$$

and therefore

$$\begin{aligned} f[y_n, z_n] &= \frac{f(y_n) - f(z_n)}{y_n - z_n}, \\ &= \frac{f'(\alpha)(d_n - \theta_n)\left[1 + C_2(d_n + \theta_n) + C_3(d_n^2 + \theta_n^2 + d_n\theta_n) + O(\theta_n^4)\right]}{(d_n - \theta_n)}, \\ &= f'(\alpha)\left[1 + C_2(d_n + \theta_n) + C_3(d_n^2 + \theta_n^2 + d_n\theta_n) + O(\theta_n^4)\right]. \end{aligned}$$

Consequently, we get

$$\begin{aligned} \frac{f(z_n)}{f[y_n, z_n]} &= \frac{f'(\alpha)\left[\theta_n + C_2\theta_n^2 + C_3\theta_n^3 + O(\theta_n^4)\right]}{f'(\alpha)\left[1 + C_2(d_n + \theta_n) + C_3(d_n^2 + \theta_n^2 + d_n\theta_n) + O(\theta_n^4)\right]} \\ &= \left(\theta_n + C_2\theta_n^2 + C_3\theta_n^3\right)\left(1 - C_2(d_n + \theta_n) - C_3(d_n^2 + \theta_n^2 + d_n\theta_n)\right) \\ &= \theta_n - C_2 d_n\theta_n - (C_2^2 + C_3)\theta_n^2 d_n - C_3 d_n^2\theta_n - C_2^2\theta_n^3, \end{aligned}$$

using which the error equation of the iterate x_{n+1} in (1.5) is obtained as

$$\begin{aligned} e_{n+1} &= C_2 d_n\theta_n + (C_2^2 + C_3)\theta_n^2 d_n + C_3 d_n^2\theta_n + C_2^2\theta_n^3 \\ &\approx C_2 d_n\theta_n, \end{aligned}$$

which by using (2.3) and (2.4) gives

$$\begin{aligned} e_{n+1} &= C_2(Ae_n{}^p)(Bd_n{}^q) \\ &= C_2(Ae_n{}^p)(B(Ae_n{}^p)^q) \\ &= C_2A^{q+1}Be_n^{pq+p}. \end{aligned} \tag{2.5}$$

and the assertion follows. □

Remark 2. *In view of Theorem 2.2, it follows that if, in (1.5), the iterates y_n and z_n are interchanged, then the order of the method becomes $pq+q$. So, in order to have a higher order of convergence one should start with the iterate having a higher order of convergence.*

3 Increasing the efficiency

We know that the standard secant method is given by

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})},$$

which, in terms of divided difference, can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_{n-1}, x_n]}. \tag{3.1}$$

Secant method is a one point method with memory having order of convergence 1.618. Only one function evaluation per iteration is required in this method and as a result its efficiency is 1.618. In (1.3) or (1.5), the strategy was to use method (3.1) once the two nodes are calculated from other methods.

Recently, in [7], Kogan et. al. used Newton's divided difference formula

$$f(x) = f(x_n) + f[x_{n-1}, x_n](x - x_n) + \cdots + f[x_0, x_n] \prod_{j=1}^{n} (x - x_j) + R_n,$$

where

$$R_n = f(x, x_n, ..., x_0) \prod_{j=0}^{n} (x - x_j)$$

and generalized the secant method (3.1) as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_{n-1}, x_n] + \sum_{i=2}^{k} f[x_{n-i}, x_n] \prod_{j=1}^{i-1} (x_n - x_{n-j})}, \qquad n = k, k+1, \cdots \tag{3.2}$$

where $k \geq 1$ is an arbitrary fixed integer and the initial k approximations $x_0, x_1, \cdots, x_k$ are known. Obviously for $k = 1$, (3.2) becomes (3.1).

