ISSN (Print): 2077–9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2018, Volume 9, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http://publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 515
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 9, Number 3 (2018), 33 – 57

ON A CLASS OF ABSTRACT DEGENERATE MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS IN LOCALLY CONVEX SPACES

V.E. Fedorov, M. Kostić

Communicated by D. Suragan

Key words: abstract time-fractional differential equations, degenerate differential equations, fractional calculus, ultra-logarithmic regions, ultradistribution semigroups.

AMS Mathematics Subject Classification: 34K30, 34A08, 35R11.

Abstract. In this paper, we consider regularized solutions for a class of abstract degenerate multi-term fractional differential equations with Caputo derivatives. Our results seem to be new even for non-degenerate differential equations under consideration.

DOI: https://doi.org/10.32523/2077-9879-2018-9-3-33-57

1 Introduction and preliminaries

The first congress on fractional calculus was held at the University of New Haven, in 1974 [34]. From then on, considerable interest in fractional calculus and fractional differential equations has been stimulated due to their numerous applications in engineering, physics, chemistry, biology and other sciences. Fairly complete information about fractional calculus and non-degenerate fractional differential equations can be obtained by consulting the references [4, 22, 23, 24, 25, 35, 36].

Various types of abstract degenerate fractional differential equations and their qualitative properties have been recently considered in [16, 17] (cf. [7, 12, 13, 14, 29, 32, 33, 38, 40] for the basic source of information on the abstract degenerate differential equations). In a joint paper with A. Debbouche [16], the first named author has analyzed the unique solvability of the Cauchy and Showalter problems for a class of degenerate fractional evolution systems by using the notion of strongly (B, p)-sectorial operators, while in the papers [17, 18], written in cooperation with D.M. Gordievskikh and M.V. Plekhanova necessary and sufficient conditions for relative p-boundedness of a pair of operators have been obtained in terms of families of resolving operators for a corresponding degenerate fractional differential equation. In this paper, we continue our previous research by considering the existence and uniqueness of regularized solutions for a class of abstract degenerate multi-term fractional differential equations with Caputo derivatives.

The organization of paper is briefly described as follows. In Theorem 2.1, we consider a Ljubich type uniqueness theorem for the initial value problem (2.5), (2.6) stated below. Although not visible at a first glance, our main structural results on the existence and uniqueness of regularized solutions of problem (2.5), (2.6), cf. Theorem 2.2, Theorem 2.3 and Theorem 3.1, are in a close connection with the corresponding results on regularization of ultradistribution semigroups and sines from our previous paper [21]; in other words, it has turned out that some ideas from the afore-mentioned paper can be applied in the analysis of an essentially larger class of abstract (degenerate) differential equations, considered in the general setting of sequentially complete locally convex spaces. A great number of various thoughts and insights about Theorem 2.2 is collected in Remark 1, which seems to us as a very compact and non-desultory but a little bit oversized. We also reconsider the old ideas of R. Beals [5, 6] for abstract degenerate relaxation equations and prove, as a by-product, some new results on the generation of fractionally regularized resolvent families (Remark 2, Remark 3). In Subsection 2.1, we provide the basic information about the possibility of extension of Theorem 2.2 and Theorem 2.3 to the non-Gevrey case, while in Section 3 we present various applications of our abstract theoretical results from Section 2.

Unless specifed otherwise, we assume that E is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. If X is also an SCLCS, then we denote by $L(E, X)$ the space consisting of all continuous linear mappings from E in X; $L(E) \equiv L(E, E)$. By $\circledast$ we denote the fundamental system of seminorms which defines the topology of E. Let $\mathcal{B}$ be the family consisting of all bounded subsets of E, and let $p_{\mathbb{B}}(T) := \sup_{x\in\mathbb{B}} p(Tx)$, $p \in \circledast$, $\mathbb{B} \in \mathcal{B}$, $T \in L(E)$. Then $p_{\mathbb{B}}(\cdot)$ is a seminorm on $L(E)$ and the system $(p_{\mathbb{B}})_{(p,\mathbb{B})\in\circledast\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(E)$. Let us recall that the space $L(E)$ is sequentially complete provided that E is barreled [31]. If E is a Banach space, then we denote by $\|x\|$ the norm of an element $x \in E$.

If A is a linear operator acting on E, then the domain and range of A will be denoted by $D(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. By I we denote the identity operator on E. If $C \in L(E)$ is injective, then we define the C-resolvent set of A, $\rho_C(A)$ for short, by $\rho_C(A) := \{\lambda \in \mathbb{C} \mid \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\}$; $\rho(A) \equiv \rho_I(A)$.

In the remaining part of this paragraph, it will be assumed that the operator A is closed. We refer the reader to [9, Definition 3.4] for the notion of an (analytic) C-regularized semigroup of growth order $r > 0$; the fractional power $(-A-\omega)_b$, appearing in Remark 3 (i), will be understood in the sense of [25, Definition 2.9.24]. For further information concerning fractional powers of almost C-sectorial operators, the reader may consult [25, Section 2.9].

Let A and B be closed linear operators acting on E. The notion of a (local) (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ with a subgenerator A will be understood in the sense of [24, Definition 2.1]; $(R(t))_{t\in[0,\tau)}$ is said to be locally equicontinuous if and only if, for every $t \in (0, \tau)$, the family $\{R(s) : s \in [0, t]\} \subseteq L(E)$ is equicontinuous. In the case $\tau = \infty$, $(R(t))_{t\geq 0}$ is said to be exponentially equicontinuous (equicontinuous) if and only if there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}R(t) : t \geq 0\} \subseteq L(E)$ is equicontinuous. If $a(t)$ is a kernel on $[0, \tau)$, then we define the integral generator $\hat{A}$ of $(R(t))_{t\in[0,\tau)}$ by setting

$$\hat{A} := \left\{(x, y) \in E \times E : R(t)x - k(t)Cx = \int_0^t a(t-s)R(s)y\,ds,\ t \in [0, \tau)\right\}.$$

For further information concerning abstract Volterra integro-differential equations in Banach and locally convex spaces, the reader may consult [25] and [35].

If V is a general topological vector space, then a function $f : \Omega \to V$, where Ω is an open subset of $\mathbb{C}$, is said to be analytic if it is locally expressible in a neighborhood of any point $z \in \Omega$ by a uniformly convergent power series with coefficients in V. We refer the reader to [25, Section 1.1] and references cited there for the basic information about vector-valued analytic functions. In our approach the space E is sequentially complete, so that the analyticity of a mapping $f : \Omega \to E$ ($\emptyset \neq \Omega \subseteq \mathbb{C}$) is equivalent with its weak analyticity.

Given $\theta \in (0, \pi]$ and $d \in (0, 1]$, define $\Sigma_\theta := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg(\lambda)| < \theta\}$, $B_d := \{\lambda \in \mathbb{C} : |\lambda| \leq d\}$ and $\Omega_{\theta,d} := \Sigma_\theta \cup B_d$. By $\Gamma_{\theta,d}$ we denote the upwards oriented boundary of $\Omega_{\theta,d}$. Further on, $\lfloor\beta\rfloor := \sup\{k \in \mathbb{Z} : k \leq \beta\}$, $\lceil\beta\rceil := \inf\{k \in \mathbb{Z} : \beta \leq k\}$ ($\beta \in \mathbb{R}$), $\mathbb{N}_n := \{1, \ldots, n\}$ and $\mathbb{N}_n^0 := \mathbb{N}_n \cup \{0\}$ ($n \in \mathbb{N}$). By $\mathbb{C}[z]$ we denote the set consisting of all complex polynomials of one variable. A scalar-valued function $k \in L^1_{\mathrm{loc}}[0,\tau)$ is said to be a kernel on $[0,\tau)$ if and only if for any scalar-valued continuous function $t \mapsto u(t)$, $t \in [0,\tau)$, the preassumption $\int_0^t k(t-s)u(s)\,ds = 0$, $t \in [0,\tau)$ implies $u(t) = 0$, $t \in [0,\tau)$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $0^\zeta := 0$ ($\zeta > 0$, $t > 0$), and $g_0(t) :=$ the Dirac δ-distribution. For a number $\zeta > 0$ given in advance, the Caputo fractional derivative $\mathbf{D}_s^\zeta u$ [4, 25] is defined for those functions $u \in C^{\lceil \zeta \rceil - 1}([0,\infty) : E)$ for which $g_{\lceil \zeta \rceil - \zeta} * (u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1}) \in C^{\lceil \zeta \rceil}([0,\infty) : E)$, by

$$\mathbf{D}_s^\zeta u(s) := \frac{d^{\lceil \zeta \rceil}}{ds^{\lceil \zeta \rceil}} \left[g_{\lceil \zeta \rceil - \zeta} * \left(u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1} \right) \right].$$

The Mittag-Leffler function $E_{\beta,\gamma}(z)$ ($\beta > 0$, $\gamma \in \mathbb{R}$) is defined by

$$E_{\beta,\gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad z \in \mathbb{C}.$$

In this place, we assume that $1/\Gamma(\beta k + \gamma) = 0$ if $\beta k + \gamma \in -\mathbb{N}_0$. Set, for short, $E_\beta(z) := E_{\beta,1}(z)$, $z \in \mathbb{C}$. The asymptotic behaviour of the entire function $E_{\beta,\gamma}(z)$ is given in the following auxiliary lemma (see e. g. [25, Section 1.3]).

Lemma 1.1. *Let $0 < \sigma < \pi/2$. Then, for every $z \in \mathbb{C} \setminus \{0\}$ and $l \in \mathbb{N} \setminus \{1\}$,*

$$E_{\beta,\gamma}(z) = \frac{1}{\beta} \sum_s Z_s^{1-\gamma} e^{Z_s} - \sum_{j=1}^{l-1} \frac{z^{-j}}{\Gamma(\gamma - \beta j)} + O\big(|z|^{-l}\big), \quad |z| \to \infty,$$

where Z_s is defined by $Z_s := z^{1/\beta} e^{2\pi i s/\beta}$ and the first summation is taken over all integers s satisfying $|\arg(z) + 2\pi s| < \beta(\sigma + \pi/2)$.

For further information about the Mittag-Leffler functions, cf. [4, 25] and references cited there.

We introduce the abstract Beurling space of (M_p) class associated to A, $E^{(M_p)}(A)$ for short, as in the Banach space case (cf. [10, 23] for more details). Put $D_\infty(A) := \bigcap_{n \in \mathbb{N}} D(A^n)$,

$$E^{(M_p)}(A) := \operatorname{projlim}_{h \to +\infty} E_h^{(M_p)}(A),$$

where for each $h > 0$,

$$E_h^{(M_p)}(A) := \left\{ x \in D_\infty(A) : \|x\|_{h,q}^{(M_p)} = \sup_{p \in \mathbb{N}_0} \frac{h^p q\big(A^p x\big)}{M_p} < \infty \text{ for all } q \in \circledast \right\}.$$

In this place, it is worth noting that for each $h > 0$ the calibration $(\|\cdot\|_{h,q}^{(M_p)})_{q \in \circledast}$ induces a Hausdorff locally convex space topology on $E_h^{(M_p)}(A)$, as well as that $E_{h'}^{(M_p)}(A) \subseteq E_h^{(M_p)}(A)$ provided $0 < h < h' < \infty$, and that the spaces $E_h^{(M_p)}(A)$ and $E^{(M_p)}(A)$ are continuously embedded in E; cf. [23]. Following the ideas of R. Beals [6], we define the space $E^{\langle M_p \rangle}(A)$ as the inductive limit of spaces $E_h^{(M_p)}(A)$ as $h \to 0+$; that is

$$E^{\langle M_p \rangle}(A) := \operatorname{indlim}_{h \to 0+} E_h^{(M_p)}(A).$$

Henceforth we shall always assume that (M_p) is a sequence of positive real numbers such that $M_0 = 1$ and the following condition is satisfied:

$$M_p^2 \leq M_{p+1}M_{p-1}, \quad p \in \mathbb{N}. \tag{$M.1$}$$

By $(M.1)$, the sequence $(m_p \equiv M_p/M_{p-1})_{p\in\mathbb{N}}$ is increasing. Any usage of the conditions:

$$M_p \leq rh^p \sup_{0\leq i\leq p} M_i M_{p-i}, \quad p \in \mathbb{N}, \text{ for some numbers } r, \; h > 1, \tag{$M.2$}$$

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty, \tag{$M.3'$}$$

and the condition

$$\sup_{p\in\mathbb{N}} \sum_{q=p+1}^{\infty} \frac{M_{q-1}M_{p+1}}{pM_pM_q} < \infty, \tag{$M.3$}$$

which is slightly stronger than $(M.3')$, will be explicitly emphasized. Let us recall that for each number $s > 1$ the Gevrey sequence $(p!^s)$ satisfies all the above conditions. The associated function of the sequence (M_p) is defined on $[0, \infty)$ by

$$M(\rho) := \sup_{p\in\mathbb{N}_0} \ln \frac{\rho^p}{M_p}, \; \rho > 0 \text{ and } M(0) := 0;$$

if $\lambda \in \mathbb{C}$, then we define $M(\lambda) := M(|\lambda|)$. It is well known that the function $t \mapsto M(t)$, $t \geq 0$ is non-negative, increasing as well as that $\lim_{\lambda\to\infty} M(\lambda) = \infty$ and that the function $M(\cdot)$ vanishes in some open neighborhood of zero. Furthermore, the mapping $t \mapsto M(t)$, $t \geq 0$ is absolutely continuous and the mapping $t \mapsto M(t)$, $t \in [0,\infty) \setminus \{m_p : p \in \mathbb{N}\}$ is continuously differentiable with $M'(t) = \frac{m(t)}{t}$, $t \in [0,\infty) \setminus \{m_p : p \in \mathbb{N}\}$. The (M_p)-ultralogarithmic region of type l

$$\Lambda_{\alpha,\beta,l} := \big\{\lambda \in \mathbb{C} : \Re\lambda \geq \alpha M(l|\Im\lambda|) + \beta\big\},$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ and $l \geq 1$, was defined for the first time by J. Chazarain in 1971 [8]. We assume that the boundary of the ultra-logarithmic region $\Lambda_{\alpha,\beta,l}$, denoted by Γ_l, is upwards oriented. If (N_p) and (R_p) are two sequences of positive real numbers, then we write $N_p \prec R_p$ if and only if for each number $\sigma > 0$ we have

$$\sup_{p\in\mathbb{N}_0} \frac{N_p\sigma^p}{R_p} < \infty.$$

2 Regularized solutions for a class of abstract degenerate multi-term fractional differential equations

Our first task will be to extend the assertions of [21, Theorem 2.1, Corollary 2.1] to abstract degenerate multi-term fractional differential equations (the Gevrey case). Throughout the section, the numbers $\zeta \in (0,1]$, $\alpha > 0$, $\beta > 0$, $l \geq 1$, $\xi \geq 0$ and $b \in (0,1)$ will be fixed. Denote by $M_v(\cdot)$ the associated function of the sequence $(p^{\frac{p}{v}})$ $(v \in (0,1))$. Then we know that $M_v(t) \sim (ve)^{-1}t^v$ as $t \to +\infty$. Suppose that

$$p^{\frac{p}{b}} \prec M_p. \tag{2.1}$$

Then, for every $\mu > 0$, there exist positive real constants $c_\mu > 0$ and $C_\mu > 0$ such that $\lim_{\mu\to 0} c_\mu = 0$ and

$$M(l\lambda) \leq M_b(\mu l\lambda) + C_\mu \leq c_\mu|\lambda|^b + C_\mu, \; \lambda \geq 0. \tag{2.2}$$

