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ON THE ASSOCIATED SPACES OF THE WEIGHTED ALTERED CESÀRO SPACE

D.V. Prokhorov

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Abstract: We study weighted altered Cesàro space $\mathrm{Ch}_{\infty,w}(I)$, which is a non-ideal enlargement of the usual Cesàro space. We prove the connection of this space with one weighted Sobolev space of the first order on real line and give characterizations of associate spaces of this space.

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1 Introduction

Let $I := (c, d) \subset \mathbb{R}$, $p \in [1, \infty]$, $p' := \frac{p}{p-1}$, $\mathcal{L}^1$ be the Lebesgue measure on I, $\mathfrak{M}(I)$ be the vector space of all $\mathcal{L}^1$-measurable functions $f : I \to [-\infty, \infty]$, and let $L^p(I)$ be the Lebesgue space. Also we put

$$L^p_{\mathrm{loc}}(I) := \Big\{ f \in \mathfrak{M}(I) : \|\chi_{(a,b)} f\|_{L^p(I)} < \infty, \ \forall a, b \in I \Big\},$$

$$L^p_{\mathrm{loc}}([c, d)) := \Big\{ f \in \mathfrak{M}(I) : \|\chi_{(c,x)} f\|_{L^p(I)} < \infty, \ \forall x \in I \Big\},$$

$$L^p_{\mathrm{loc}}((c, d]) := \Big\{ f \in \mathfrak{M}(I) : \|\chi_{(x,d)} f\|_{L^p(I)} < \infty, \ \forall x \in I \Big\}.$$

Let

$$w \in \mathfrak{M}(I), \ w > 0 \ \mathcal{L}^1\text{-almost everywhere on } I, \ w \in L^p_{\mathrm{loc}}((c, d]) \tag{1.1}$$

and (if the measure in the integral is omitted, then the integral is taken with respect to the measure $\mathcal{L}^1$)

$$\rho(f) := \begin{cases} \left(\int_I \left|w(x) \int_c^x f\right|^p dx\right)^{\frac{1}{p}}, & p \in [1, \infty), \\ \mathcal{L}^1\text{-ess sup}_{x \in I} w(x) \left|\int_c^x f\right|, & p = \infty; \end{cases}$$

$$\mathrm{Cs}_{p,w}(I) := \Big\{ f \in L^1_{\mathrm{loc}}([c, d)) \,\Big|\, \|f\|_{\mathrm{Cs}_{p,w}(I)} < \infty \Big\}, \ \|f\|_{\mathrm{Cs}_{p,w}(I)} := \rho(|f|),$$

$$\mathrm{Ch}_{p,w}(I) := \Big\{ f \in L^1_{\mathrm{loc}}([c, d)) \,\Big|\, \|f\|_{\mathrm{Ch}_{p,w}(I)} < \infty \Big\}, \ \|f\|_{\mathrm{Ch}_{p,w}(I)} := \rho(f).$$

It is clear that $\mathrm{Cs}_{p,w}(I)$ is embedded in $\mathrm{Ch}_{p,w}(I)$. Since w satisfies condition (1.1) then $f \in \mathfrak{M}(I)$ with compact support belongs to the space $\mathrm{Cs}_{p,w}(I)$. The space $(\mathrm{Cs}_{p,w}(I), \|\cdot\|_{\mathrm{Cs}_{p,w}(I)})$ is called weighted Cesàro space, it has been actively studied (see [6, 3] and the survey [1]). We call the space $(\mathrm{Ch}_{p,w}(I), \|\cdot\|_{\mathrm{Ch}_{p,w}(I)})$ weighted altered Cesàro space. This space has been studied in the works [9, 10, 14].

Let $(X, \|\cdot\|)$ be the normed space of elements of $\mathfrak{M}(I)$. We define the "strong" associated space (Köthe dual space) of X by

$$X'_{\mathrm{s}} := (X, \|\cdot\|)'_{\mathrm{s}} := \left\{ g \in \mathfrak{M}(I) \,\middle|\, \|g\|_{X'_{\mathrm{s}}} := \sup_{h \in X \setminus \{0\}} \frac{\int_I |hg|}{\|h\|} < \infty \right\}$$

and the "weak" associated space of X by

$$X'_{\mathrm{w}} :=(X, \|\cdot\|)'_{\mathrm{w}} := \left\{ g \in \mathfrak{M}(I) \,\middle|\, fg \in L^1(I)\, \forall f \in X \;\&\; \|g\|_{X'_{\mathrm{w}}} := \sup_{h \in X \setminus \{0\}} \frac{|\int_I hg|}{\|h\|} < \infty \right\},$$

which is isomorphic to the subspace of the set X^* of all continuous functionals of the form $f \mapsto \int_I fg$, $f \in X$. It is clear that $X'_{\mathrm{s}} \subset X'_{\mathrm{w}}$.