Remark 3. *For the later use, let us mention that (see [7]), based on* $k+1$ *initial approximations* $x_0, x_1, \cdots, x_k$, *the error equation corresponding to the method (3.2) is given by*

$$e_{n+1} = C_k \prod_{j=0}^{k} e_{n-j} + O(\prod_{j=0}^{k+1} e_{n-j}). \tag{3.3}$$

In the light of above discussion, we propose a multipoint general Aitken method of the type (1.3) as follows:

$$\begin{aligned}
x_n^{(0)} &= \phi(x_n), \\
x_n^{(1)} &= \psi(x_n^{(0)}), \\
x_n^{(2)} &= x_n^{(1)} - \frac{f(x_n^{(1)})}{f[x_n^{(0)}, x_n^{(1)}]}, \\
&\;\;\vdots
\end{aligned}$$

$$x_{n+1} = x_n^{(k)} - \frac{f(x_n^{(k)})}{f[x_n^{(k-1)}, x_n^{(k)}] + \sum_{i=2}^{k} f[x_n^{(k-i)}, x_n^{(k)}] \prod_{j=1}^{i-1} (x_n^{(k)} - x_n^{(k-j)})}, \qquad k = 1, 2, 3, \cdots \tag{3.4}$$

with the initial approximation x_0. Clearly, for $k = 1$, method (3.4) becomes method (1.5). We shall prove that as k increases, not only the order of convergence but also the efficiency of (3.4) increases. Precisely, we prove the following:

Theorem 3.1. *Let f be a sufficiently differentiable function in a neighbourhood of α which is a simple zero of f. If ϕ and ψ are iterative functions such that the methods*

$$x_{n+1} = \phi(x_n)$$

and

$$x_{n+1} = \psi(x_n)$$

have order of convergences p and q, with number of function evaluations per iterations as n_1 and n_2, respectively. Let $O(k)$ and $EI(k)$ denote, respectively, the order of convergence and efficiency index of (3.4) for $k = 1, 2, 3, \cdots$. Then

(a) $O(k) = (pq + p) \times 2^{k-1}$,

(b) $EI(k) = [(pq + p) \times 2^{k-1}]^{\frac{1}{n_1+n_2+k}}$,

(c) EI(k) is strictly increasing,

(d) $EI(k) \to 2$ *as* $k \to \infty$.

Proof. We only prove (a) and (b). It is straightforward to verify (c) and (d).

(a) Let e_n, e_{n+1} and d_n denote the errors in the iterates x_n, x_{n+1} and $x_n^{(0)}$. For the intermediate steps, let $e_{n,k}$ denote the errors in $x_n^{(k)}$, $k = 0, 1, 2, 3, \cdots$. Since $x_n^{(0)}$ and $x_n^{(1)}$ are of order p and q, respectively, the corresponding errors are given by

$$d_n = e_{n,0} \approx A_1 e_n^p \tag{3.5}$$

$$e_{n,1} \approx d_n^q \approx A_2 e_{n,0}^q \approx A_1^q A_2 e_n^{pq}, \tag{3.6}$$

where A_1 and A_2 are appropriate constants. For the intermediate steps, for $k = 2, 3, 4, 5, \cdots$, the corresponding error equations, in view of (3.3), are given by

$$e_{n,k+1} \approx C_{k+1} \prod_{j=1}^{k} e_{n,k+1-j} \tag{3.7}$$

and once $k = 1, 2, 3, \cdots$ is fixed, we shall write

$$e_{n+1} = e_{n,k+1}.$$

We shall prove by induction that the order of convergence of the method (3.4) is $(pq+p) \times 2^{k-1}$ for $k = 1, 2, 3, \cdots$