Set

$$\Lambda^{\zeta}_{\alpha,\beta,l} := \{\lambda^{\zeta} : \lambda \in \Lambda_{\alpha,\beta,l}\} \text{ and } \Omega := \mathbb{C} \setminus \Lambda^{\zeta}_{\alpha,\beta,l}.$$

By $\mathcal{A}$ we denote the class consisting of all continuous functions $f : \overline{\Omega} \to \mathbb{C}$ that are analytic in Ω and satisfy the following condition: there exist numbers $a_1 > 0$ and $a_2 > \xi$ such that

$$|f(\lambda)| \leq a_1 e^{-a_2|\lambda|^{b/\zeta}}, \quad \lambda \in \overline{\Omega}. \tag{2.3}$$

Suppose that $f \in \mathcal{A}$, $f \neq 0$. Then we define $F(\cdot)$ by

$$F(\lambda) := f\Big(-\lambda^{\frac{\pi-(\zeta\pi/2)}{\pi/2}}\Big), \quad \lambda \in \overline{\Sigma_{\pi/2}}.$$

The function $F(\cdot)$ can be analytically extended to an open neighborhood of the region $\overline{\Sigma_{\pi/2}}$ and satisfies the condition:

$$|F(\lambda)| \leq a_1 e^{-a_2|\lambda|^{\frac{\pi-(\zeta\pi/2)}{\pi/2}\frac{b}{\zeta}}}, \quad \lambda \in \overline{\Sigma_{\pi/2}}.$$

Now we can apply the Phragmén–Lindelöf type theorems (see e. g. [30, p. 40]) in order to see that the inequality $\frac{\pi-(\zeta\pi/2)}{\pi/2}\frac{b}{\zeta} \geq 1$ implies $f = 0$ identically. Hence, one has to assume that $\frac{\pi-(\zeta\pi/2)}{\pi/2}\frac{b}{\zeta} < 1$, i. e., that

$$\frac{1}{2-\zeta} > \frac{b}{\zeta} \tag{2.4}$$

in order to ensure the non-triviality of the class $\mathcal{A}$ (observe that $1/(2-\zeta) \in (1/2, 1]$ for $\zeta \in (0, 1]$, so that (2.4) automatically implies $b < \zeta$). Suppose now that (2.4) holds. Then the class $\mathcal{A}$ is non-trivial. Indeed, this can be proved in the following way. Put $\theta := \arctan(\cos(\frac{b}{\zeta}(\pi - \pi\zeta/2)))$. Then the function

$$f(\lambda) = f_t(\lambda) := e^{-t(-\lambda+\omega)^{b/\zeta}}, \quad \lambda \in \overline{\Omega}$$

belongs to $\mathcal{A}$ provided $t = t_1 + it_2 \in \Sigma_{\theta}$, $\omega > \beta^{\zeta}$ and $t_1 \tan\theta - |t_2| > \xi$, because $\arg(-\lambda^{\zeta} + \omega) \to \pi - \pi\zeta/2$ as $|\lambda| \to \infty$, $\lambda \in \Gamma_l$ and there exists $R > 0$ such that, for every $t = t_1 + it_2 \in \Sigma_{\theta}$,

$$\begin{aligned}\Big|e^{-t(-\lambda+\omega)^{b/\zeta}}\Big| &= e^{-t_1|-\lambda+\omega|^{b/\zeta}\cos(\frac{b}{\zeta}\arg(-\lambda+\omega))+t_2|-\lambda+\omega|^{b/\zeta}\sin(\frac{b}{\zeta}\arg(-\lambda+\omega))} \\ &\leq e^{-(t_1\cos(\frac{b}{\zeta}\arg(-\lambda+\omega))-|t_2|)|-\lambda+\omega|^{b/\zeta}} \leq e^{-(t_1\tan\theta-|t_2|)|-\lambda+\omega|^{b/\zeta}}, \quad \lambda \in \overline{\Omega},\ |\lambda| \geq R.\end{aligned}$$

It is clear that $f \cdot g,\ f + g,\ P \cdot f \in \mathcal{A}$, provided $f,\ g \in \mathcal{A}$ and $P \in \mathbb{C}[z]$.

Further on, let $n \in \mathbb{N}$, $\mathbb{N}_n = \{1, 2, \ldots, n\}$, $\mathbb{N}^0_n = \{0, 1, \ldots, n\}$, let $p_0, p_1, \cdots, p_n$ and $q_0, q_1, \cdots, q_n$ be given non-negative integers satisfying $p_0 = q_0 = 0$ and $0 < p_1 + q_1 \leq p_2 + q_2 \leq \cdots \leq p_n + q_n$. Let $A_0, A_1, \cdots, A_{n-1}, A_n$ be closed linear operators acting on E. Set $A_0 := A$, $A_n := B$, $T_i u(s) := (\mathbf{D}^{\zeta}_s)^{p_i} A_i (\mathbf{D}^{\zeta}_s)^{q_i} u(s)$, $s \geq 0$, $i \in \mathbb{N}^0_n$, $S_l := \{i \in \mathbb{N}_n : q_i \geq 1\}$, $S_r := \{i \in \mathbb{N}_n : p_i \geq 1\}$,

$$P_{\lambda} := \lambda^{(p_n+q_n)\zeta} B + \sum_{i=0}^{n-1} \lambda^{(p_i+q_i)\zeta} A_i, \quad \lambda \in \mathbb{C} \setminus \{0\},$$

and conventionally, $\max(\emptyset) := \emptyset$, $\mathbb{N}^0_{\emptyset} := \emptyset$.

Under consideration is the following abstract degenerate multi-term Cauchy problem:

$$\sum_{i=0}^{n} T_i u(s) = 0, \quad s \geq 0, \tag{2.5}$$

with the following initial conditions:

$$\begin{cases} \Big(\big(\mathbf{D}^{\zeta}_s\big)^j u(s)\Big)_{s=0} = u_j,\ j \in \mathbb{N}^0_{\max\{q_i-1 : i \in S_l\}}, \\ \Big(\big(\mathbf{D}^{\zeta}_s\big)^j A_i \big(\mathbf{D}^{\zeta}_s\big)^{q_i} u(s)\Big)_{s=0} = u_{i,j}\ \big(i \in S_r,\ j \in \mathbb{N}^0_{p_i-1}\big). \end{cases} \tag{2.6}$$

Before going any further, we would like to point out that the choice of initial values (2.6), which will be considered as an only possible option in the sequel, may be non-optimal because there exist some very natural situations where we cannot expect the existence of solutions of problem (2.5), (2.6), in general. On the other hand, accompanying the fractional differential equation (2.5) by the initial conditions of form (2.6) will enable us to integrate equation (2.5) $((p_n+q_n)\zeta)$-times and obtain the corresponding integral equation associated to problem (2.5), (2.6), which will be of crucial importance in the proof of Theorem 2.1 below. Observe also that, in our concrete situation $0<\zeta\leq 1$, the following fractional Sobolev problems:

$$(\mathrm{DFP})_R : \begin{cases} \mathbf{D}_s^{\zeta} Bu(s) + Au(s) = 0, \quad s \geq 0, \\ Bu(0) = Bx, \end{cases}$$

and

$$(\mathrm{DFP})_L : \begin{cases} B\mathbf{D}_s^{\zeta} u(s) + Au(s) = 0, \quad s \geq 0, \\ u(0) = x, \end{cases}$$

are special cases of problem (2.5), (2.6), with $n=1$, $q_1=0$, $p_1=1$ and $u_{1,0}=Bx$, in the case of problem $(\mathrm{DFP})_R$, and $n=1$, $q_1=1$, $p_1=0$, $u_1=x$, in the case of problem $(\mathrm{DFP})_L$.

The notion of a strong solution of problem (2.5), (2.6) is introduced in the following definition.

Definition 1. A function $u\in C([0,\infty):E)$ is said to be a strong solution to problem (2.5), (2.6) if and only if the term $T_i u(s)$ is well defined and continuous for any $s\geq 0$, $i\in\mathbb{N}_n^0$, and (2.5), (2.6) holds identically on $[0,\infty)$.

In the following theorem, we state a Ljubich type uniqueness theorem for the problem (2.5), (2.6).

Theorem 2.1. *Suppose that an operator $C\in L(E)$ is injective, $CA_i\subseteq A_iC$, $i\in\mathbb{N}_n^0$ and there exists a number $\omega>0$ such that the operator P_λ^{-1} is injective and $D(P_\lambda^{-1}C)=E$ for $\lambda>\omega$. Let the following condition hold:*

(H) *For every $p\in\circledast$ and $i\in\mathbb{N}_n^0$, there exist numbers $\lambda_{p,i}$, $\sigma_{p,i}>0$, a seminorm $q_{p,i}\in\circledast$ and a function $h_{p,i}:(\lambda_{p,i},\infty)\to(0,\infty)$ such that:*

$$p\big(P_\lambda^{-1}CA_ix\big)\leq\big[q_{p,i}(x)+q_{p,i}\big(A_ix\big)\big]h_{p,i}(\lambda),\quad \lambda>\lambda_{p,i},\ x\in D\big(A_i\big),$$

and

$$\lim_{\lambda\to+\infty}e^{-\lambda\sigma_{p,i}}h_{p,i}(\lambda)=0.$$

Then there exists at most one strong solution of problem (2.5), (2.6).

Proof. Clearly, it suffices to show the uniqueness of a strong solution to problem (2.5), (2.6) with all initial values chosen to be zeroes. Let a function $u\in C([0,\infty):E)$ be a strong solution to this problem. Then we can integrate equation (2.5) $((p_n+q_n)\zeta)$-times; taking into account the equality [4, (1.21)] and an elementary argumentation, we get that

$$Bu(s)+\sum_{i=0}^{n-1}A_i\big(g_{((p_n+q_n)-(p_i+q_i))\zeta}*u\big)(s)=0,\quad s\geq 0. \tag{2.7}$$

Convoluting the function $u(\cdot)$ with $g_\delta(\cdot)$, for a sufficiently large number $\delta>0$, we may assume without loss of generality that, for every $i\in\mathbb{N}_n^0$, the mapping $s\mapsto A_iu(s)$, $s\geq 0$ is well-defined and continuous. Set, for every $s\geq 0$ and $\delta>0$, $v_{s,\delta}(\lambda):=(g_\delta*e^{\lambda\cdot})(s)-\lambda^{-\delta}e^{s\lambda}$, $\lambda>0$; $v_{s,0}(\lambda):=0$ $(s\geq 0,\ \lambda>0)$. Then the mapping $s\mapsto v_{s,\delta}(\lambda)$ is continuous in $s\geq 0$, for the numbers $\delta\geq 0$

and $\lambda > 0$ fixed in advance; furthermore, [39, Lemma 1.5.5, p. 23] implies that, for every $s > 0$ and $\delta > 0$, we have

$$\left|v_{s,\delta}(\lambda)\right| = O\Big(\big(1+s\big)^{\delta-1}\lambda^{-1}\big(1+\lambda^{1-\delta}\big) + s^{\delta-1}\lambda^{-1}\Big), \quad \lambda > 0. \tag{2.8}$$

Set, for every index $i \in \mathbb{N}_n^0$, $\beta_i := (p_i + q_i)\zeta$. Keeping in mind (2.7) and the assumption $CA_i \subseteq A_iC$, $i \in \mathbb{N}_n^0$, we have that, for every $s \geq 0$, $\lambda > 0$ and $i \in \mathbb{N}_n^0$,

$$\begin{aligned}
&\lambda^{\beta_i-\beta_n}\int_0^s e^{\lambda(s-r)}A_iCu(r)\,dr + \int_0^s v_{s-r,\beta_n-\beta_i}(\lambda)A_iCu(r)\,dr \\
&= C\int_0^s e^{\lambda(s-r)}\big(g_{\beta_n-\beta_i} * A_iu\big)(r)\,dr = (-C)\sum_{v\in\mathbb{N}_n^0\setminus\{i\}}\int_0^s e^{\lambda(s-r)}\big(g_{\beta_n-\beta_v} * A_vu\big)(r)\,dr \\
&= -\sum_{v\in\mathbb{N}_n^0\setminus\{i\}}\left[\lambda^{\beta_v-\beta_n}\int_0^s e^{\lambda(s-r)}A_vCu(r)\,dr + \int_0^s v_{s-r,\beta_n-\beta_v}(\lambda)A_vCu(r)\,dr\right],
\end{aligned}$$

which clearly implies that, for every $\lambda > \omega$, $\sigma > 0$, $s \geq 0$ and $i \in \mathbb{N}_n^0$, the following equality holds:

$$\begin{aligned}
e^{-\lambda\sigma}\int_0^s e^{\lambda(s-r)}Cu(r)\,dr &= -\lambda^{\beta_n}e^{-\lambda\sigma}P_\lambda^{-1}CA_i\int_0^s v_{s-r,\beta_n-\beta_i}(\lambda)u(r)\,dr \\
&- \lambda^{\beta_n}e^{-\lambda\sigma}\sum_{v\in\mathbb{N}_n^0\setminus\{i\}}P_\lambda^{-1}CA_v\int_0^s v_{s-r,\beta_n-\beta_v}(\lambda)u(r)\,dr.
\end{aligned} \tag{2.9}$$

Making use of condition (H) and (2.8), (2.9), we obtain that, for every $p \in \circledast$, there exists a sufficiently large number $\sigma_p > 0$ such that $\lim_{\lambda\to+\infty} e^{-\lambda\sigma_p}p((e^{\lambda\cdot} * Cu)(s)) = 0$, $s \geq 0$. By the Dominated Convergence Theorem, it readily follows that for each $p \in \circledast$ we have: $\lim_{\lambda\to+\infty} p(\int_0^{s-\sigma} e^{\lambda(s-r-\sigma)}Cu(r)\,dr) = 0$, $s \geq \sigma > \sigma_p$. Therefore,

$$\lim_{\lambda\to+\infty}\int_0^s e^{\lambda(s-r)}Cu(r)\,dr = 0, \quad s \geq 0.$$

Since C is injective, we can apply [25, Lemma 2.1.33(iii)] to conclude that $u(s) = 0$, $s \geq 0$. □

Now we are ready to formulate the following extension of [21, Theorem 2.1].