The classic Cesàro space $\mathrm{Cs}_{p,w_0}(I_0)$ (where $I_0 := (0, \infty)$ and $w_0(x) := \frac{1}{x}$, $x \in I_0$) has been studied since 1970s. For $p \in (1, \infty)$ both spaces $\mathrm{Cs}_{p,w_0}(I_0)$ and $\mathrm{Ch}_{p,w_0}(I_0)$ appeared [11, 12] when solving the problem of describing the associated spaces with order one weighted Sobolev space on the real line, defined as

$$W^1_p(I_0) := \{f \in L^1_{\mathrm{loc}}(I_0) : Df \in L^1_{\mathrm{loc}}(I_0) \,\&\, \|f\|_{W^1_p(I_0)} < \infty\}, \tag{1.2}$$

where $\|f\|_{W^1_p(I_0)} := \|f\|_{L^p(I_0)} + \|\frac{1}{w_0} Df\|_{L^p(I_0)}$. As proved in [9, Theorem 3.3]

$$\begin{aligned} &(W^1_p(I_0))'_{\mathrm{s}} = \mathrm{Cs}_{p',w_0}(I_0), \\ &(W^1_p(I_0))'_{\mathrm{w}} = (\mathrm{Cs}_{p',w_0}(I_0), \|\cdot\|_{\mathrm{Ch}_{p',w_0}(I_0)}), \\ &(X, \|\cdot\|_{W^1_p(I_0)})'_{\mathrm{w}} = \mathrm{Ch}_{p',w_0}(I_0), \end{aligned}$$

where

$$X := \{f \in AC(I_0) \,|\, \exists f(0+),\ \exists b \in I_0 :\ \chi_{(b,\infty)} f = 0\}.$$

Note that X differs from

$$\overset{\circ\circ}{W}{}^1_p(I_0) := \{f \in AC(I_0) \,|\, \operatorname{supp} f \text{ is a compact in } I_0\}, \tag{1.3}$$

which plays an important role in the results of [11, 12]. The example in Section 2 shows that $(\overset{\circ\circ}{W}{}^1_p(I_0), \|\cdot\|_{W^1_p(I_0)})'_{\mathrm{w}} \neq \mathrm{Ch}_{p',w_0}(I_0)$. The key difference is the fact that the space $(\overset{\circ\circ}{W}{}^1_p(I_0), \|\cdot\|_{W^1_p(I_0)})'_{\mathrm{w}}$ contains functions that are not integrable at the left end of the segment I_0.

From the definition of associated spaces it follows that $(\mathrm{Cs}_{p,w}(I))'_{\mathrm{s}} = (\mathrm{Cs}_{p,w}(I))'_{\mathrm{w}}$ and $\|g\|_{(\mathrm{Cs}_{p,w}(I))'_{\mathrm{s}}} = \|g\|_{(\mathrm{Cs}_{p,w}(I))'_{\mathrm{w}}}$ for $g \in (\mathrm{Cs}_{p,w}(I))'_{\mathrm{s}}$ and $p \in [1, \infty]$. For $p \in [1, \infty)$ the space $\mathrm{Cs}_{p,w}(I)$ is an order ideal and it has an absolutely continuous norm. Then for $\Lambda \in (\mathrm{Cs}_{p,w}(I))^*$ there exists $g \in (\mathrm{Cs}_{p,w}(I))'_{\mathrm{s}}$ such that $\|\Lambda\|_{(\mathrm{Cs}_{p,w}(I))^*} = \|g\|_{(\mathrm{Cs}_{p,w}(I))'_{\mathrm{s}}}$ and $\Lambda f = \int_I fg$, $f \in \mathrm{Cs}_{p,w}(I)$ (see [2, Chapter 1, Theorem 4.1]).

The problem of describing the associated spaces of $\mathrm{Cs}_{p,w}(I)$ was solved in [3] with the help of an essential $\int_x^d w^p$-concave majorant (see [3, Definition 2.11]), and in [15] with the help of a monotone majorant.