For $k = 1$, (3.7) becomes

$$e_{n,2} \approx C_3 e_{n,1} . e_{n,0}$$

which by using (3.5) and (3.6) gives

$$e_{n,2} \approx C_3 A_1^{q+1} A_2 e_n^{(pq+p)},$$

i.e.,

$$e_{n+1} \approx C_3 A_1^{q+1} A_2 e_n^{(pq+p)}.$$

Therefore, the assertion holds for $k = 1$. Assume that it holds for k, i.e.,

$$e_{n+1} = e_{n,k+1} \approx D_k e_n^{(pq+p)\times 2^{k-1}}, \tag{3.8}$$

where D_k is some constant. Note that by (3.7)

$$e_{n,k+1} \approx C_{k+1}.e_{n,k}.e_{n,k-1}.e_{n,k-2}.\dots.e_{n,0}. \tag{3.9}$$

For k replaced by $k+1$, (3.7) gives

$$e_{n,k+2} \approx C_{k+2} \prod_{j=1}^{k+1} e_{n,k+2-j}$$

which by using (3.8) and (3.9) gives

$$\begin{aligned} e_{n,k+2} &\approx C_{k+2}.e_{n,k+1}.e_{n,k}.e_{n,k-1}.e_{n,k-2}\dots.e_{n,0} \\ &\approx \frac{C_{k+2}}{C_{k+1}} e_{n,k+1}^2 \\ &\approx \frac{C_{k+2}}{C_{k+1}} D_k^2 e_n^{(pq+p)\times 2^{k-1}\times 2} \\ &= \frac{C_{k+2}}{C_k+1} D_k^2 e_n^{(pq+p)\times 2^{k-2}} \end{aligned}$$

and the assertion follows.
(b) In method (3.4), the first two steps are the iterates that require per iteration n_1 and n_2 functions evaluation respectively. Thus for these two steps, a total of n_1+n_2 function evaluations per iteration are required. After third step onward, the method requires only one function evaluation per iteration since it uses the previously calculated values. Thus for $k = 1, 2, 3, \cdots$, a total of $n_1 + n_2 + k$ functions need to be evaluated per iteration. Combining this information with the order of convergence of the method, the result follows. □

Example 1. *If we consider*

$$\begin{aligned} w_n &= x_n + \beta f(x_n), \\ \phi(x_n) &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \end{aligned}$$

and

$$\begin{aligned} w_n^* &= x_n - \beta f(x_n), \\ \psi(x_n) &= x_n - \frac{f(x_n)}{f[x_n, w_n^*]}, \end{aligned}$$

then method (3.4) becomes the following:

$$
\begin{aligned}
w_n &= x_n + \beta f(x_n),\\
x_n^{(0)} &= x_n - \frac{f(x_n)}{f[x_n, w_n]},\\
w_n^* &= x_n^{(0)} - \beta f(x_n^{(0)}),\\
x_n^{(1)} &= x_n^{(0)} - \frac{f(x_n^{(0)})}{f[x_n^{(0)}, w_n^*]},\\
x_n^{(2)} &= x_n^{(1)} - \frac{f(x_n^{(1)})}{f[x_n^{(0)}, x_n^{(1)}]},\\
&.\\
&.\\
&.
\end{aligned}
$$

$$
x_{n+1} = x_n^{(k)} - \frac{f(x_n^{(k)})}{f[x_n^{(k-1)}, x_n^{(k)}] + \sum_{i=2}^{k} f[x_n^{(k-i)}, x_n^{(k)}] \prod_{j=1}^{i-1}(x_n^{(k)} - x_n^{(k-j)})}, \quad k = 1, 2, 3, \cdots \tag{3.10}
$$

In the view of Theorem 3.1, we can prove the following theorem.

Theorem 3.2. *Let f be a sufficiently differentiable function in a neighbourhood of α which is a simple zero of $f(x) = 0$. Let O(k) and EI(k) denote, respectively, the order of convergence and efficiency index of* (3.10) *for $k = 1, 2, 3, \cdots$. Then*

(a) $O(k) = 6 \times 2^{k-1}$,

(b) $EI(k) = (6 \times 2^{k-1})^{\frac{1}{k+4}}$,

(c) EI(k) is strictly increasing,

(d) $EI(k) \to 2$ *as* $k \to \infty$.