Theorem 2.2. *Suppose that (M_p) satisfies $(M.1)$, $b \in (0,1)$, $\zeta \in (0,1]$ and (2.1) holds. Let $\nu > -1$, $\xi \geq 0$, $\alpha > 0$, $\beta > 0$, $l \geq 1$, and let (2.4) hold. Suppose, further, that the operator P_λ is injective for all $\lambda \in \Lambda_{\alpha,\beta,l}$, as well as that $P_\lambda^{-1}C \in L(E)$, $\lambda \in \Lambda_{\alpha,\beta,l}$, the mapping $\lambda \mapsto P_\lambda^{-1}Cx$, $\lambda \in \Lambda_{\alpha,\beta,l}$ is continuous for every fixed element $x \in E$, and the operator family*

$$\big\{(1+|\lambda|)^{-\nu}e^{-\xi|\lambda|^b}P_\lambda^{-1}C : \lambda \in \Lambda_{\alpha,\beta,l}\big\} \subseteq L(E)$$

is equicontinuous. Set, for every function $f \in \mathcal{A}$,

$$S_f(s)x := \frac{\zeta}{2\pi i}\int_{\Gamma_l} f\big(\lambda^\zeta\big)\lambda^{\zeta-1}E_\zeta\big(s^\zeta\lambda^\zeta\big)P_\lambda^{-1}Cx\,d\lambda, \quad s \geq 0,\ x \in E. \tag{2.10}$$

Then $(S_f(s))_{s\geq 0} \subseteq L(E)$ is strongly continuous, the mapping $s \mapsto S_f(s) \in L(E)$, $s \geq 0$ ($s \mapsto S_f(s) \in L(E)$, $s > 0$) is infinitely differentiable provided $\zeta = 1$, $f \in \mathcal{A}$ ($\zeta \in (0,1)$, $f \in \mathcal{A}$) and, for every $p \in \mathbb{N}_0$ and $f \in \mathcal{A}$, the mapping $s \mapsto (\mathbf{D}_s^\zeta)^p S_f(s) \in L(E)$, $s \geq 0$ is well-defined, with

$$\big(\mathbf{D}_s^\zeta\big)^p S_f(s)x := \frac{\zeta}{2\pi i}\int_{\Gamma_l} f\big(\lambda^\zeta\big)\lambda^{\zeta-1}\lambda^{p\zeta}E_\zeta\big(s^\zeta\lambda^\zeta\big)P_\lambda^{-1}Cx\,d\lambda, \quad s\geq 0,\ x\in E. \tag{2.11}$$

Furthermore, the following statements hold.

(i) *Suppose that there exists $i \in \mathbb{N}_n^0$ such that the mappings $\lambda \mapsto A_jP_\lambda^{-1}Cx$, $\lambda \in \Lambda_{\alpha,\beta,l}$ are continuous for some $x \in E$ ($j \in \mathbb{N}_n^0 \setminus \{i\}$) and for each seminorm $p \in \circledast$ the set $\big\{(1+|\lambda|)^{-\nu}e^{-\xi|\lambda|^b}p(A_jP_\lambda^{-1}Cx) : \lambda \in \Lambda_{\alpha,\beta,l},\ j \in \mathbb{N}_n^0 \setminus \{i\}\big\}$ is bounded.*

Then we have

$$\big(\mathbf{D}_s^\zeta\big)^p A_i\big(\mathbf{D}_s^\zeta\big)^q S_f(s)x = \frac{\zeta}{2\pi i}\int_{\Gamma_l} f\big(\lambda^\zeta\big)\lambda^{\zeta-1}\lambda^{(p+q)\zeta}E_\zeta\big(s^\zeta\lambda^\zeta\big)A_iP_\lambda^{-1}Cx\,d\lambda, \tag{2.12}$$

for any $x \in E$, $s \geq 0$, $i \in \mathbb{N}_n^0$ and p, $q \in \mathbb{N}_0$. Moreover, the mapping $s \mapsto u(s) := S_f(s)x$, $s \geq 0$ is a strong solution of problem (2.5), (2.6), *with the initial value u_j obtained by plugging $p = j$ and $s = 0$ in the right-hand side of* (2.11), *for $j \in \mathbb{N}^0_{\max\{q_i-1:i\in S_l\}}$, and the initial value $u_{i,j}$ obtained by plugging $p = j$, $q = q_i$ and $s = 0$ in the right-hand side of* (2.12), *for $i \in S_r$ and $j \in \mathbb{N}^0_{p_i-1}$ ($f \in \mathcal{A}$). If $CA_i \subseteq A_iC$ for all $i \in \mathbb{N}_n^0$, then there exists at most one strong solution of problem* (2.5), (2.6).

(ii) *Suppose that $f \in \mathcal{A}$, $q \in \circledast$, $\mathbb{B}$ is a bounded subset of E and K is a compact subset of* $[0,\infty)$.

Then there exists $h_0 > 0$ such that

$$\sup_{p\in\mathbb{N}_0, s\in K, x\in\mathbb{B}} \frac{(h_0)^p q\big((\mathbf{D}_s^\zeta)^p S_f(s)x\big)}{p^{p\zeta/b}} < \infty. \tag{2.13}$$

Proof. We will basically follow the proof of [21, Theorem 2.1]. Let $f \in \mathcal{A}$ be such that (2.3) holds with some numbers $a_1 > 0$ and $a_2 > \xi$. In order to prove that $S_f(s) \in L(E)$ for all $s \geq 0$, observe that Lemma 1.1 in combination with (2.2) and the equicontinuity of the operator family $\{(1+|\lambda|)^{-\nu}e^{-\xi|\lambda|^b}P_\lambda^{-1}C : \lambda \in \Lambda_{\alpha,\beta,l}\}$ (cf. also the asymptotic expansion formulae [4, (1.26)–(1.28)]) implies that for each $p \in \circledast$ there exist $c_p > 0$ and $q \in \circledast$ such that, for any sufficiently small number $\mu > 0$, the following holds with an appropriate constant $M_\mu > 0$:

$$\begin{aligned}&\Big|f(\lambda^\zeta)\lambda^{\zeta-1}E_\zeta\big(s^\zeta\lambda^\zeta\big)p\big(P_\lambda^{-1}Cx\big)\Big| \\ &\qquad\leq a_1M_\mu c_p e^{-(a_2-\xi)|\lambda|^b}e^{s(\beta+c_\mu|\lambda|^b)}(1+|\lambda|)^{\nu+\zeta}q(x),\ \lambda\in\Gamma_l,\ |\lambda|\geq R,\ x\in E.\end{aligned} \tag{2.14}$$

Keeping in mind that $\lim_{\mu\to 0} c_\mu = 0$, we obtain from (2.14) that $S_f(s) \in L(E)$ for all $s \geq 0$, and that the operator family $(S_f(s))_{s\geq 0} \subseteq L(E)$ is strongly continuous. The infinite differentiability of mapping $s \mapsto S_f(s) \in L(E)$, $s \geq 0$ for $\zeta = 1$ and $f \in \mathcal{A}$ can be easily proved.

In order to prove that the mapping $s \mapsto S_f(s) \in L(E)$, $s > 0$ is infinitely differentiable for $\zeta < 1$ and $f \in \mathcal{A}$, we need to recall the well known fact that, for every $l \in \mathbb{N}$, there exist real numbers $(c_{j,\zeta})_{1\leq j\leq l}$ and $(c_{j,l,\zeta})_{1\leq j\leq l}$ such that

$$\frac{d^l}{ds^l}E_\zeta(zs^\zeta) = \sum_{j=1}^{l} c_{j,\zeta}s^{j\zeta-l}E_\zeta^{(j)}(zs^\zeta), \quad s>0,\ z\in\mathbb{C}$$

and

$$\frac{d^l}{dz^l}E_\zeta(z)=\sum_{j=1}^{l}c_{j,l,\zeta}E_{\zeta,\zeta l-(l-j)}(z),\quad z\in\mathbb{C}$$

(cf. [25, Section 1.3]). This implies that, for every $l\in\mathbb{N}$, and for every sufficiently small $h>0$, we have:

$$\frac{E_\zeta^{(l)}\big((s+h)^\zeta\lambda^\zeta\big)-E_\zeta^{(l)}\big(s^\zeta\lambda^\zeta\big)}{h}-\frac{d^{l+1}}{ds^{l+1}}E_\zeta\big(s^\zeta\lambda^\zeta\big)$$
$$=\frac{1}{h}\sum_{j=1}^{l+2}\sum_{i=1}^{j}\int_s^{s+h}\int_s^r c_{j,\zeta}c_{i,j,\zeta}\tau^{j\zeta-(l+2)}E_{\zeta,\zeta j-(i-j)}\big(\tau^\zeta\lambda^\zeta\big)\,d\tau\,dr,\quad s>0,\ \lambda\in\Gamma_l. \tag{2.15}$$

An application of Lemma 1.1 yields that, for every $l\in\mathbb{N}$, there exists a constant $\delta>0$ satisfying that, for every $j\in\mathbb{N}$ with $j\le l+2$, and for every $i\in\mathbb{N}$ with $i\le j$, we have

$$\Big|E_{\zeta,\zeta j-(i-j)}\big(\tau^\zeta\lambda^\zeta\big)\Big|\le\delta\Big[1+\big(\tau\lambda\big)^{(1+(i-j)-\zeta j)/\zeta}e^{\tau\Re\lambda}\Big],\quad \tau>0,\ \lambda\in\Gamma_l.$$

Combining this estimate with (2.15), it readily follows that the mapping $s\mapsto S_f(s)\in L(E)$, $s>0$ is l-times continuously differentiable, with

$$\frac{d^l}{ds^l}S_f(s)x=\frac{\zeta}{2\pi i}\int_{\Gamma_l}f\big(\lambda^\zeta\big)\lambda^{\zeta-1}\frac{d^l}{ds^l}\big[E_\zeta\big(s^\zeta\lambda^\zeta\big)\big]P_\lambda^{-1}Cx\,d\lambda,\quad s>0,\ x\in E. \tag{2.16}$$

Using the identity $\lambda^\zeta(g_{\lceil\zeta\rceil}*E_\zeta(\cdot^\zeta\lambda^\zeta))(s)=(g_{\lceil\zeta\rceil-\zeta}*[E_\zeta(\cdot^\zeta\lambda^\zeta)-1])(s)$, $s\ge0$, $\lambda\in\Gamma_l$ (see e.g. [4, (1.25)] and the proof of [25, Lemma 3.3.1]) and a straightforward integral computation, it is checked at once that for each $x\in E$ and $s\ge0$ we have:

$$\Big[g_{\lceil\zeta\rceil-\zeta}*\big(S_f(\cdot)x-S_f(0)x\big)\Big](s)=\left[g_{\lceil\zeta\rceil}*\frac{\zeta}{2\pi i}\int_{\Gamma_l}f\big(\lambda^\zeta\big)\lambda^{\zeta-1}\lambda^\zeta E_\zeta\big(\cdot^\zeta\lambda^\zeta\big)P_\lambda^{-1}Cx\,d\lambda\right](s).$$

This implies the validity of (2.11) with $p=1$. Inductively, we obtain that (2.11) holds for any integer $p\in\mathbb{N}$ by repeating literally the above arguments.

Suppose now that the requirements of (i) hold for some element $x\in E$. Using the resolvent equation, we obtain that the mappings $\lambda\mapsto A_iP_\lambda^{-1}Cx$, $\lambda\in\Lambda_{\alpha,\beta,l}$ are continuous for all $i\in\mathbb{N}_n^0$ and that there exists a number $\nu'>0$ such that for each seminorm $p\in\circledast$ the set

$$\{(1+|\lambda|)^{-\nu'}e^{-\xi|\lambda|^b}p(A_iP_\lambda^{-1}Cx):\lambda\in\Lambda_{\alpha,\beta,l},\ i\in\mathbb{N}_n^0\}$$

is bounded, which clearly implies that the mapping $s\mapsto A_iS_f(s)x$, $s\ge0$ is well defined for any $x\in E$ and $i\in\mathbb{N}_n^0$. Hence (2.12) holds for any $x\in E$, $s\ge0$, $i\in\mathbb{N}_n^0$ and $p,\ q\in\mathbb{N}_0$. Using the substitution $z=\lambda^\zeta$, Lemma 1.1 and the Cauchy formula, we get that

$$\int_{\Gamma_l}f\big(\lambda^\zeta\big)\lambda^{\zeta-1}E_\zeta\big(s^\zeta\lambda^\zeta\big)\,d\lambda=0,\quad s\ge0. \tag{2.17}$$

By (2.12), (2.17), it readily follows that the mapping $s\mapsto u(s)=S_f(s)x$, $s\ge0$ is a strong solution to problem (2.5), (2.6) with the prescribed set of initial values.