For $p \in [1, \infty)$ characterizations of dual spaces of weighted altered Cesàro space are given in [10]. The key step of the proof was the approximation of an element of the space $\mathrm{Ch}_{p,w}(I)$ by elements with compact support. For $p = \infty$ there is no such approximation but it is possible (see Section 3) to describe the associated spaces of $\mathrm{Ch}_{\infty,w}(I)$ with a weight satisfying the conditions

$$w(x) = \left[\int_c^x v\right]^{-1} \in (0, \infty),\ x \in I,\ v \in \mathfrak{M}(I),\quad v \in L^1_{\mathrm{loc}}([c, d)),\ \lim_{b \to d-} \int_c^b v = \infty. \tag{1.4}$$

Throughout this article, $A \lesssim B$ and $B \gtrsim A$ mean that $A \le cB$, where the constant c depends only on p and may be different in different places. If both $A \lesssim B$ and $A \gtrsim B$ hold, then we write $A \approx B$. $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ is the set of all real numbers, Df is the weak derivative of $f \in \mathfrak{M}(I)$. The space of all locally absolutely continuous functions $f : I \to \mathbb{R}$ is denoted by $AC_{\mathrm{loc}}(I)$, $AC(I)$ is the space of all absolutely continuous functions. The symbol $BPV(I)$ denotes the space of all functions $f : I \to \mathbb{R}$ that have bounded pointwise variation (see [5, §2.1]). For any Borel measure λ defined on Borel subsets of I, the symbol $\|\lambda\|$ means $|\lambda|(I)$, where $|\lambda|$ is the total variation of λ. If $f \in BPV(I)$, then λ_f denotes the unique real Borel measure such that $\lambda_f((a,b]) = f(b+) - f(a+)$ for all $a, b \in I$, with $a \le b$ (see [5, Theorem 5.13]). $C_c^1(I)$ is the space of all real-valued continuously differentiable functions with compact support in I; $C_0(I)$ is the space of all real-valued continuous functions on I that vanish at infinity (see [13, 3.16]).

2 Connection with a Sobolev space

Let $W_p^1(I_0)$ be as defined in (1.2) and $\overset{\circ\circ}{W}{}_p^1(I_0)$ be as defined in (1.3). We start with an example showing that $(\overset{\circ\circ}{W}{}_p^1(I_0), \|\cdot\|_{W_p^1(I_0)})'_{\mathrm{w}} \ne \mathrm{Ch}_{p',w_0}(I_0)$.

Example. According to [12, Remark 5.1] the following relation holds

$$g \in (\overset{\circ\circ}{W}{}_p^1(I_0), \|\cdot\|_{W_p^1(I_0)})'_{\mathrm{w}} \quad \Leftrightarrow \quad (g \in L^1_{\mathrm{loc}}(I_0) \ \& \ [\mathbb{G}(g) + \mathfrak{G}(g)] < \infty),$$

where for $g \in L^1_{\mathrm{loc}}(I_0)$

$$\mathbb{G}(g) \approx \left(\int_0^\infty \frac{1}{t^{p'}} \left| \int_{\frac{t}{2}}^t g \right|^{p'} dt \right)^{\frac{1}{p'}},$$

$$\begin{aligned}\mathfrak{G}(g) &\approx \left(\int_0^\infty \frac{1}{t^{p'(2-p')}} \left| \int_t^{2t} y^{-p'} \left[\int_{\frac{y}{2}}^t g \right] dy \right|^{p'} dt \right)^{\frac{1}{p'}} \\ &= \left(\int_0^\infty \frac{1}{t^{p'(2-p')}} \left| \int_{\frac{t}{2}}^t g(x) \left[\int_t^{2x} y^{-p'}\, dy \right] dx \right|^{p'} dt \right)^{\frac{1}{p'}}.\end{aligned}$$

Further,

$$\begin{aligned}\mathfrak{G}(g) &\approx \left(\int_0^\infty \frac{1}{t^{p'(2-p')}} \left| \int_{\frac{t}{2}}^t g(x) \left[\frac{t^{1-p'} - (2x)^{1-p'}}{p'-1} \right] dx \right|^{p'} dt \right)^{\frac{1}{p'}} \\ &= \frac{1}{p'-1} \left(\int_0^\infty \frac{1}{t^{p'}} \left| \int_{\frac{t}{2}}^t g(x) \left[1 - \left(\frac{2x}{t} \right)^{1-p'} \right] dx \right|^{p'} dt \right)^{\frac{1}{p'}} \\ &= \frac{1}{p'-1} \left(\int_0^\infty \frac{1}{t^{p'}} \left| \int_{\frac{t}{2}}^t g - \left(\frac{2}{t} \right)^{1-p'} \int_{\frac{t}{2}}^t g(x) x^{1-p'}\, dx \right|^{p'} dt \right)^{\frac{1}{p'}}.\end{aligned}$$