4 Examples

Example 2. *We consider the equation*

$$f(x) = x^3 - e^{-x}$$

and implement three derivative free methods (1.1), (1.2) and (1.4) which have order of convergence 2, 4 and 6 respectively, and compare the results. The initial approximation is taken as $x_0 = 1.5$ and $\beta = 1$. The corresponding iterates are shown in Table 1.

Table 1

n	Method (1.1)	Method (1.2)	Method (1.4)
1	1.3981146700	0.9688659395	0.6364988523
2	1.2879323790	0.7780301104	0.7726261774
3	1.1698167750	0.7728829635	0.7728829591
4	1.0466098090	0.7728829591	Division by 0
5	0.9271575466	Division by 0	
6	0.8311949465		
7	0.7824232093		
8	0.7731543614		
9	0.7728831811		
10	0.7728829591		
11	0.7728829591		

5 Dynamics of the method

Let $p(z)$ be a polynomial having simple roots and defined on the Riemann sphere $\hat{\mathbb{C}}$. We apply the sixth order method (1.4) presented in this paper on the complex polynomial $p(z)$ and correspondingly define the operator:

$$\left.\begin{aligned} y(z) &= z - \frac{\beta p^2(z)}{p(z+\beta p(z)) - p(z)}, \\ w(z) &= y(z) - \frac{\beta p^2(y(z))}{p(y(z)) - p(y(z) - \beta p(y(z)))}, \\ M(z) &= w(z) - \frac{p(w(z))(w(z)-y(z))}{p(w(z)) - p(y(z))}. \end{aligned}\right\} \tag{5.1}$$

The fixed points of the method are obtained from the equation $M(z) = z$ and critical points of the method are obtained from the equation $M'(z) = 0$. For the second degree polynomials $p(z) = z^2 - 1$ and $p(z) = z^2 + 1$ with $\beta = 1$, the fixed and critical points of the method $M(z)$ are presented in the Table 2.

As the degree of a polynomial increases, the number of the fixed and critical points of the method increases very rapidly. It has been worked out that for the third degree polynomial $p(z) = z^3 - 1$, the number of fixed points of method (5.1) is 104 and the number of critical points is 181. The roots of the polynomials are always the fixed points as well as the critical points. Fixed points and critical points, which are not the roots of the polynomial involved, are called extraneous fixed points and extraneous critical points respectively.

The existence of extraneous fixed points of any operator may complicate the root finding procedure. The fixed points may be (super)attractive, repulsive or neutral (see [1], [2], [5] etc). As attractive fixed points, they may trap an iteration sequence, giving erroneous results for a root α of the polynomial $p(z)$. Even as the repulsive or neutral fixed points, however, they may alter the structure of the basin of attraction for the roots [13]. Generally, increasing the order of convergence of any multipoint method, increases the number of extraneous fixed points. This may adversely affect the basin of attraction of the method, *i.e.*, increasing number of extraneous fixed points reduces the attraction basins [8]. Therefore, large number of extraneous fixed and critical points for the higher degree polynomial make the method less stable.

Table 2: Fixed and Critical points

Polynomial	Fixed Points		Critical Points	
	Roots	Extraneous	Roots	Extraneous
$p(z) = z^2 - 1$	-1, 1	-3.15242, -2.08799, 0.59003, 2.84026, $-1.72787 - 0.157967i$, $-1.72787 + 0.157967i$, $0.223615 - 0.686001i$, $0.223615 + 0.686001i$, $0.409311 - 0.252665i$, $0.409311 + 0.252665i$ Total: 10	-1, 1	-2, 0, -3.73205, -0.267949, -2.13302, -1.87785, 0.521913, 3.43398, $0.527489 - 0.590424i$, $0.527489 + 0.590424i$ Total: 10
$p(z) = z^2 + 1$	$-i$, i	-2.545, -0.115835, $-0.92611 - 0.857748i$, $-0.92611 + 0.857748i$, $-0.703422 - 1.1787i$, $-0.703422 + 1.1787i$, $-0.52876 - 0.769406i$, $-0.52876 + 0.769406i$, $1.48871 - 0.613581i$, $1.48871 + 0.613581i$ Total: 10	$-i$, i	-3, $-1 - i$, $-1 + i$, $-0.847445 - 0.788837i$, $-0.847445 + 0.788837i$, $-0.713402 - 0.747803i$, $-0.713402 + 0.747803i$, $2.06085 - 0.541034i$, $2.06085 + 0.541034i$ Total: 9