If $CA_i\subseteq A_iC$ for all $i\in\mathbb{N}_n^0$, then the uniqueness of a strong solution to associated integral equation (2.7) is an immediate corollary of Theorem 2.1, finishing in a routine manner the proof of (i). The existence of a number $h_0>0$ in (ii) and the proof of inequality (2.13) follows from (2.2), (2.11) and a simple computation. □

Remark 1. (i) In the case $\zeta < 1$, Theorem 2.2 seems to be new and not considered elsewhere (even in the case $B = I$). If $\zeta = 1$, then there exist two possibilities: $n = 1$ and $n > 1$. If $n = 1$ and $\zeta = 1$, then the assertion of Theorem 2.2 seems to be new in the case in which E is not a Banach space and $B \neq I$, or $B \neq I$ and $C \neq I$ (cf. [24, Theorem 3.16, Example 4.5] for some results in locally convex spaces, with $B = I$). If $n = 1$, $\zeta = 1$ and E is a Banach space, then it is worth noting that A. Favini [13] was the first who considered R. Beals's type regularization process [5, 6] for seeking solutions to degenerate equations of the first order, provided in addition that $C = I$ (cf. also [14, Section 5.4] for the case $B \neq I$, as well as [23, Section 1.4], [25, Section 2.9], [39, Section 4.4], [20, 21, 37] for more details concerning the case $B = I$). If $n > 1$ and $\zeta = 1$, then the assertion of Theorem 2.2 seems to be considered only in the case in which $C = I$, $\xi = 0$, $p_i = 0$ for all $i \in \mathbb{N}_n^0$, and E is a Banach space (cf. [13, Application 2, Assumption H.10] and compare with our assumptions made in (i) of Theorem 2.2). Finally, it is needless to say that the usual converting of higher-order (degenerate) differential equations to first order matrix (degenerate) differential equations, used in numerous papers on higher-order abstract differential equations and, in particular, in the above-mentioned Application 2 of [13], cannot offer significant help in the analysis of problem (2.5), (2.6), in general.

(ii) Let $v \in \mathbb{Z}$, let $f \in \mathcal{A}$, and let an element $x \in E$ satisfy the requirements of (i). Define

$$S_{f,v}(s)x := \frac{\zeta}{2\pi i} \int_{\Gamma_l} f\left(\lambda^\zeta\right) \lambda^{\zeta-1} \lambda^{v\zeta} E_\zeta\left(s^\zeta \lambda^\zeta\right) P_\lambda^{-1} C x \, d\lambda, \quad s \geq 0, \ x \in E.$$

Then the mapping $s \mapsto S_{f,v}(s)x$, $s \geq 0$ is a strong solution to problem (2.5), with initial values (2.6) endowed similarly as in the formulation of (i).

(iii) In the formulation of [21, Theorem 2.1], it has been additionally assumed that the sequence (M_p) satisfies condition $(M.2)$. The proof of Theorem 2.2 shows that we can completely neglect this condition from our analysis.

(iv) It is worth noting that the term $\mathbf{D}_s^{2\zeta} u(s)$ need not be defined for some functions $s \mapsto u(s)$, $s \geq 0$ for which the term $(\mathbf{D}_s^\zeta)^2 u(s)$ is defined (for example, in the case $\zeta = 1/2$, $r > 0$ and $u(s) = E_{1/2}(r^{1/2} s^{1/2})$, $s \geq 0$). Even in the case in which both terms exist, they can be completely different so that we have to make a strict distinction between the operator $(\mathbf{D}_s^\zeta)^p$ and the operator $\mathbf{D}_s^{\zeta p}$. As explained in [26, Remark 2(iii)], the method proposed in Theorem 2.2 cannot be used for proving the existence of a strong solution to (non-degenerate) problem

$$B \mathbf{D}_s^{\alpha_n} u(s) + \sum_{i=0}^{n-1} A_i \mathbf{D}_s^{\alpha_i} u(s) = 0, \quad s \geq 0, \tag{2.18}$$

provided that $n > 1$ and there exists an index $i \in \mathbb{N}_n^0$ such that the order α_i of the Caputo fractional derivative $\mathbf{D}_s^{\alpha_i} u(s)$ does not belong to $\mathbb{N}_0$. Here, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n$.

(v) It can be easily verified that

$$\int_0^s g_\zeta(s-r)(-A) S_f(r) x \, dr = B S_f(s) x - B S_f(0) x,$$

provided that $n = 1$, $s \geq 0$, $f \in \mathcal{A}$ and $x \in E$ satisfies the requirements of Theorem 2.2 (i).

(vi) It is well known that the notion of an abstract Beurling space plays an important role in the theory of ultradistribution semigroups in Banach spaces (cf. Theorem 2.2 (i) with $n = 1$ and $B = I$). Unfortunately, it is very difficult to introduce a similar concept for degenerate differential equations of the first order, especially in the case in which the operator B is not injective. For the purpose of illustration of Theorem 2.2 (i), we shall present two examples in which we use the abstract Beurling spaces:

(vi.1) Suppose that $n=1$, $x \in D_\infty(B)$, the element $B^p x$ satisfies the requirements of Theorem 2.2 (i) for all $p \in \mathbb{N}_0$, and

$$B(zB+A)^{-1}CB^p x = (zB+A)^{-1}CB^{p+1}x, \quad p \in \mathbb{N}_0, \quad z \in \Lambda^{\zeta}_{\alpha,\beta,l}.$$

Then

$$A^p S_f(s)x = \frac{(-1)^p \zeta}{2\pi i}\int_{\Gamma_l} f(\lambda^\zeta)\lambda^{\zeta-1}\lambda^{p\zeta}E_\zeta(s^\zeta\lambda^\zeta)P_\lambda^{-1}CB^p x\, d\lambda,\ s\geq 0,\ p\in\mathbb{N},\ f\in\mathcal{A}.$$

This, in turn, implies

$$\bigcup_{s\geq 0, f\in\mathcal{A}} \{S_f(s)x\} \subseteq E^{\langle p^{p\zeta/b}\rangle}(A),$$

provided that the orbit $\{B^p x : p \in \mathbb{N}_0\}$ is bounded, and

$$\bigcup_{s\geq 0, f\in\mathcal{A}} \{S_f(s)x\} \subseteq E^{\langle p^{2p\zeta/b}\rangle}(A),$$

provided that $Bx \in E^{\langle p^{p\zeta/b}\rangle}(A)$.

(vi.2) (cf. also Remark 2) Suppose that $n=1$, B is injective and an element $x \in E$ satisfies the requirements of Theorem 2.2 (i). Then B^{-1} is closed and we can inductively prove that

$$(B^{-1}A)^p S_f(s)x = (-1)^p(\mathbf{D}_s^\zeta)^p S_f(s)x, \quad s\geq 0, \quad p\in\mathbb{N}, \quad f\in\mathcal{A}.$$

Taking into account (2.13), the above implies that

$$\bigcup_{s\geq 0, f\in\mathcal{A}} \{S_f(s)x\} \subseteq E^{\langle p^{p\zeta/b}\rangle}(B^{-1}A).$$

(vii) Let $f \in \mathcal{A}$, let $\varepsilon > 0$, and let $g : \mathbb{C}\setminus\Lambda^{\zeta}_{\alpha,\beta,l+\varepsilon} \to \mathbb{C}$ be continuous in $D(g)$ and analytic in $\operatorname{int}(D(g))$. Suppose, further, that there exist constants $a_1' > 0$ and $a_2' > \xi$ such that (2.3) holds with $f=g$, $a_1 = a_1'$, $a_2 = a_2'$, $\lambda \in D(g)$, as well as that $n=1$ and the family

$$\{(1+|\lambda|)^{-\nu}e^{-\xi|\lambda|^b}BP_\lambda^{-1}Cx : \lambda \in \Lambda_{\alpha,\beta,l}\} \subseteq L(E)$$

is both equicontinuous and strongly continuous. Let

$$CB\big(zB+A\big)^{-1}C = B\big(zB+A\big)^{-1}C^2, \quad z \in \Lambda^{\zeta}_{\alpha,\beta,l}, \tag{2.19}$$

and let Γ_l^ζ ($\Gamma_{l,\varepsilon}^\zeta$) denote the upwards oriented boundary of $\Lambda^{\zeta}_{\alpha,\beta,l}$ ($\Lambda^{\zeta}_{\alpha,\beta,l+\varepsilon}$). Then, for every $z,\ z' \in \Lambda^{\zeta}_{\alpha,\beta,l}$ and $x \in E$, the resolvent equation

$$\big(zB+A\big)^{-1}C^2x - \big(z'B+A\big)^{-1}C^2x = (z'-z)\big(zB+A\big)^{-1}CB\big(z'B+A\big)^{-1}Cx, \tag{2.20}$$

holds, which implies that the mapping $z \mapsto B(zB+A)^{-1}C^2x$, $z \in \operatorname{int}(\Lambda^{\zeta}_{\alpha,\beta,l})$ is analytic ($x \in E$). Using (2.19) and the arguments from [11, Remark 2.7], the above implies that the mapping $z \mapsto B(zB+A)^{-1}Cx$, $z \in \operatorname{int}(\Lambda^{\zeta}_{\alpha,\beta,l})$ is analytic, as well ($x \in E$). Applying the substitution $z=\lambda^\zeta$ and the Cauchy formula, we then get that

$$BS_g(0)x = (2\pi i)^{-1}\int_{\Gamma^\zeta_{l,\varepsilon}} g(z)B(zB+A)^{-1}Cx\, dz, \quad x \in E.$$

Proceeding as in the proof of [6, Lemma 4.2], it readily follows that

$$S_f(0)BS_g(0)x = S_{fg}(0)Cx, \quad x \in E.$$

If $\xi = 0$, $f(\lambda) = f_t(\lambda)$ and $g(\lambda) = f_s(\lambda)$, with $t,\ s \in \Sigma_\theta$, the above means that

$$T(t)BT(s) = T(t+s)C.$$

(viii) If $B = I$, $n = 1$, $P_\lambda^{-1}C$ exists and is polynomially bounded on the region $\Lambda_{\alpha,\beta,l}$ (with the clear meaning), then it might be surprising that we must impose condition (2.4) in order to ensure the existence of a strong solution to problem $(\mathrm{DFP})_R$ with the initial value $x \neq 0$. If we replace condition (2.4) with the condition $\frac{1}{2-\zeta} \leq \frac{b}{\zeta}$ (which clearly implies $\zeta < 1$ and the triviality of the class $\mathcal{A}$), and accept all the remaining assumptions from the formulation of this theorem, with $B = I$, $n = 1$ and $\xi = 0$, then it is not clear whether there exist a Hilbert space (Banach space, sequentially complete locally convex space) E and a closed linear operator A acting on E such that the problem $(\mathrm{DFP})_R$ has no local strong solutions unless $x = 0$ (cf. [5, Theorem 2, Theorem 2'] for more details concerning the case $\zeta = 1$). This is a very interesting open problem which we would like to address to our readers.

(ix) If the assumptions of Theorem 2.2 hold with the region $\Lambda_{\alpha,\beta,l}$ replaced by the right half-plane $RHP_{\bar\omega} \equiv \{z \in \mathbb{C} : \Re z > \bar\omega\}$ (and with the set Ω replaced by the set $\mathbb{C} \setminus (RHP_{\bar\omega})^\zeta$), then for each $p \in \mathbb{N}_0$ and $f \in \mathcal{A}$ the operator family $\{e^{-\bar\omega s}(\mathbf{D}_s^\zeta)^p S_f(s) : s \geq 0\}$ is equicontinuous ($\bar\omega > 0$); cf. [3] for corresponding examples. It is also worth noting that we can consider, instead of the region $\Lambda_{\alpha,\beta,l}$ considered above, a region of the form $\Omega(\omega) = \{\lambda \in \mathbb{C} : \Re\lambda \geq \max(x_0, \omega(|\Im\lambda|))\}$, where $x_0 > 0$, $\omega : [0,\infty) \to [0,\infty)$ is a continuous, concave, increasing function satisfying

$$\lim_{t\to\infty} \omega(t) = \infty,\ \lim_{t\to\infty} \frac{\omega(t)}{t} = 0 \text{ and } \int_1^\infty \frac{\omega(t)}{t^2}\,dt < \infty$$

(cf. [5, 6, 21, 23], and [24, Example 4.5]), or the exponential region

$$E(a,b) = \{\lambda \in \mathbb{C} : \Re\lambda \geq b,\ |\Im\lambda| \leq e^{a\Re\lambda}\} \quad (a,\ b > 0),$$

introduced for the first time by W. Arendt, O. El-Mennaoui and V. Keyantuo in [2] (cf. also C. Foias [19] for a very similar notion of the logarithmic region $\Lambda(\alpha,\beta,\omega)$ which can also be used here). It would take too long to go into further details concerning these questions here.

(x) Suppose $x \in \bigcap_{v=0}^n D(A_v)$, $i \in \mathbb{N}_n^0$, $j \in \mathbb{N}_0$, $(f_\epsilon(\lambda))_{\epsilon>0}$ is a net of functions in $\mathcal{A}$ and $CA_v \subseteq A_vC$, $v \in \mathbb{N}_n^0$. Denote $u_{i,\epsilon}^j := ((\mathbf{D}_s^\zeta)^j S_{f_\epsilon}(s)A_ix)_{s=0}$ ($\epsilon > 0$). Then the following equality holds:

$$P_\lambda^{-1}CA_ix = \lambda^{-(p_i+q_i)\zeta}[Cx - \sum_{v\in\mathbb{N}_n^0\setminus\{i\}} \lambda^{(p_v+q_v)\zeta} P_\lambda^{-1}CA_vx], \quad \lambda \in \Lambda_{\alpha,\beta,l},$$

which implies that

$$u_{i,\epsilon}^j = \frac{1}{2\pi i}\int_{\Gamma_l^\zeta} f_\epsilon(\lambda)\lambda^{j-(p_i+q_i)}\left[Cx - \sum_{v\in\mathbb{N}_n^0\setminus\{i\}} \lambda^{(p_v+q_v)} P_{\lambda^{1/\zeta}}^{-1}CA_vx\right] d\lambda.$$

If we impose some additional conditions on the net $(f_\epsilon(\lambda))_{\epsilon>0}$ (for example, the condition that $f_\epsilon(0) \neq 0$, $\epsilon > 0$, as well as $f_\epsilon^{(p_i+q_i-j-1)}(0) \to z_0^{i,j}$ as $\epsilon \to 0$, provided $p_i + q_i - j - 1 \geq 0$, and the limit equality $f_\epsilon(\lambda) \to 1$ as $\epsilon \to 0$ ($\lambda \in \Gamma_l^\zeta$), uniformly on compacts of Γ_l^ζ, at least) and if we suppose that the operator family $\{(1+|\lambda|)^{-\nu'}P_\lambda^{-1}C : \lambda \in \Lambda_{\alpha,\beta,l}\} \subseteq L(E)$ is both equicontinuous and strongly continuous for a sufficiently large negative number $\nu' < 0$ (cf. also [39, Theorem 4.2, p. 168], where it has been assumed that $C = I$), then we may apply the Dominated Convergence