Hence, for $g \in L^1_{\mathrm{loc}}(I_0)$ the inequality $[\mathbb{G}(g) + \mathfrak{G}(g)] < \infty$ is equivalent to

$$\left(\int_0^\infty \frac{1}{t^{p'}} \left| \int_{\frac{t}{2}}^t g \right|^{p'} dt \right)^{\frac{1}{p'}} + \left(\int_0^\infty \frac{1}{t^{p'(2-p')}} \left| \int_{\frac{t}{2}}^t g(x) x^{1-p'}\, dx \right|^{p'} dt \right)^{\frac{1}{p'}} < \infty.$$

Now let $p = p' = 2$, $g(x) := \frac{1}{x}\sin\frac{1}{x}\chi_{(0,1]}(x)$, $x \in I_0$. Then

$$\int_0^1 |g(x)|\,dx = \int_0^1 \frac{|\sin\frac{1}{x}|}{x}\,dx = \int_1^\infty \frac{|\sin y|}{y}\,dy = \infty,$$

$$\int_1^\infty \left|\int_{\frac{t}{2}}^t \frac{g(x)}{x}\,dx\right|^2 dt = \int_1^2 \left|\int_{\frac{t}{2}}^1 \frac{\sin\frac{1}{x}}{x^2}\,dx\right|^2 dt \le 4,$$

$$\int_0^1 \left|\int_{\frac{t}{2}}^t \frac{g(x)}{x}\,dx\right|^2 dt = \int_0^1 \left|\int_{\frac{t}{2}}^t \frac{\sin\frac{1}{x}}{x^2}\,dx\right|^2 dt = \int_0^1 \left|\int_{\frac{1}{t}}^{\frac{2}{t}} \sin y\,dy\right|^2 dt$$
$$= \int_1^\infty \frac{1}{x^2}\left|\int_x^{2x} \sin y\,dy\right|^2 dx < \infty,$$

$$\int_1^\infty \frac{1}{t^2}\left|\int_{\frac{t}{2}}^t g\right|^2 dt = \int_1^2 \frac{1}{t^2}\left|\int_{\frac{t}{2}}^1 \frac{\sin\frac{1}{x}}{x}\,dx\right|^2 dt \le 1.$$

Moreover, from

$$\left|\int_y^{2y} \frac{\sin t}{t}\,dt\right| = \left|\int_y^{2y} \frac{d\cos t}{t}\right| = \left|\frac{\cos t}{t}\Big|_y^{2y} + \int_y^{2y} \frac{\cos t}{t^2}\,dt\right| \le \frac{5}{2y},$$

we have the estimates

$$\int_0^1 \frac{1}{t^2}\left|\int_{\frac{t}{2}}^t g\right|^2 dt = \int_0^1 \frac{1}{t^2}\left|\int_{\frac{t}{2}}^t \frac{\sin\frac{1}{x}}{x}\,dx\right|^2 dt = \int_0^1 \frac{1}{t^2}\left|\int_{\frac{1}{t}}^{\frac{2}{t}} \frac{\sin y}{y}\,dy\right|^2 dt$$
$$= \int_1^\infty \left|\int_x^{2x} \frac{\sin y}{y}\,dy\right|^2 dx \le \frac{25}{4}\int_1^\infty \frac{dy}{y^2} < \infty.$$

Therefore, $g \in L^1_{\text{loc}}(I_0) \setminus L^1_{\text{loc}}([0,\infty))$ and $[\mathbb{G}(g) + \mathfrak{G}(g)] < \infty$, that is

$$g \in (\overset{\circ\circ}{W}{}^1_2(I_0), \|\cdot\|_{W^1_2(I_0)})'_{\text{w}} \setminus \text{Ch}_{2,w_0}(I_0).$$

Now we show that in the case of a decreasing weight w satisfying condition (1.4) the spaces $\text{Cs}_{\infty,w}(I)$ and $\text{Ch}_{\infty,w}(I)$ are associated spaces of the space $W^1_1(I)$ defined in formula (2.1). In particular, the theorem contains a criterion for the embedding of $W^1_1(I)$ into the Lebesgue space $L^1_g(I)$ with arbitrary weight g and thereby complements the results obtained in [4], [7, Chapter III], [8].