5.1 Basins of attraction

It is known that the Steffensen type methods do not satisfy "scaling theorem" (see [3], [4] and references there) and as a result, the dynamics of such methods can not be studied on the basis of a class of polynomials. In fact different polynomials of the same degree may have different dynamics. So the basins of attraction of the methods may vary within polynomials of the same degree.

To understand the dynamical behavior of the methods visually, we show the basins of attraction of the method $M(z)$ for second and third degree polynomials and for different values of β in the figures that follow.

In our work, we use Mathematica 9.0 for the calculations as well as to determine the basins of attraction of the method. We divide the complex plane into 250×250 initial points in the domain $[-2, 2] \times [-2, 2]$ to determine the basins of attraction of the roots of the polynomials. Different colors are used for basins of attraction of each root [12]. Light color specifies the region where initial points require less iterations to converge to the particular root. As the color gets darker and darker, it means that the number of iteration increases to approximate the root. We use black color for the initial points which do not converge to any of the roots within the maximum limit of 40 iterations. However, if the whole region is black, it does not mean that the initial points in that region never converge to any of the roots with method (5.1). It only shows that within our predetermined criteria, those points do not converge to the roots. If we change our criteria, e.g., the number of initial points, domain or maximum of limit of iterations, then the basin of attraction may alter.

From Figures 1, and 2, it is clear that the basins of attraction of the new method changes with the change of value of β and polynomials also. For quadratic polynomials, the attraction basins is smooth as β decreases. As shown in Figure 1, it is quite smooth and wider when $\beta = 0.001$ for both the quadratic polynomials. But, for cubic polynomials, as shown in Figure 2, basins of attraction are more smooth when $\beta = 0.01$.