Theorem and the Residue Theorem in order to see that $\lim_{\epsilon\to 0} u^j_{i,\epsilon}$ equals 0, if $j \geq p_i + q_i$, and $[(p_i + q_i - j - 1)!]^{-1} z_0^{i,j} Cx$, otherwise. If we use the net of functions of the form

$$f_\epsilon(\lambda) = e^{-\epsilon(-\lambda+\omega)^{b/\zeta}} \ (\epsilon > 0),$$

then we have that $z_0^{i,j} = 1$ if $p_i + q_i - j - 1 = 0$, and $z_0^{i,j} = 0$ if $p_i + q_i - j - 1 > 0$ [37, 39]. Suppose now that $x_w \in \bigcap_{v=0}^n D(A_v)$ for all $w \in \mathbb{N}^0_{q_n-1}$, the elements Bx_w satisfy the assumption (i) of Theorem 2.2 for all $w \in \mathbb{N}^0_{q_n-1}$, $i = n$, $\nu' < -\zeta q_{n-1}$, the function $f_\epsilon(\lambda)$ is chosen as above, and $p_v = 0$, $v \in \mathbb{N}^0_n$ (with the exception of problem $(\mathrm{DFP})_R$, the analysis becomes very difficult in the case in which there exists $v_0 \in \mathbb{N}^0_n$ such that $p_{v_0} > 0$). By the foregoing arguments, we have that the function

$$s \mapsto \sum_{w=0}^{q_n-1} \frac{1}{2\pi i} \int_{\Gamma_l^\zeta} f_\epsilon(\lambda) \lambda^{-w-1+(p_n+q_n)} E_\zeta\big(s^\zeta \lambda\big) P^{-1}_{\lambda^{1/\zeta}} CBx_w \, d\lambda, \quad s \geq 0$$

is a strong solution to problem (2.5) with the initial values $(u_0^\epsilon, \cdots, u^\epsilon_{q_n-1})$, converging to $(Cx_0, \cdots, Cx_{q_n-1})$ as $\epsilon \to 0+$. Hence, the set $\mathfrak{W}$ consisting of all initial values $(y_0, \cdots, y_{q_n-1}) \in E^{q_n}$ subjected to some strong solution $s \mapsto u(s)$, $s \geq 0$ of problem (2.5) is dense in $(C(\bigcap_{v=0}^n D(A_v)))^{q_n}$ (cf. Example 2 below for an interesting application of this result, with C not being the identity operator). Generally, it is very difficult to say anything else about the set $\mathfrak{W}$ in the case $n > 1$.

(xi) Following the method employed in the proof of Theorem 2.2, one can extend the assertions of [22, Theorem 2.1, Theorem 2.2] to abstract degenerate (multi-term) fractional differential equations, thus proving some results on the C-wellposedness of problem (2.5), (2.6) in the case $\zeta > 2$ ([22, Theorem 2.1]) and $2 \geq \zeta > 1$ ([22, Theorem 2.2]). Consider, for example, the case $2 \geq \zeta > 1$. Let $\vartheta \in (\pi(2-\zeta)/2, \pi/2)$, let $b \in (1/\zeta, \pi/(2(\pi - \vartheta)))$ and let $z \in \Sigma_{\vartheta'}$, where $\vartheta' := \arctan(\cos(b(\pi - \vartheta)))$. If there exist $d \in (0,1]$ and $\nu > -1$ such that the operator family

$$\{(1 + |\lambda|)^{-\nu} P_\lambda^{-1} C : \lambda \in \Sigma_{\vartheta/\zeta} \cup B_d\} \subseteq L(E)$$

is both equicontinuous and strongly continuous (for the sake of simplicity, we shall only consider the case $\xi = 0$), then for each number $s \geq 0$ we can define the bounded linear operator $S(s)$ by

$$S(s)x := \frac{1}{2\pi i} \int_{\Gamma_{\zeta,d}} e^{-z(-\lambda)^b} E_\zeta\big(s^\zeta \lambda\big) P^{-1}_{\lambda^{1/\zeta}} Cx \, d\lambda, \quad x \in E, \ s \geq 0,$$

where $c \in (0, \vartheta)$ is chosen so that $b \in (1/\zeta, \pi/(2(\pi - c)))$ and the inequality $\vartheta < \arctan(\cos(b(\pi - c)))$ is valid (cf. (2.10) and apply the substitution $\lambda \mapsto \lambda^\zeta$). Suppose, further, that there exists $i \in \mathbb{N}^0_n$ such that the mappings $\lambda \mapsto A_j P_\lambda^{-1} Cx$, $\lambda \in \Sigma_{\vartheta/\zeta} \cup B_d$ are continuous for some $x \in E$ $(j \in \mathbb{N}^0_n \setminus \{i\})$ and for each seminorm $p \in \circledast$ the set

$$\{(1 + |\lambda|)^{-\nu} p(A_j P_\lambda^{-1} Cx) : \lambda \in \Sigma_{\vartheta/\zeta} \cup B_d, \ j \in \mathbb{N}^0_n \setminus \{i\}\}$$

is bounded. Then the final conclusions stated in Theorem 2.2 continue to hold after some obvious modifications. In the situation of [22, Theorem 2.1] (the case $\zeta > 2$), which is very specific, we can assume that the operators $P_\lambda^{-1} C$ exist on a certain region of the complex plane which does not contain any acute angle. The interested reader may try to carry out details concerning the transmitting our previous results and comments from the items (i)–(x) of this remark to the case in which $\zeta > 1$. The method proposed in [22], [39, Section 4.4, pp. 167–175] as well as in the parts (x), (xi) of this remark can serve to prove some results on the existence of entire solutions of degenerate multi-term differential equations with integer order derivatives. For more details, see [15].

The proof of following extension of [21, Corollary 2.1] is omitted because of similarity to the previous proof.

Theorem 2.3. *Suppose that* $0 < c < b < \zeta \leq 1$, $\sigma > 0$, $\nu > -1$, $\xi \geq 0$, $\varsigma > 0$ *and* (2.4) *holds. Denote*

$$\Pi_{c,\sigma,\varsigma} := \{\lambda \in \mathbb{C} : \Re\lambda \geq \sigma|\Im\lambda|^c + \varsigma\},\ \Pi^\zeta_{c,\sigma,\varsigma} := \{\lambda^\zeta : \lambda \in \Pi_{c,\sigma,\varsigma}\},\ \Omega' := \mathbb{C} \setminus \Pi^\zeta_{c,\sigma,\varsigma}.$$

Let $f : \overline{\Omega'} \to \mathbb{C}$ *be a continuous function that is analytic in* Ω' *and satisfy the following condition: there exist numbers* $a_1 > 0$ *and* $a_2 > \xi$ *such that*

$$|f(\lambda)| \leq a_1 e^{-a_2|\lambda|^{b/\zeta}}, \quad \lambda \in \overline{\Omega'}.$$

Suppose, further, that the operator P_λ *is injective for all* $\lambda \in \Pi_{c,\sigma,\varsigma}$, *as well as that* $P_\lambda^{-1}C \in L(E)$, $\lambda \in \Pi_{c,\sigma,\varsigma}$, *the mapping* $\lambda \mapsto P_\lambda^{-1}Cx$, $\lambda \in \Pi_{c,\sigma,\varsigma}$ *is continuous for every fixed element* $x \in E$, *and the operator family*

$$\{(1+|\lambda|)^{-\nu} e^{-\xi|\lambda|^b} P_\lambda^{-1}C : \lambda \in \Pi_{c,\sigma,\varsigma}\} \subseteq L(E)$$

is equicontinuous. Set

$$T_f(s)x := \frac{\zeta}{2\pi i} \int_{\Gamma_c} f(\lambda^\zeta)\lambda^{\zeta-1} E_\zeta(s^\zeta\lambda^\zeta) P_\lambda^{-1}Cx\, d\lambda, \quad s \geq 0,\ x \in E, \tag{2.21}$$

where Γ_c *denotes the upwards oriented boundary of* $\Pi_{c,\sigma,\varsigma}$.

Then $(T_f(s))_{s\geq 0} \subseteq L(E)$ *is strongly continuous, the mapping* $s \mapsto T_f(s) \in L(E)$, $s \geq 0$ ($s \mapsto T_f(s) \in L(E)$, $s > 0$) *is infinitely differentiable provided* $\zeta = 1$ ($\zeta \in (0,1)$) *and, for every* $p \in \mathbb{N}_0$, *the mapping* $s \mapsto (\mathbf{D}_s^\zeta)^p T_f(s) \in L(E)$, $s \geq 0$ *is well-defined. Furthermore,* (2.11) *holds with* $(S_f(s))_{s\geq 0}$ *and* Γ_l *replaced respectively by* $(T_f(s))_{s\geq 0}$ *and* Γ_c, *and the following statements hold.*

(i) *Suppose that there exists* $i \in \mathbb{N}_n^0$ *such that the mappings* $\lambda \mapsto A_j P_\lambda^{-1}Cx$, $\lambda \in \Pi_{c,\sigma,\varsigma}$ *are continuous for some* $x \in E$ ($j \in \mathbb{N}_n^0 \setminus \{i\}$) *and for each seminorm* $p \in \circledast$ *the set*

$$\{(1+|\lambda|)^{-\nu} e^{-\xi|\lambda|^b} p(A_j P_\lambda^{-1}Cx) : \lambda \in \Pi_{c,\sigma,\varsigma},\ j \in \mathbb{N}_n^0 \setminus \{i\}\}$$

is bounded.

Then (2.12) *holds with* $(S_f(s))_{s\geq 0}$ *and* Γ_l *replaced respectively by* $(T_f(s))_{s\geq 0}$ *and* Γ_c, *the mapping* $s \mapsto u(s) := T_f(s)x$, $s \geq 0$ *is a strong solution to problem* (2.5), (2.6), *with the initial value* u_j *obtained by plugging* $p = j$ *and* $s = 0$ *into the right-hand side of* (2.11), *for* $j \in \mathbb{N}^0_{\max\{q_i-1 : i\in S_l\}}$, *and the initial value* $u_{i,j}$ *obtained by plugging* $p = j$, $q = q_i$ *and* $s = 0$ *into the right-hand side of* (2.12), *for* $i \in S_r$ *and* $j \in \mathbb{N}^0_{pi-1}$ (*with the obvious replacements described above*). *If* $CA_i \subseteq A_iC$ *for all* $i \in \mathbb{N}_n^0$, *then there exists at most one strong solution of problem* (2.5), (2.6).

(ii) *Suppose that* $q \in \circledast$, $\mathbb{B}$ *is a bounded subset of* E *and* K *is a compact subset of* $[0,\infty)$.

Then there exists $h_0 > 0$ *such that* (2.13) *holds.*

In the following remark, we shall clarify a few important facts closely linked with the assertions of [6, Lemma 1, Lemma 4] and Remark 1 (ii), (vi.1).

Remark 2. Consider the situation of Theorem 2.3 with $\xi = 0$ and $n = 1$ (the final conclusions continue to hold in the case of consideration of Theorem 2.2; after the replacement of the region $\Pi^\zeta_{c,\sigma,\varsigma}$ by $\Lambda^\zeta_{\alpha,\beta,l}$, one just has to make some obvious terminological changes). Suppose, additionally, that $b/\zeta < 1/(2-\zeta)$, $\omega > \varsigma^\zeta$, $CA \subseteq AC$, $CB \subseteq BC$, B is injective,

$$B^{-1}A(zB+A)^{-1}Cx = (zB+A)^{-1}CB^{-1}Ax, \quad x \in D_\infty(B^{-1}A),\ z \in \Pi^\zeta_{c,\sigma,\varsigma}, \tag{2.22}$$

the family

$$\{(1+|z|)^{-\nu}B(zB+A)^{-1}C : z \in \Pi^{\zeta}_{c,\sigma,\varsigma}\} \subseteq L(E)$$

is both equicontinuous and strongly continuous, $y \in D(B)$ satisfies that $Cy \in D(A)$, $BACy = ACBy$ and there exists $h_0 > 0$ such that the set $\{h_0^p p^{(-p\zeta)/b}(B^{-1}A)^p By : p \in \mathbb{N}_0\}$ is bounded. Then it is checked at once that $C(zB+A)^{-1}C = (zB+A)^{-1}C^2$, $z \in \Pi^{\zeta}_{c,\sigma,\varsigma}$ and that the mapping $z \mapsto (zB+A)^{-1}Cx$, $z \in \operatorname{int}(\Pi^{\zeta}_{c,\sigma,\varsigma})$ is analytic ($x \in E$); cf. Remark 1 (vii). Using the Cauchy formula and the foregoing arguments, we have that

$$S_{f_t}(0)x = \frac{1}{2\pi i}\int_{\Gamma^{\zeta}_c-\omega} e^{-t(-\lambda)^{b/\zeta}}((\lambda+\omega)B+A)^{-1}Cx\,d\lambda \quad (x \in E,\ t > 0),$$

where Γ^{ζ}_c denotes the upwards oriented boundary of $\Pi^{\zeta}_{c,\sigma,\varsigma}$. Let the curve Γ' be sufficiently close to Γ^{ζ}_c, on the right of Γ^{ζ}_c, and let the curve $\Gamma'_\omega := \Gamma' - \omega$ be upwards oriented. Modifying slightly the second part of the proof of [6, Lemma 4] (the proof of this lemma contains some typographical mistakes but the essence and final conclusions are true; we can apply Stirling's formula here), and keeping in mind the boundedness of the set $\{h_0^p p^{(-p\zeta)/b}(B^{-1}A)^p By : p \in \mathbb{N}_0\}$, we get that there exists a number $\delta > 0$ such that for each integer $p \in \mathbb{N}$ there exists an integer $n(p) \in \mathbb{N} \cap \big(\frac{bp}{\zeta}+\nu+2, \frac{bp}{\zeta}+\nu+3\big]$ such that the series $\sum_{p=0}^{\infty} x_p$ and $\sum_{p=0}^{\infty} Bx_p$ are convergent, where

$$x_p := \frac{\delta^p}{2\pi i p!}\int_{\Gamma'_\omega} (-\lambda)^{bp/\zeta}(\lambda+\omega)^{-n(p)}\big((\lambda+\omega)B+A\big)^{-1}C\big(B^{-1}A\big)^{n(p)}By\,d\lambda.$$