Theorem 2.1. *Let w satisfy condition* (1.4), *$v > 0$ $\mathcal{L}^1$-almost everywhere on I,*

$$X := \{f \in AC(I), \,|\, \exists f(c+),\ \exists b \in I:\ \chi_{(b,d)}f = 0\}$$

and

$$W^1_1(I) := \{f \in L^1_{\text{loc}}(I) : Df \in L^1_{\text{loc}}(I) \,\&\, \|f\|_{W^1_1(I)} < \infty\}, \tag{2.1}$$

where $\|f\|_{W^1_1(I)} := \|vf\|_{L^1(I)} + \|\frac{1}{w}Df\|_{L^1(I)}$. Then

$$(W^1_1(I))'_{\text{s}} = \text{Cs}_{\infty,w}(I), \tag{2.2}$$
$$(W^1_1(I))'_{\text{w}} = (\text{Cs}_{\infty,w}(I), \|\cdot\|_{\text{Ch}_{\infty,w}(I)}), \tag{2.3}$$
$$(X, \|\cdot\|_{W^1_1(I)})'_{\text{w}} = \text{Ch}_{\infty,w}(I). \tag{2.4}$$

Proof. We fix an arbitrary element $f \in W^1_1(I)$. Then there exists an $AC_{\rm loc}(I)$ representative $\tilde f$ of f. For any $x, y \in I$ such that $x > y$ we have

$$|\tilde f(x) - \tilde f(y)| \le \int_y^x |Df| \le \|\chi_{(y,d)} w\|_{L^\infty(I)} \|\chi_{(y,d)} \tfrac{1}{w} Df\|_{L^1(I)}.$$

Hence, there exists the limit $\tilde f(d-)$. Since $\|vf\|_{L^1(I)} < \infty$ and $v \notin L^1_{\rm loc}((c,d])$ then $\tilde f(d-) = 0$. In addition, $w \in L^\infty_{\rm loc}((c,d])$ implies $Df \in L^1_{\rm loc}((c,d])$. Consequently, $f(x) = -\int_x^d Df$ for $\mathcal{L}^1$-almost all $x \in I$.

Further, for an arbitrary $h \in L^1(I)$ since $w \in L^\infty_{\rm loc}((c,d])$ then $wh \in L^1_{\rm loc}((c,d])$, and for $f_h(y) := \int_y^d wh$, $y \in I$, we have

$$\|vf_h\|_{L^1(I)} = \int_I \left| v(y) \int_y^d wh \right| dy \le \|h\|_{L^1(I)}, \quad \|\tfrac{1}{w} Df_h\|_{L^1(I)} = \|h\|_{L^1(I)},$$

that is $f_h \in W^1_1(I)$ and $\|f_h\|_{W^1_1(I)} \le 2\|h\|_{L^1(I)}$.

If $g \in \mathrm{Cs}_{\infty,w}(I)$ then for any $f \in W^1_1(I) \setminus \{0\}$

$$\frac{\int_I |fg|}{\|f\|_{W^1_1(I)}} \le \frac{\int_I |(Df)(x)| \left(\int_c^x |g|\right) dx}{\|\frac{1}{w} Df\|_{L^1(I)}} \le \|g\|_{\mathrm{Cs}_{\infty,w}(I)}, \tag{2.5}$$

that is $g \in (W^1_1(I))'_{\rm s}$ and $\|g\|_{(W^1_1(I))'_{\rm s}} \le \|g\|_{\mathrm{Cs}_{\infty,w}(I)}$.

Now let $g \in (W^1_1(I))'_{\rm s}$. Since (see [2, Lemma 2.8]) for $g \in \mathfrak{M}(I)$ the equalities $\|g\|_{(L^1(I))'_{\rm s}} = \|g\|_{(L^1(I))'_{\rm w}} = \|g\|_{L^\infty(I)}$ hold, we get the estimate

$$\begin{aligned} \|g\|_{(W^1_1(I))'_{\rm s}} &\ge \sup_{h \in L^1(I)\setminus\{0\}} \frac{\int_I |f_{|h|} g|}{\|f_{|h|}\|_{W^1_1(I)}} \ge \sup_{h \in L^1(I)\setminus\{0\}} \frac{\int_I |g(y)| \left(\int_y^d w|h|\right) dy}{2\|h\|_{L^1(I)}} \\ &= \sup_{h \in L^1(I)\setminus\{0\}} \frac{\int_I |h(x)| w(x) \left(\int_c^x |g|\right) dx}{2\|h\|_{L^1(I)}} = \frac{1}{2}\|g\|_{\mathrm{Cs}_{\infty,w}(I)}, \end{aligned} \tag{2.6}$$

and (2.2) is proved.

By [11, Theorem 2.5] the equalities $(W^1_1(I))'_{\rm w} = (W^1_1(I))'_{\rm s} = \mathrm{Cs}_{\infty,w}(I)$ hold. Besides that, for any $g \in \mathrm{Cs}_{\infty,w}(I)$, $f \in W^1_1(I)$ we have

$$\int_I fg = \int_I g(x) \left(\int_x^d Df\right) dx = \int_I (Df)(y) \left(\int_c^y g\right) dy. \tag{2.7}$$

Hence, similarly to (2.5) and (2.6) we get $\|g\|_{(W^1_1(I))'_{\rm w}} \approx \|g\|_{\mathrm{Ch}_{\infty,w}(I)}$, and (2.3) is proved.