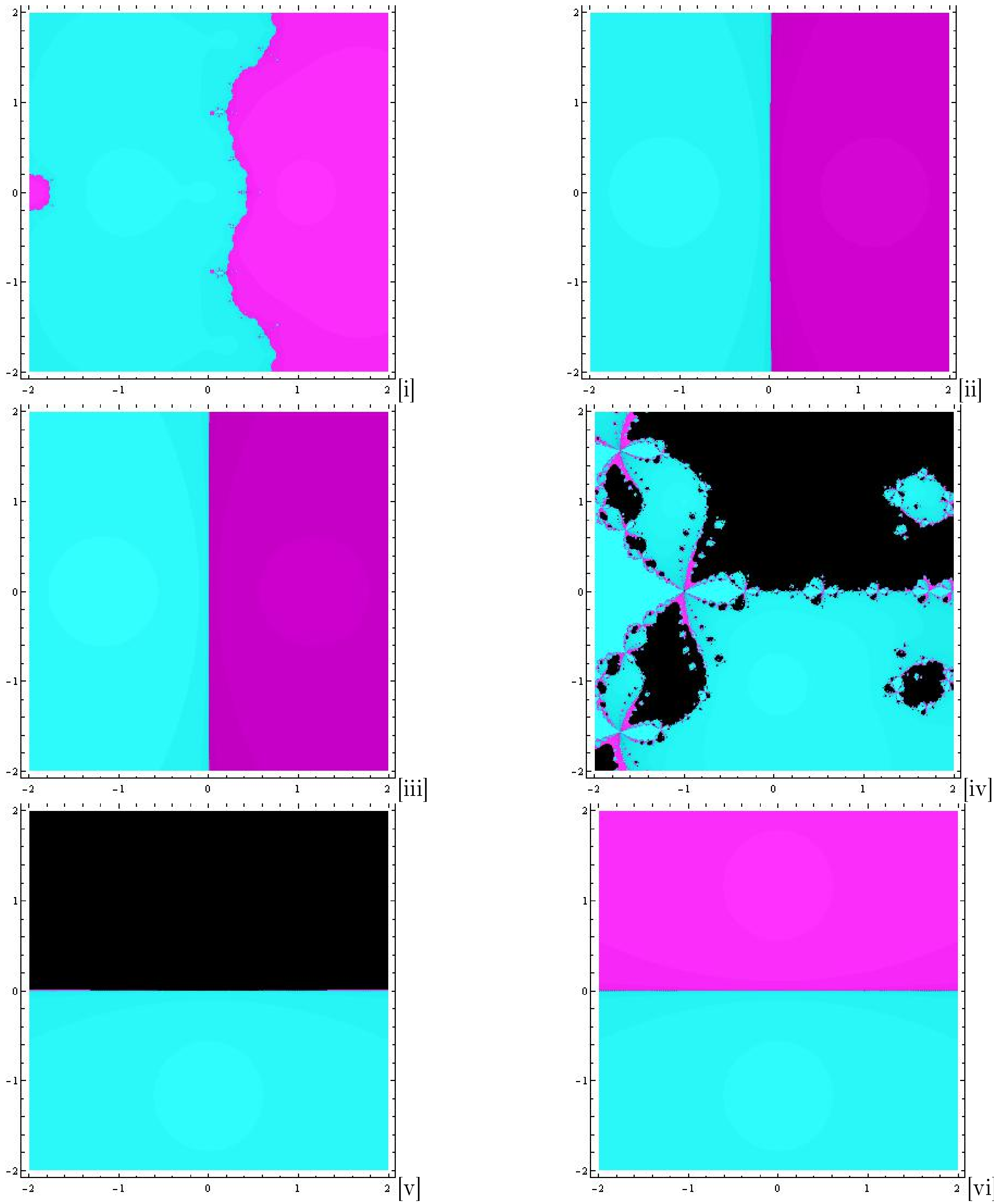

Figure 1: Basin of attraction of New method M for the quadratic polynomial $p_1(z) = z^2 - 1$ and $p_2(z) = z^2 + 1$ in the region $[-2, 2] \times [-2, 2]$ in xy-plane is divided into 250×250 points. $[i]$ $p_1(z)$ with $\beta = 1$, $[ii]$ $p_1(z)$ with $\beta = 0.01$, $[iii]$ $p_1(z)$ with $\beta = 0.001$, $[iv]$ $p_2(z)$ with $\beta = 1$, $[v]$ $p_2(z)$ with $\beta = 0.01$, $[vi]$ $p_2(z)$ with $\beta = 0.001$.