Let $x = \sum_{p=0}^{\infty} x_p$ and $Bx = \sum_{p=0}^{\infty} Bx_p$; arguing as in Remark 1 (vii), we obtain with the help of equation (2.22), the Cauchy formula and the resolvent equation that

$$S_{f_t}(0)Bx = \frac{1}{2\pi i}\int_{\Gamma'_\omega} e^{-(t-\delta)(-\lambda)^{b/\zeta}}\big((\lambda+\omega)B+A\big)^{-1}C^2By\,d\lambda, \quad t > \delta. \tag{2.23}$$

Let $\lambda_0 \in \mathbb{C}$ be on the right of Γ^{ζ}_c, and simultaneously, on the left of Γ'. Making use of the identity [24, (3.16)], with the operator A replaced by $-B^{-1}A$ therein (we only need the linearity of operator $B^{-1}A$, not its closedness), we get that

$$\begin{aligned} CBy = &\sum_{j=0}^{\lceil\nu\rceil+2}\frac{(-1)^j}{((\lambda+\omega)-\lambda_0)^{j+1}}\big((\lambda+\omega)+B^{-1}A\big)CBy \\ &+ (-1)^{\lceil\nu\rceil+1}\frac{C\big(\lambda_0 I + B^{-1}A\big)^{\lceil\nu\rceil+3}By}{((\lambda+\omega)-\lambda_0)^{\lceil\nu\rceil+3}}, \quad \lambda \in \Gamma'_\omega. \end{aligned} \tag{2.24}$$

Since $Cy \in D(A)$, $BACy = ACBy$ and $CB \subseteq BC$, we have that $((\lambda+\omega)+B^{-1}A)CBy = ((\lambda+\omega)B+A)Cy$, $\lambda \in \Gamma'_\omega$. Applying the operator $((\lambda+\omega)B+A)^{-1}C$ to the both sides of (2.24), the above implies

$$\begin{aligned} \big((\lambda+\omega)B+A\big)^{-1}C^2By = &\sum_{j=0}^{\lceil\nu\rceil+2}\frac{(-1)^j}{((\lambda+\omega)-\lambda_0)^{j+1}}C^2y \\ &+ (-1)^{\lceil\nu\rceil+1}\frac{\big((\lambda+\omega)B+A\big)^{-1}C}{((\lambda+\omega)-\lambda_0)^{\lceil\nu\rceil+3}}\big(\lambda_0 I + B^{-1}A\big)^{\lceil\nu\rceil+3}CBy, \quad \lambda \in \Gamma'_\omega. \end{aligned}$$

Inserting this expression in (2.23), and using after that the limit equality [37, Lemma 2.7; p. 543], as well as the Residue Theorem and the Dominated Convergence Theorem, we obtain that $S_{f_\delta}(0)Bx = \lim_{t\to\delta+} S_{f_t}(0)Bx = C^2y$. Keeping in mind Remark 1 (vi.2), the above implies

$$C^2\big(E(A;B)\big) \subseteq \bigcup_{t>0} R(S_{f_t}(0)B) \subseteq \bigcup_{t>0} R(S_{f_t}(0)) \subseteq E^{\langle p^{p\zeta/b}\rangle}(B^{-1}A), \tag{2.25}$$

where

$$E(A;B) := \Big\{ y \in B^{-1}\big(E^{\langle p^{p\zeta/b}\rangle}(B^{-1}A)\big) : Cy \in D(A),\ BACy = ACBy \Big\}.$$

We do not know, in the present situation, whether equation (2.25) continues to hold if we replace the term $C^2(E(A;B))$ with $C(E(A;B))$. It is also worth noting that the inclusions stated in (2.25) are completely new provided that $C \neq I$ or $\zeta < 1$.

Remark 3. Consider the case $B = I$, $n = 1$ and $\zeta \in (0,1)$. As before, we assume that (M_p) is a sequence of positive real numbers satisfying $M_0 = 1$ and $(M.1)$, as well as that there exist numbers $l \geq 1$, $\alpha > 0$, $\beta > 0$, $\nu > -1$ and $\xi \geq 0$ such that $\Lambda^{\zeta}_{\alpha,\beta,l} \subseteq \rho_C(-A)$. Suppose that the operator family $\{(1+|\lambda|)^{-\nu}e^{-M(\xi|\lambda|)}(\lambda^{\zeta}+A)^{-1}C : \lambda \in \Lambda^{\zeta}_{\alpha,\beta,l}\}$ is both equicontinuous and strongly continuous. We note that the question

(P) In which cases does there exist an injective operator $C' \in L(E)$ such that the operator $-A$ generates a global locally equicontinuous (g_{ζ}, C')-regularized resolvent $(S(t))_{t\geq 0}$ on E?

is very difficult to answer in general. Here we shall shortly explain how one can solve problem (P) in the affirmative provided that (2.1) holds with some $b \in (0,1)$, as well as that $\xi = 0$ and $1/(2-\zeta) > b/\zeta$ (cf. Theorem 2.2, Theorem 2.3 and Remark 1 (v)). Then $(S_{f_t}(0))_{t\in\Sigma_\theta}$ is an analytic C-regularized semigroup of growth order $(\nu+1)\zeta/b$, consisting of injective operators, with the closed linear operator $-(-A-\omega)_{b/\zeta}$ being its integral generator ($\omega > 0$ is a sufficiently large real number; cf. [9, Theorem 3.5, Theorem 3.7]), and the following holds.

(a) $(S_{f_t}(s))_{s\geq 0}$ is a locally equicontinuous $(g_{\zeta}, S_{f_t}(0))$-regularized resolvent family generated by $-A$ $(t \in \Sigma_\theta)$. If $q \in \circledast$, $\mathbb{B}$ is a bounded subset of E and K is a compact subset of $[0,\infty)$, then there exists $h_0 > 0$ such that (2.13) holds with $f = f_t$ $(t \in \Sigma_\theta)$.

(b) Suppose that $0 < c < b$, $\sigma > 0$, $\nu > -1$, $\varsigma > 0$, $\Pi^{\zeta}_{c,\sigma,\varsigma} \subseteq \rho_C(-A)$, and the operator family

$$\{(1+|\lambda|)^{-\nu}(\lambda^{\zeta}+A)^{-1}C : \lambda \in \Pi_{c,\sigma,\varsigma}\} \subseteq L(E)$$

is both equicontinuous and strongly continuous. Then the conclusions stated in (a) continue to hold.

Therefore, a great number of multiplication and (pseudo-)differential operators in L^p-spaces can serve as examples of the integral generators of fractional C-regularized resolvent families. Although the applications of theoretical results in statements (a)–(b) and Remark 2 can be also made to (pseudo-)differential operators with empty resolvent set, and to the operators considered in certain classes of Fréchet function spaces, we shall present only one illustrative example of application of the results in (b) and Remark 2 to abstract non-degenerate fractional differential equations (cf. Example 1 below).

3 The Non-Gevrey case

As before, in this subsection we assume that $\alpha > 0$, $\beta > 0$, $l \geq 1$, $0 < \zeta \leq 1$, as well as that (M_p) is a sequence of positive real numbers such that $M_0 = 1$ and the condition $(M.1)$ is satisfied for (M_p). Recall that $\Omega = \mathbb{C} \setminus \Lambda^{\zeta}_{\alpha,\beta,l}$. If $g : [0,\infty) \to [0,\infty)$ is a monotonically increasing, continuous function satisfying

$$\lim_{t\to+\infty} (1+t)^v e^{\sigma M(st)-g(t)} = 0, \quad v \in \mathbb{N},\ s \geq 0,\ \sigma > 0, \tag{3.1}$$

then we denote by $\mathcal{A}_g$ the class consisting of all continuous functions $f : \overline{\Omega} \to \mathbb{C}$ that are analytic in Ω and satisfy the inequality:

$$|f(z)| \leq \text{const} \cdot e^{-g(|z|^{1/\zeta})}, \quad z \in \overline{\Omega}.$$

Having this notion in mind, we can formulate the following non-Gevrey analogue of Theorem 2.2 and Theorem 2.3; the proof is very similar to that of Theorem 2.2 and therefore omitted.

Theorem 3.1. *Suppose that (M_p) satisfies $(M.1)$, as well as that there exists a monotonically increasing, continuous function $g : [0,\infty) \to [0,\infty)$ satisfying (3.1) and that the class $\mathcal{A}_g$ is non-trivial. Let $0 < \zeta \leq 1$, $\nu > -1$, $\xi \geq 0$, $\alpha > 0$, $\beta > 0$ and $l \geq 1$. Suppose, further, that the operator P_λ is injective for all $\lambda \in \Lambda_{\alpha,\beta,l}$, as well as that $P_\lambda^{-1}C \in L(E)$, $\lambda \in \Lambda_{\alpha,\beta,l}$, the mapping $\lambda \mapsto P_\lambda^{-1}Cx$, $\lambda \in \Lambda_{\alpha,\beta,l}$ is continuous for every fixed element $x \in E$, and the operator family*

$$\left\{(1+|\lambda|)^{-\nu}e^{-M(\xi\lambda)}P_\lambda^{-1}C : \lambda \in \Lambda_{\alpha,\beta,l}\right\} \subseteq L(E)$$

is equicontinuous. Define, for every function $f \in \mathcal{A}_g$, the bounded linear operator $S_f(s)$ $(s \geq 0)$ by (2.10).

Then $(S_f(s))_{s\geq 0} \subseteq L(E)$ is strongly continuous, the mapping $s \mapsto S_f(s) \in L(E)$, $s \geq 0$ $(s \mapsto S_f(s) \in L(E)$, $s > 0)$ is infinitely differentiable provided that $\zeta = 1$, $f \in \mathcal{A}_g$ $(\zeta \in (0,1)$, $f \in \mathcal{A}_g)$ and, for every $p \in \mathbb{N}_0$ and $f \in \mathcal{A}_g$, the mapping $s \mapsto (\mathbf{D}_s^\zeta)^p S_f(s) \in L(E)$, $s \geq 0$ is well-defined. Furthermore, (2.11) and the following statements hold.

(i) *Suppose that there exists $i \in \mathbb{N}_n^0$ such that the mappings $\lambda \mapsto A_jP_\lambda^{-1}Cx$, $\lambda \in \Lambda_{\alpha,\beta,l}$ are continuous for some $x \in E$ $(j \in \mathbb{N}_n^0 \setminus \{i\})$ and for each seminorm $p \in \circledast$ the set*

$$\{(1+|\lambda|)^{-\nu}e^{-M(\xi\lambda)}p(A_jP_\lambda^{-1}Cx) : \lambda \in \Lambda_{\alpha,\beta,l},\ j \in \mathbb{N}_n^0 \setminus \{i\}\}$$

is bounded.

Then (2.12) holds for any $x \in E$, $s \geq 0$, $i \in \mathbb{N}_n^0$ and $p, q \in \mathbb{N}_0$. Moreover, the mapping $s \mapsto u(s) := S_f(s)x$, $s \geq 0$ is a strong solution to problem (2.5), (2.6), with the initial value u_j obtained by plugging $p = j$ and $s = 0$ into the right-hand side of (2.11), for $j \in \mathbb{N}^0_{\max\{q_i-1: i\in S_l\}}$, and the initial value $u_{i,j}$ obtained by plugging $p = j$, $q = q_i$ and $s = 0$ into the right-hand side of (2.12), for $i \in S_r$ and $j \in \mathbb{N}^0_{p_i-1}$ $(f \in \mathcal{A})$. If $CA_i \subseteq A_iC$ for all $i \in \mathbb{N}_n^0$, then there exists at most one strong solution to problem (2.5), (2.6).

(ii) *Let $(N_p)_{p\in\mathbb{N}_0}$ be a sequence of positive real numbers satisfying $N_0 = 1$, $(M.1)$ and the following property: for each $v \in \mathbb{N}$, $s \geq 0$ and $\sigma > 0$ there exists $h > 0$ such that*

$$\lim_{t\to+\infty}(1+t)^v e^{\sigma M(st)+N(ht^\zeta)-g(t)} = 0.$$

Suppose that $f \in \mathcal{A}$, $q \in \circledast$, $\mathbb{B}$ is a bounded subset of E and K is a compact subset of $[0,\infty)$.

Then there exists $h_0 > 0$ such that (2.13) holds with the sequence $(p^{p\zeta/b})$ replaced by (N_p).

Remark 4. Theorem 3.1 is closely linked with the assertion of [21, Theorem 3.3], where we have considered the regularization of ultradistribution semigroups in Banach spaces ($B = I$, $n = 1$, $\zeta = 1$, $\xi \geq 0$, $-A$ generates an ultradistribution semigroup of (M_p)-class; cf. [23] for the notion, as well as [10, 21, 27, 28, 32], for more details concerning ultradistribution semigroups). If the corresponding sequence (M_p) satisfies conditions $(M.1)$, $(M.2)$ and $(M.3)$, then we have proved in the afore-mentioned theorem that there exist two functions, $g(\cdot)$ and $f \in \mathcal{A}_g$, such that the operator $-A$ generates a global locally equicontinuous C-regularized semigroup $(S_f(s))_{s\geq 0}$, with $C = S_f(0)$ being injective, satisfying additionally that the mapping $s \mapsto S_f(s) \in L(E)$, $s \geq 0$ is infinitely differentiable and $E^{(M_p)}(A) \subseteq C(D_\infty(A))$. The proof of this fact is based on the existence of a sequence (N_p) of positive real numbers satisfying $N_0 = 1$, $(M.1)$, $(M.2)$, $(M.3)$ and $N_p \prec M_p$ (cf. [21, Lemma 3.2]), and by putting $f(\cdot) = 1/\omega_{l',(N_p)}(-\cdot)$ ($l' \in \mathbb{N}$ sufficiently large), where

$$\omega_{l',(N_p)}(\lambda) = \prod_{p=1}^{\infty}\left(1 + \frac{l'\lambda N_{p-1}}{N_p}\right), \quad \lambda \in \mathbb{C} \quad (l' > 0).$$

It is worth noting that we have considered a slightly different growth rate of $P_\lambda^{-1}C$ in Theorem 2.2 (Theorem 2.3), and that one has to assume that for each $v \in \mathbb{N}$, $s \geq 0$ and $\sigma > 0$ there exists $h > 0$ such that

$$\lim_{t\to+\infty}(1+t)^v e^{\sigma M(st)+\xi|\lambda|^b+N(ht^\zeta)-g(t)} = 0 \quad \big(\lim_{t\to+\infty}(1+t)^v e^{st^c+\xi|\lambda|^b+N(ht^\zeta)-g(t)} = 0\big)$$

in order to deduce Theorem 2.2 (Theorem 2.3) from Theorem 3.1. Observe also that the comments in Remark 1 can be reformulated in the case in which the assumptions of Theorem 2.3 or Theorem 3.1 hold.