Further, for any $a \in I$ there exists a function $f \in X$ such that $\chi_{(c,a)} f = \chi_{(c,a)}$, and this implies $(X, \|\cdot\|_{W^1_1(I)})'_{\rm w} \subset L^1_{\rm loc}([c,d))$. Therefore, for any $a \in I$, $f \in X$, $g \in L^1_{\rm loc}([c,d))$, taking into account the decrease of the function w we have

$$\begin{aligned} \left|\int_a^d fg\right| &= \left|\int_a^d g(x) \left(\int_x^d Df\right) dx\right| = \left|\int_a^d (Df)(y) \left(\int_a^y g\right) dy\right| \\ &\le \|f\|_{W^1_1(I)} \sup_{y \in [a,d)} \left| w(y) \left[\int_c^y g - \int_c^a g\right]\right| \le 2\|g\|_{\mathrm{Ch}_{\infty,w}(I)} \|f\|_{W^1_1(I)}. \end{aligned}$$

Passing to the limit as $a \to c+$, we obtain $\|g\|_{(X, \|\cdot\|_{W^1_1(I)})'_{\rm w}} \le 2\|g\|_{\mathrm{Ch}_{\infty,w}(I)}$.

If $g \in (X, \|\cdot\|_{W_1^1(I)})'_{\mathrm{w}}$ and $h \in L^1(I)$ with $\operatorname{supp} h \subset (c,b]$ for some $b \in I$, equalities (2.7) hold with $f := f_h$. Therefore,

$$\begin{aligned}
\|g\|_{(X,\|\cdot\|_{W_1^1(I)})'_{\mathrm{w}}} &\geq \sup_{b\in I} \sup_{h \in L^1(I)\setminus\{0\}, \operatorname{supp} h \subset (c,b]} \frac{\left|\int_c^b g(y)\left(\int_y^b wh\right)\,dy\right|}{2\|h\|_{L^1(I)}} \\
&\geq \sup_{b\in I} \sup_{h \in L^1(I)\setminus\{0\}, \operatorname{supp} h \subset (c,b]} \frac{\left|\int_c^b h(x)w(x)\left(\int_c^x g\right)\,dx\right|}{2\|h\|_{L^1(I)}} \\
&= \frac{1}{2}\sup_{b\in I}\sup_{x\in(c,b]}\left|w(x)\int_c^x g\right| = \frac{1}{2}\|g\|_{\mathrm{Ch}_{\infty,w}(I)},
\end{aligned}$$

and (2.4) follows. □

3 Associated spaces of $\mathrm{Ch}_{\infty,w}(I)$

As in the case $p < \infty$ the "strong" associated space of $\mathrm{Ch}_{\infty,w}(I)$ is the null space. This follows from Lemma 3.1, the proof of which is similar to the proof of [10, Lemma 2.2].

Lemma 3.1. *Let w satisfy condition* (1.1), $[a,b] \subset I$ *and* $h \in L^1([a,b])$. *For any $\varepsilon > 0$ there exists $f \in \mathfrak{M}(I)$ such that* $\operatorname{supp} f \subset [a,b]$, $|f| = |h|$ *on* (a,b) *and* $\|f\|_{\mathrm{Ch}_{\infty,w}(I)} < \varepsilon$.

The next two lemmas contain the key constructions for obtaining a criterion for an element to belong to the "weak" associated space of $\mathrm{Ch}_{\infty,w}(I)$.

Lemma 3.2. *Let w satisfy condition* (1.4).

1. *If $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$ then $vg \in L^1(I)$. If $g \in (\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}$ and $v > 0$ $\mathcal{L}^1$-almost everywhere on I then*

$$\|vg\|_{L^1(I)} \leq \|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}}. \tag{3.1}$$

2. *Let (a) $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$ and $A_g := \|g\|_{(\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}}$, or (b) $v > 0$ $\mathcal{L}^1$-almost everywhere on I, $g \in (\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}$ and $A_g := \|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}}$. Then there exists a $BPV(I)$ representative $\tilde{g}$ of $\frac{g}{w}$ and the estimate*

$$\|\lambda_{\tilde{g}}\| \leq \|vg\|_{L^1(I)} + A_g \tag{3.2}$$

holds.