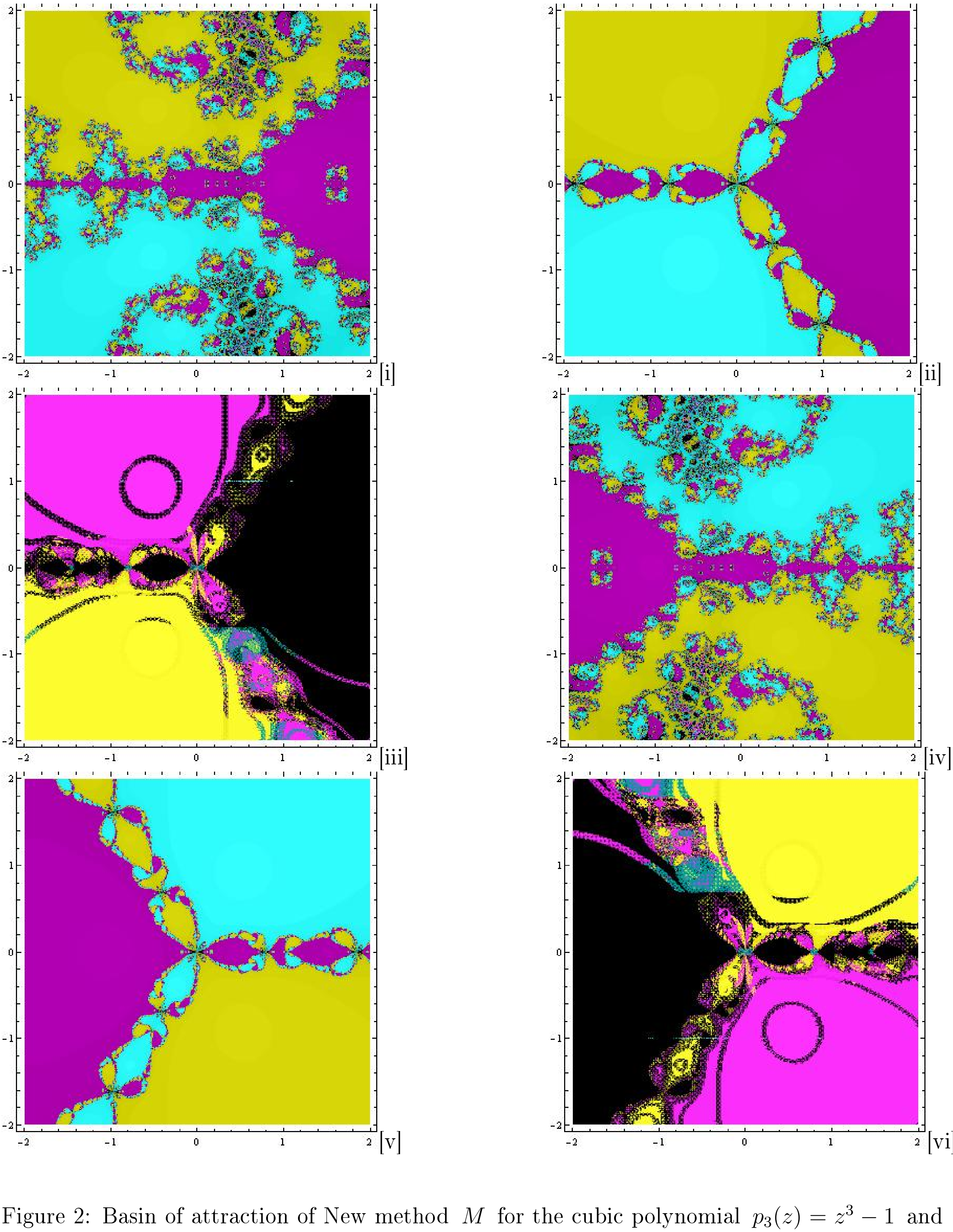

Figure 2: Basin of attraction of New method M for the cubic polynomial $p_3(z) = z^3 - 1$ and $p_4(z) = z^3 + 1$ in the region $[-2, 2] \times [-2, 2]$ in xy-plane is divided into 250×250 points. $[i]$ $p_3(z)$ with $\beta = 1$, $[ii]$ $p_3(z)$ with $\beta = 0.01$, $[iii]$ $p_3(z)$ with $\beta = 0.001$, $[iv]$ $p_4(z)$ with $\beta = 1$, $[v]$ $p_4(z)$ with $\beta = 0.01$, $[vi]$ $p_4(z)$ with $\beta = 0.001$.

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Pankaj Jain, Prem Bahadur Chand, Kriti Sethi
Department of Mathematics
South Asian University
Akbar Bhawan, Chanakya Puri, New Delhi-110021, India
E-mails: pankaj.jain@sau.ac.in, pankajkrjain@hotmail.com
premchand13@gmail.com
kritisethi90@gmail.com, kritisethi@students.sau.ac.in

Received: 26.09.2017