4 Examples and applications

Example 1. Assume that $0 < c < b < 1$, $1/(2-\zeta) > b/\zeta$, $\sigma > 0$, $\varsigma > 0$, $p \in [1,\infty)$, $m > 0$, $\rho \in [0,1]$, $r > 0$, $a \in S^m_{\rho,0}$ satisfies (H_r), the inequality

$$n\Big|\frac{1}{2}-\frac{1}{p}\Big|\frac{m-r-\rho+1}{r} < 1 \tag{4.1}$$

holds, $E = L^p(\mathbb{R}^n)$ or $E = C_0(\mathbb{R}^n)$ (in the last case, we assume that (4.1) holds with $p = \infty$), and $A := -\mathrm{Op}_E(a)$ (cf. [1, Chapter 8] for the notion and terminology). If $\operatorname{dist}(a(\mathbb{R}^n), \Pi^\zeta_{c,\sigma,\varsigma}) > 0$, then there exists a number $\nu > -1$ such that the operator family

$$\{(1+|\lambda|)^{-\nu}(\lambda^\zeta + A)^{-1} : \lambda \in \Pi_{c,\sigma,\varsigma}\} \subseteq L(E)$$

is both equicontinuous and strongly continuous ($C = I$), so that $(S_{f_t}(s))_{s\geq 0}$ is a global $(g_\zeta, S_{f_t}(0))$-regularized resolvent family generated by $-A$ ($t \in \Sigma_\theta$); furthermore, if K is a compact subset of $[0,\infty)$ and $t \in \Sigma_\theta$, then there exists $h_0 > 0$ such that

$$\sup_{p\in\mathbb{N}_0, s\in K}\frac{(h_0)^p\big\|(\mathbf{D}^\zeta_s)^p S_{f_t}(s)\big\|}{p^{p\zeta/b}} < \infty.$$

The proof of (2.25) implies that $\bigcup_{t>0} S_{f_t}(0)(D(A)) = E^{\langle p^{p\zeta/b}\rangle}(A)$, so that the problem $(\mathrm{DFP})_R$, with $B = I$, has a unique strong solution for all $x \in E^{\langle p^{p\zeta/b}\rangle}(A)$, given by $u(s) := S_{f_t}(s)S_{f_t}(0)^{-1}x$, $s \geq 0$, where $t > 0$ satisfies $x \in S_{f_t}(0)(D(A))$. A concrete example can be simply constructed. Suppose that $\zeta = 1 - c > c(1+c)$. This, in turn, implies $1/(2-\zeta) > b/\zeta > c/\zeta$ for some $c < b < 1$. Since

$$(x+ix^{1/c})^\zeta = (x^2+x^{2/c})^{\zeta/2}[\cos(\zeta\arctan x^{(1/c)-1}) + i\sin(\zeta\arctan x^{(1/c)-1})], \quad x > 0,$$

an elementary calculus shows that

$$\Re\big((x+ix^{1/c})^\zeta\big)/\Im\big((x+ix^{1/c})^\zeta\big) \sim 1/\tan(\zeta\pi/2) \text{ as } x \to +\infty,$$

and

$$\Re\big((x+ix^{1/c})^\zeta\big) - (\tan(\zeta\pi/2))^{-1}\Im\big((x+ix^{1/c})^\zeta\big) \\ \sim \zeta(\sin(\zeta\pi/2))^{-1}x^{((\zeta-1)/c)+1} = \zeta(\sin(\zeta\pi/2))^{-1} \text{ as } x \to +\infty$$

(similar formulae hold if we consider the term $(x-ix^{1/c})^\zeta$ in place of $(x+ix^{1/c})^\zeta$). Using these asymptotic formulae, it readily follows that for each number $d \in (0, \zeta/\sin(\zeta\pi/2))$ there exists a sufficiently large number $r_d > 0$ such that the inequality $\operatorname{dist}\big(a(\mathbb{R}^n), \Pi^\zeta_{c,\sigma_d,\varsigma_d}\big) > 0$ is true for suitably chosen numbers $\sigma_d > 0$ and $\varsigma_d > 0$, provided that $a(x) = d + (r_d + P(x))e^{\pm i\pi\zeta/2}$, where $P(x)$ is a positive real elliptic polynomial in n variables, of order m (then (4.1) holds with $m = r$ and $\rho = 1$).

In our recent papers (cf. [22, 23, 24, 25]), we have considered the polynomials of the operator $A := -d/ds,\ D(A) := \{f \in E\ ;\ f' \in E,\ f(0) = 0\}$, acting on the Banach space

$$E := \left\{ f \in C^{\infty}[0,1]\ ;\ \|f\| := \sup_{p \geq 0} \frac{\|f^{(p)}\|_{\infty}}{p!^s} < \infty \right\} \quad (s > 1).$$

For instance, we have proved that there exist numbers $b > 0$, $c > 0$ and $\eta > 0$ such that the following estimates hold:

$$\left\| \bigl(\lambda P_2(A) - P_1(A)\bigr)^{-1} \right\| = O\Bigl(e^{b|\lambda|^{1/(N_1-N_2)s} + c|\lambda|^{1/(N_1-N_2)}}\Bigr),\ \lambda \in \mathbb{C},$$

and

$$\left\| \bigl(\lambda P_2(A) - P_1(A)\bigr)^{-1} P_2(A) f \right\| \leq \eta \|f\| e^{b|\lambda|^{1/(N_1-N_2)s} + c|\lambda|^{1/(N_1-N_2)}},$$

for any $\lambda \in \mathbb{C},\ f \in D(P_2(A))$, with $P_1(z)$ and $P_2(z)$ being two complex non-zero polynomials satisfying $N_1 = \mathrm{dg}(P_1) > 1 + \mathrm{dg}(P_2) = 1 + N_2$. The interested reader may try to prove some upper bounds of the growth rate of the term

$$\left\| \left(\lambda^{(p_n+q_n)\zeta} P_n(A) + \sum_{i=0}^{n-1} \lambda^{(p_i+q_i)\zeta} P_i(A) \right)^{-1} \right\|,$$

where $P_i(z)$ is a complex non-zero polynomial $(1 \leq i \leq n)$, thus providing certain applications of Theorem 2.2 and Theorem 2.3.

Observe also that the method proposed by R. Beals in [5, Section 5] and A. Guzman in [20, Section 3] can be used for successful applications of Theorem 2.3 to some systems of linear PDEs that are degenerate in the time-variable. In the remaining part of paper, we will illustrate the obtained theoretical results with some other instructive examples.

Example 2. Suppose that E is a general SCLCS, $b \in (0,1)$, (M_p) satisfies $(M.1)$ and (2.1), $\zeta = 1$, $p_i = 0$ for all $i \in \mathbb{N}_n^0$, $q_n > q_{n-1}$, $\alpha > 0$, $\beta > 0$, $l \geq 1$, $\emptyset \neq \Omega \subseteq \mathbb{C}$, $N \in \mathbb{N}$, A is a densely defined closed linear operator in E satisfying that $\Omega \subseteq \rho(A)$ and the operator family

$$\{(1 + |\lambda|)^{-N} (\lambda - A)^{-1} : \lambda \in \Omega\} \subseteq L(E)$$

is equicontinuous. Suppose, further, that $P_i(z)$ is a complex polynomial $(i \in \mathbb{N}_n^0)$, $P_n(z) \not\equiv 0$, $\lambda_0 \in \rho(A) \setminus \{z \in \mathbb{C} : P_n(z) = 0\}$, $\mathrm{dist}(\lambda_0, \Omega) > 0$, as well as that for each $\lambda \in \Lambda_{\alpha,\beta,l}$ all roots of the polynomial $z \mapsto \lambda^{q_n} P_n(z) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(z)$, $z \in \mathbb{C}$ belong to Ω. Set $B := P_n(A)$ and $A_i := P_i(A)$ $(i \in \mathbb{N}_{n-1}^0)$. Then there exist $M \in \mathbb{N}$, λ-polynomials $F_0(\lambda), \ldots, F_M(\lambda)$ and not necessarily distinct numbers $f_1(\lambda) \in \Omega, \ldots, f_M(\lambda) \in \Omega$ such that

$$\lambda^{q_n} P_n(z) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(z) = F_M(\lambda) z^M + \cdots + F_1(\lambda) z + F_0(\lambda) =$$

$$= (-1)^M F_M(\lambda) (f_M(\lambda) - z) \cdots (f_1(\lambda) - z)$$

for all $\lambda \in \Lambda_{\alpha,\beta,l} \setminus \mathcal{P}$ and $z \in \mathbb{C}$, where $\mathcal{P} \equiv \{\lambda \in \mathbb{C} : F_M(\lambda) = 0\}$; furthermore, for each $\lambda \in \Lambda_{\alpha,\beta,l} \setminus \mathcal{P}$ the following equality holds:

$$\left(\lambda^{q_n} P_n(A) + \sum_{i=0}^{n-1} \lambda^{q_i} P_i(A) \right)^{-1} = (-1)^M \bigl(F_M(\lambda)\bigr)^{-1} \bigl(f_M(\lambda) - A\bigr)^{-1} \cdots \bigl(f_1(\lambda) - A\bigr)^{-1}.$$

Using the generalized resolvent equation [25, (6)], it readily follows that for any integer $Q \geq N+2$ the operator family

$$\{(f_i(\lambda)-\lambda_0)(f_i(\lambda)-A)^{-1}(\lambda_0-A)^{-Q} : \lambda \in \Lambda_{\alpha,\beta,l} \setminus \mathcal{P}\} \subseteq L(E)$$

is equicontinuous ($1 \leq i \leq M$). This implies that there exists a sufficiently large integer $Q' \geq N+2$ such that for each seminorm $p \in \circledast$ there exist $c_p > 0$ and $q \in \circledast$ such that, for every $j \in \mathbb{N}^0_{n-1}$, $\lambda \in \Lambda_{\alpha,\beta,l} \setminus \mathcal{P}$ and $x \in E$,

$$\begin{aligned}
& p\Biggl(\Bigl(\lambda^{q_n}P_n(A)+\textstyle\sum_{i=0}^{n-1}\lambda^{q_i}P_i(A)\Bigr)^{-1}\bigl(\lambda_0-A\bigr)^{-Q'}x\Biggr) \\
& +p\Biggl(P_j(A)\Bigl(\lambda^{q_n}P_n(A)+\textstyle\sum_{i=0}^{n-1}\lambda^{q_i}P_i(A)\Bigr)^{-1}\bigl(\lambda_0-A\bigr)^{-Q'}x\Biggr) \\
&= p\Biggl(\bigl(F_M(\lambda)\bigr)^{-1}\bigl(f_M(\lambda)-A\bigr)^{-1}\cdots\bigl(f_1(\lambda)-A\bigr)^{-1}\bigl(\lambda_0-A\bigr)^{-Q'}x\Biggr) \\
& +p\Biggl(P_j(A)\bigl(F_M(\lambda)\bigr)^{-1}\bigl(f_M(\lambda)-A\bigr)^{-1}\cdots\bigl(f_1(\lambda)-A\bigr)^{-1}\bigl(\lambda_0-A\bigr)^{-Q'}x\Biggr) \\
& \leq c_p\bigl|F_M(\lambda)\bigr|^{-1}\Bigl|\bigl(f_M(\lambda)-\lambda_0\bigr)\cdots\bigl(f_1(\lambda)-\lambda_0\bigr)\Bigr|^{-1}q(x) \\
&= c_p\bigl|F_M(\lambda)\bigr|^{-1}\bigl|F_M(\lambda)\bigr|\Bigl|F_M(\lambda)\lambda_0^m+\cdots+F_1(\lambda)\lambda_0+F_0(\lambda)\Bigr|^{-1}q(x) \\
&= c_p\Biggl|\lambda^{q_n}P_n\bigl(\lambda_0\bigr)+\textstyle\sum_{i=0}^{n-1}\lambda^{q_i}P_i\bigl(\lambda_0\bigr)\Biggr|^{-1}q(x) \sim c_p\bigl|P_n\bigl(\lambda_0\bigr)\bigr|^{-1}|\lambda|^{-q_n}q(x) \text{ as } |\lambda|\rightarrow\infty.
\end{aligned}$$

Therefore, there exists a sufficiently large number $\beta' > \beta$ such that the operator families

$$\left\{(1+|\lambda|)^{q_n}(\lambda^{q_n}P_n(A)+\sum_{i=0}^{n-1}\lambda^{q_i}P_i(A))^{-1}(\lambda_0-A)^{-Q'} : \lambda \in \Lambda_{\alpha,\beta',l}\right\} \subseteq L(E)$$

and

$$\left\{(1+|\lambda|)^{q_n}P_j(A)(\lambda^{q_n}P_n(A)+\sum_{i=0}^{n-1}\lambda^{q_i}P_i(A))^{-1}(\lambda_0-A)^{-Q'} : \lambda \in \Lambda_{\alpha,\beta',l}\right\} \subseteq L(E)$$

are equicontinuous ($j \in \mathbb{N}^0_{n-1}$). Since $q_n > q_{n-1}$ and $P(A)$ is dense in E for any complex polynomial $P(z) \in \mathbb{C}[z]$, the analysis contained in Remark 1 (x), with $C \equiv (\lambda_0-A)^{-Q'}$, shows that for each $(x_0,\cdot\cdot\cdot,x_{q_n-1}) \in E^{q_n}$ there exists a net $(u_\epsilon(t))_{\epsilon>0}$ of strong solutions of problem (2.5) with the subjected initial values $(u_0^\epsilon,\cdot\cdot\cdot,u^\epsilon_{q_n-1})$, converging to $(x_0,\cdot\cdot\cdot,x_{q_n-1})$ as $\epsilon \rightarrow 0+$ (in the topology of E^{q_n}).