Proof. 1. Since $v \in \mathrm{Ch}_{\infty,w}(I)$ then $vg \in L^1(I)$ for $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$. Now let $v > 0$ $\mathcal{L}^1$-almost everywhere on I. Then for any $f \in \mathfrak{M}(I)$

$$\|f\|_{\mathrm{Cs}_{\infty,w}(I)} \leq \|\tfrac{f}{v}\|_{L^\infty(I)},$$

and for $g \in (\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}$ the relations

$$\|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}} \geq \sup_{f:\frac{f}{v}\in L^\infty(I)\setminus\{0\}} \frac{\left|\int_I \frac{f}{v} vg\right|}{\|\frac{f}{v}\|_{L^\infty(I)}} = \sup_{h \in L^\infty(I)\setminus\{0\}} \frac{\left|\int_I hvg\right|}{\|h\|_{L^\infty(I)}} = \|vg\|_{L^1(I)}$$

hold.

2. We fix an arbitrary function $\phi \in C_c^1(I)$ and put $f := D(\frac{1}{w}\phi) = v\phi + \frac{1}{w}D\phi$. Then $f \in L^1(I)$ and $\|f\|_{\mathrm{Ch}_{\infty,w}(I)} = \max_{x\in I}|\phi(x)|$. If $v > 0$ $\mathcal{L}^1$-almost everywhere on I then $f \in \mathrm{Cs}_{\infty,w}(I)$. From $vg \in L^1(I)$ we have $\int_I |\phi v g| < \infty$. Hence, $\int_I |\frac{1}{w} g D\phi| < \infty$ and

$$\frac{|\int_I \frac{1}{w} g D\phi|}{\max\limits_{x\in I}|\phi(x)|} \le \frac{|\int_I \phi v g|}{\max\limits_{x\in I}|\phi(x)|} + \frac{|\int_I fg|}{\|f\|_{\mathrm{Ch}_{\infty,w}(I)}} \le \|vg\|_{L^1(I)} + A_g.$$

For $\phi \in C_c^1(I)$ we put $\Lambda\phi := \int_I \frac{1}{w} g D\phi$. By the Hahn– Banach theorem there exists an extension $\tilde{\Lambda} \in (C_0(I))^*$ of the functional Λ for which the estimate

$$\|\tilde{\Lambda}\|_{(C_0(I))^*} \le \|vg\|_{L^1(I)} + A_g$$

holds.

By the Riesz theorem [13, 6.19] on the representation of a linear continuous functional on $C_0(I)$ there exists a unique regular real Borel measure λ such that $\|\lambda\| = \|\tilde{\Lambda}\|_{(C_0(I))^*}$ and $\tilde{\Lambda}\varphi = \int_I \varphi\, d\lambda$ for any $\varphi \in C_0(I)$.

We define $h_g(x) := \lambda(I \cap (-\infty, x])$, $x \in I$. Then $h_g \in BPV(I)$ and applying [5, Corollary 5.41], we have

$$\int_I \frac{1}{w} g D\phi = \tilde{\Lambda}\phi = \int_I \phi\, d\lambda = -\int_I h_g D\phi$$

for any $\phi \in C_c^1(I)$. Hence, $\frac{1}{w}g + h_g$ $\mathcal{L}^1$-almost everywhere on I coincides with a constant function. Therefore, there exists a $BPV(I)$ representative $\tilde{g}$ of $\frac{g}{w}$, and $\lambda_{\tilde{g}} = \lambda_{h_g} = \lambda$ are valid (see [5, Remark 5.14]). $\square$

Lemma 3.3. *Let w satisfy condition* (1.4), *$f \in \mathrm{Ch}_{\infty,w}(I)$, $g \in L^\infty_{\mathrm{loc}}(I)$, $vg \in L^1(I)$, $\int_I |fg| < \infty$, $\frac{g}{w}$ has an $BPV(I)$ representative $\tilde{g}$. Then*

$$\left|\int_I fg\right| \le 2\left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde{g}}\|\right)\|f\|_{\mathrm{Ch}_{\infty,w}(I)}. \tag{3.3}$$

Proof. We fix $\gamma \in (0,1)$. For $n \in \mathbb{N}$ we define

$$b_n := \sup\Big\{x \in I : \frac{1}{w(x)} \le n\Big\}, \quad a_n := \inf\Big\{x \in I : \frac{1}{w(x)} \ge \frac{1}{n}\Big\}.$$