Example 3. By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the Fourier transform on $\mathbb{R}^n$ and its inverse transform, respectively. Assume that $n \in \mathbb{N}$ and $iA_j,\ 1 \leq j \leq n$ are commuting generators of bounded C_0-groups on a Banach space E. Set $\mathbb{A} := (A_1,\ldots,A_n)$, $\mathbb{A}^\eta := A_1^{\eta_1}\cdots A_n^{\eta_n}$ for any $\eta = (\eta_1,\cdots,\eta_n) \in \mathbb{N}^0_n$, and denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. Let $k = 1+\lfloor n/2\rfloor$. For every $\xi = (\xi_1,\cdot\cdot\cdot,\xi_n) \in \mathbb{R}^n$ and $u \in \mathcal{F}L^1(\mathbb{R}^n) = \{\mathcal{F}f : f \in L^1(\mathbb{R}^n)\}$, we set $|\xi| := (\sum_{j=1}^n \xi_j^2)^{1/2}$, $(\xi,\mathbb{A}) := \sum_{j=1}^n \xi_jA_j$ and

$$u(\mathbb{A})x := \int_{\mathbb{R}^n} \mathcal{F}^{-1}u(\xi)e^{-i(\xi,\mathbb{A})}x\,d\xi,\ x \in E.$$

Then $u(\mathbb{A}) \in L(E)$, $u \in \mathcal{F}L^1(\mathbb{R}^n)$ and there exists a finite constant $M \geq 1$ such that

$$\|u(\mathbb{A})\| \leq M\big\|\mathcal{F}^{-1}u\big\|_{L^1(\mathbb{R}^n)}, \ u \in \mathcal{F}L^1\big(\mathbb{R}^n\big).$$

Let $N \in \mathbb{N}$, and let $P(x) = \sum_{|\eta| \leq N} a_\eta x^\eta$, $x \in \mathbb{R}^n$ be a complex polynomial. Then we define

$$P(\mathbb{A}) := \sum_{|\eta| \leq N} a_\eta \mathbb{A}^\eta \quad \text{and} \quad E_0 := \big\{\phi(\mathbb{A})x : \phi \in \mathcal{S}(\mathbb{R}^n), \ x \in E\big\}.$$

We know that the operator $P(\mathbb{A})$ is closable and that the following properties hold:

($\triangleright$) $\overline{E_0} = E$, $E_0 \subseteq \bigcap_{\eta \in \mathbb{N}_0^n} D(\mathbb{A}^\eta)$, $\overline{P(\mathbb{A})_{|E_0}} = \overline{P(\mathbb{A})}$ and $\phi(\mathbb{A})P(\mathbb{A}) \subseteq P(\mathbb{A})\phi(\mathbb{A}) = (\phi P)(\mathbb{A})$, $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Assuming that E is a function space on which translations are uniformly bounded and strongly continuous, the obvious choice for iA_j is $-i\partial/\partial x_j$ (notice also that E can consist of functions defined on some bounded domain). If $P(x) = \sum_{|\eta| \leq N} a_\eta x^\eta$, $x \in \mathbb{R}^n$ and E is such a space (for example, $L^p(\mathbb{R}^n)$ with $p \in [1, \infty)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$), then it is not difficult to prove that $\overline{P(\mathbb{A})}$ is nothing else but the operator

$$\sum_{|\eta| \leq N} a_\eta (-i)^{|\eta|} \frac{\partial^{|\eta|}}{\partial x_1^{\eta_1} \cdots \partial x_n^{\eta_n}} \equiv \sum_{|\eta| \leq N} a_\eta D^\eta,$$

acting with its maximal distributional domain. Recall that $P(x)$ is called r-coercive ($0 < r \leq N$) if there exist $M, \ L > 0$ such that $|P(x)| \geq M|x|^r$, $|x| \geq L$; by a corollary of the Seidenberg–Tarski theorem, the equality $\lim_{|x| \to \infty} |P(x)| = \infty$ implies in particular that $P(x)$ is r-coercive for some $r \in (0, N]$ (cf. [1, Remark 8.2.7]). For further information concerning the functional calculus for commuting generators of C_0-groups, see [25].

Assume, further, that $0 < \delta < 2$, $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials, $N_1 = \mathrm{dg}(P_1(x))$, $N_2 = \mathrm{dg}(P_2(x))$, $P_2(x) \neq 0$, $x \in \mathbb{R}^n$, $0 < c < b < 1$, $0 < \zeta \leq 1$, $1/(2-\zeta) > c/\zeta$, $\sigma > 0$, $\varsigma > 0$, $A := -\overline{P_1(\mathbb{A})}$, $B := \overline{P_2(\mathbb{A})}$ and

$$\mathrm{dist}\Big(\big\{-P_1(x)P_2(x)^{-1} : x \in \mathbb{R}^n\big\}, \Pi^\zeta_{c,\sigma,\varsigma}\Big) > 0. \tag{4.2}$$

Then there exist sufficiently large numbers $\beta' \geq 0$ and $\nu \geq 0$ (the proof of [25, Theorem 2.5.2] can give more detailed and accurate information about β' and ν; we leave the reader to make this precise) such that

$$\left(\frac{1}{\lambda^\zeta P_2(x) + P_1(x)}\big(1 + |x|^2\big)^{-\beta'/2}\right)(\mathbb{A}) = \big(\lambda^\zeta B + A\big)^{-1}C, \quad \lambda \in \Pi_{c,\sigma,\varsigma},$$

$$\left(\frac{P_2(x)}{\lambda^\zeta P_2(x) + P_1(x)}\big(1 + |x|^2\big)^{-\beta'/2}\right)(\mathbb{A}) = B\big(\lambda^\zeta B + A\big)^{-1}C, \quad \lambda \in \Pi_{c,\sigma,\varsigma},$$

and the operator families

$$\{(1 + |\lambda|)^{-\nu}(\lambda^\zeta B + A)^{-1}C : \lambda \in \Pi_{c,\sigma,\varsigma}\} \subseteq L(E)$$

and

$$\{(1 + |\lambda|)^{-\nu}B(\lambda^\zeta B + A)^{-1}C : \lambda \in \Pi_{c,\sigma,\varsigma}\} \subseteq L(E)$$

are both equicontinuous and strongly continuous ($C := ((1+|x|^2)^{-\beta'/2})(\mathbb{A}))$, so that Theorem 2.3 can be applied. Although equation (2.25) of Remark 2 holds in our concrete situation, we should say that Theorem 2.3, Remark 2 certainly have some disadvantages in degenerate case because it is very difficult to say whether an element $x \in E$ belongs to the space $E^{\langle p^{p\zeta/b}\rangle}(B^{-1}A)$ or not, with the exception of some very special cases.

Suppose now that the operators A_k and B_k are defined by

$$A_k := -\overline{P_{1,k}(\mathbb{A})}, \quad B_k := \overline{P_{2,k}(\mathbb{A})},$$

and that estimate (4.2) holds with the polynomials $P_1(x)$ and $P_2(x)$ replaced respectively by the polynomials $P_{1,k}(x)$ and $P_{2,k}(x)$. Then Theorem 2.3 can be applied to a large class of multi-term (non-)degenerate differential equations of form (2.5), where

$$P_\lambda = \left(\lambda^\zeta B_1 + A_1\right)\left(\lambda^\zeta B_2 + A_2\right)\cdots\left(\lambda^\zeta B_k + A_k\right).$$

The choice of regularizing operator C is essentially the same as above but we must eventually increase the value of β'. Observe, finally, that a similar analysis can be carried out in E_l-type spaces [39].

Acknowledgments

The first named author is supported by Act 211 of the Government of the Russian Federation, contract 02.A03.21.0011, and by the Ministry of Education and Science of the Russian Federation, task No 1.6462.2017/BCh. The second named author is partially supported by grant 174024 of the Ministry of Science and Technological Development, Republic of Serbia.

References

[1] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems.* Birkhäuser Verlag, Basel, 2011.

[2] W. Arendt, O. El-Mennaoui, V. Keyantuo. *Local integrated semigroups: evolution with jumps of regularity.* J. Math. Anal. Appl. 186 (1994), 572–595.

[3] W. Arendt, A. Favini. *Integrated solutions to implicit differential equations.* Rend. Sem. Mat. Univ. Poi. Torino. 51 (1993), 315–329.

[4] E. Bazhlekova. *Fractional evolution equations in Banach spaces.* Thesis, Eindhoven University of Technology, Eindhoven, 2001.

[5] R. Beals. *Semigroups and abstract Gevrey spaces.* J. Funct. Anal. 10 (1972), 281–299.

[6] R. Beals. *On the abstract Cauchy problem.* J. Funct. Anal. 10 (1972), 300–308.

[7] R.W. Carroll, R.E. Showalter. *Singular and degenerate Cauchy problems.* Academic Press, New York, 1976.

[8] J. Chazarain. *Probléme de Cauchy abstraites et applications á quelques problémes mixtes.* J. Funct. Anal. 7 (1971), 386–446.

[9] C. Chen, M. Kostić, M. Li, M. Žigić. *Complex powers of C-sectorial operators. Part I.* Taiwanese J. Math. 17 (2013), 465–499.

[10] I. Ciorănescu. *Beurling spaces of class (M_p) and ultradistribution semi-groups.* Bull. Sci. Math. 102 (1978), 167–192.

[11] R. deLaubenfels, F. Yao, S. W. Wang. *Fractional powers of operators of regularized type.* J. Math. Anal. Appl. 199 (1996), 910–933.

[12] G.V. Demidenko, S.V. Uspenskii. *Partial differential equations and systems not solvable with respect to the highest-order derivative.* Vol. 256 of Pure and Applied Mathematics Series, CRC Press, New York, 2003.

[13] A. Favini. *An operational method for abstract degenerate evolution equations of hyperbolic type.* J. Funct. Anal. 76 (1988), 432–456.

[14] A. Favini, A. Yagi. *Degenerate differential equations in Banach spaces.* Pure and Applied Mathematics, Chapman and Hall / CRC, New York, 1998.

[15] V.E. Fedorov. *Strongly holomorphic groups of linear equations of Sobolev type in locally convex spaces.* Differential Equations 40 (2004), no. 5, 753–765.

[16] V.E. Fedorov, A. Debbouche. *A class of degenerate fractional evolution systems in Banach spaces.* Differential Equations 49 (2013), no. 12, 1569–1576.

[17] V.E. Fedorov, D.M. Gordievskikh. *Resolving operators of degenerate evolution equations with fractional derivative with respect to time.* Russian Mathematics (Izvestiya VUZ). 59 (2015), no. 1, 60–70.

[18] V.E. Fedorov, D.M. Gordievskikh, M.V. Plekhanova. *Equations in Banach spaces with a degenerate operator under a fractional derivative.* Differential Equations. 51 (2015), no. 10, 1367–1375.

[19] C. Foias. *Remarques sur les semi-groupes distributions d'opérateurs normaux.* Port. Math. 19 (1960), 227–242.

[20] A. Guzman. *Further study of growth of fractional-power semigroups.* J. Funct. Anal. 29 (1978), 133–141.

[21] M. Kostić. *Regularization of some classes of ultradistribution semigroups and sines.* Publ. Inst. Math., Nouv. Sér. 87 (2010), 9–37.

[22] M. Kostić. *Abstract time-fractional equations: existence and growth of solutions.* Fract. Calculus Appl. Anal. 14 (2011), 301–316.

[23] M. Kostić. *Generalized semigroups and cosine functions.* Mathematical Institute SANU, Belgrade, 2011.

[24] M. Kostić. *Abstract Volterra equations in locally convex spaces.* Sci. China Math. 55 (2012), 1797–1825.

[25] M. Kostić. *Abstract Volterra integro-differential equations.* Taylor and Francis Group / CRC Press / Science Publishers, Boca Raton, New York, 2015.

[26] M. Kostić. *$\mathcal{D}$-Hypercyclic and $\mathcal{D}$-topologically mixing properties of degenerate multi-term fractional differential equations.* Azerbaijan J. Math. 5 (2015), 78–95.

[27] M. Kostić, S. Pilipović, D. Velinov. *On the exponential ultradistribution semigroups in Banach spaces.* Funct. Anal. Appr. Comp. 7 (2015), 59–66.

[28] M. Kostić, S. Pilipović, D. Velinov. *Structural theorems for ultradistribution semigroups.* Siberian Math. J. 56 (2015), 83–91.

[29] A.I. Kozhanov. *On boundary value problems for some classes of higher-order equations that are not solved with respect to the highest derivative.* Siberian Math. J. 35 (1994), 324–340.

[30] B.Ya. Levin. *Lectures on entire functions.* Translations of Mathematical Monographs, Vol. 150, American Mathematical Society, 1996.

[31] R. Meise, D. Vogt. *Introduction to functional analysis.* Translated from the German by M. S. Ramanujan and revised by the authors. Oxf. Grad. Texts Math., Clarendon Press, New York, 1997.

[32] I.V. Melnikova, A.I. Filinkov. *Abstract Cauchy problems: three approaches.* Chapman and Hall / CRC, Boca Raton, 2001.

[33] V. Obukhovskii, P. Zecca. *On boundary value problems for degenerate differential inclusions in Banach spaces.* Abstr. Appl. Anal. 13 (2003), 769–784.

[34] E.C. de Oliveira, J.A.T. Machado. *A review of definitions for fractional derivatives and integrals.* Mathematical Problems in Engineering. 2014, article ID 238459, 6 pp.

[35] J. Prüss. *Evolutionary integral equations and applications.* Birkhäuser-Verlag, Basel, 1993.

[36] S. G. Samko, A. A. Kilbas, O. I. Marichev. *Fractional derivatives and integrals: theory and applications.* Gordon and Breach, New York, 1993.

[37] B. Straub. *Fractional powers of operators with polynomially bounded resolvent and the semigroups generated by them.* Hiroshima Math. J. 24 (1994), 529–548.

[38] G.A. Sviridyuk, V.E. Fedorov. *Linear Sobolev type equations and degenerate semigroups of operators.* Inverse and Ill-Posed Problems (Book 42), VSP, Utrecht, Boston, 2003.

[39] T.-J. Xiao, J. Liang. *The Cauchy problem for higher-order abstract differential equations.* Springer-Verlag, Berlin, 1998.

[40] T.-J. Xiao, J. Liang. *Higher order degenerate Cauchy problems in locally convex spaces.* Math. Comp. Modelling 41 (2005), 837–847.

Vladimir Evgenievich Fedorov
Department of Mathematical Analysis
Chelyabinsk State University
129 Kashirin Brothers St.
454001 Chelyabinsk, Russia
E-mail: kar@csu.ru;
and
Functional Materials Laboratory
South Ural State University
76 Lenin Av.
454080 Chelyabinsk, Russia

Marko Kostić
Faculty of Technical Sciences
University of Novi Sad
6 Trg D.Obradovića
21125 Novi Sad, Serbia
E-mail: marco.s@verat.net

Received: 11.05.2016