Since $\frac{1}{w}$ is a continuous function, $\lim_{x\to d-}\frac{1}{w(x)} = \infty$ and $\lim_{x\to c+}\frac{1}{w(x)} = 0$, then $\frac{1}{w(b_n)} = n$, $\frac{1}{w(a_n)} = \frac{1}{n}$. Moreover, since $\{x \in I : \frac{1}{w(x)} \le n\} \subset \{x \in I : \frac{1}{w(x)} \le n+1\}$, then $b_n \le b_{n+1}$. If $b := \lim_{n\to\infty} b_n < d$ then $v \in L^1_{\mathrm{loc}}([c,d))$ implies $\infty = \lim_{n\to\infty}\frac{1}{w(b_n)} = \frac{1}{w(b)} < \infty$ and we get a contradiction. Hence, $\lim_{n\to\infty} b_n = d$. Analogously, $a_n \downarrow c$ as $n \to \infty$.

Let $n_0 \in \mathbb{N}$ be such that $a_{n_0} < b_{n_0}$. For $n \ge n_0$ we define $\alpha_n \in [a_n, b_n]$ such that $\frac{1}{w(\alpha_n)} = \min_{x\in[a_n,b_n]}\frac{1}{w(x)}$. Then $\frac{1}{w(\alpha_n)} > 0$ and $\alpha_n < b_n$. We claim that $\lim_{n\to\infty}\alpha_n = c$. We fix an arbitrary $a > c$. Since $a_n \downarrow c$ as $n \to \infty$ there exists $n_1 > n_0$ such that $a_{n_1} < a$. Let $n_2 > n_1$ and $\frac{1}{n_2} < \frac{1}{w(\alpha_{n_1})}$. Then for $n > n_2$ we have $\alpha_n \in [a_n, a_{n_1}]$ because of $\frac{1}{w(x)} \ge n_1 > \frac{1}{w(a_{n_1})}$ for $x \ge b_{n_1}$. Hence, $\alpha_n < a$.

Since $\int_{\alpha_n}^{b_n}|f| < \infty$ then by [13, 3.14] for $n \ge n_0$ there exists a function $\bar{f}_n \in C_c((\alpha_n, b_n))$ such that $\int_{\alpha_n}^{b_n}|f - \bar{f}_n| \le \frac{1}{w(\alpha_n)n}(1 + \|g\chi_{[\alpha_n,b_n]}\|_{L^\infty(I)})^{-1}$. Now we choose $\beta_n \in (b_n, d)$, $\theta_n \in \{-1, 1\}$ so that the equality $\theta_n\int_{b_n}^{\beta_n} w^\gamma v + \int_{\alpha_n}^{b_n}\bar{f}_n = 0$ holds. For

$$f_n := \bar{f}_n\chi_{[\alpha_n,b_n]} + \theta_n w^\gamma v\chi_{[b_n,\beta_n]},\ n \ge n_0$$

we have

$$\int_c^x f_n = 0, \quad x \in (c, \alpha_n] \cup [\beta_n, d),$$

$$\sup_{x\in(\alpha_n,b_n]} w(x)\left|\int_c^x f_n\right| = \sup_{x\in(\alpha_n,b_n]} w(x)\left|\int_{\alpha_n}^x (\bar{f}_n - f) + \int_c^x f - \int_c^{\alpha_n} f\right|$$
$$\le \frac{1}{n} + 2\sup_{x\in[\alpha_n,b_n]} w(x)\left|\int_c^x f\right| \le 2\|f\|_{\mathrm{Ch}_{\infty,w}(I)} + \frac{1}{n}$$

and

$$\sup_{x\in(b_n,\beta_n)} w(x)\left|\int_c^x f_n\right| \le \sup_{x\in(b_n,\beta_n)} w(x)\left[\left|\int_{\alpha_n}^{b_n}(\bar{f}_n - f) + \int_c^{b_n} f - \int_c^{\alpha_n} f\right| + \int_{b_n}^x w^\gamma v\right]$$
$$\le 2\|f\|_{\mathrm{Ch}_{\infty,w}(I)} + \frac{1}{n} + \sup_{x\in(b_n,\beta_n)} \frac{w(x)[w(x)^{\gamma-1} - w(b_n)^{\gamma-1}]}{(1-\gamma)}$$
$$\le 2\|f\|_{\mathrm{Ch}_{\infty,w}(I)} + \frac{1}{n} + \frac{1}{(1-\gamma)n^\gamma}.$$

Moreover,

$$\left|\int_I fg - \int_I f_n g\right| \le \int_c^{\alpha_n}|fg| + \|g\chi_{[\alpha_n,b_n]}\|_{L^\infty(I)}\int_{\alpha_n}^{b_n}|f-\bar{f}_n| + \int_{b_n}^\infty |fg| + \left|\int_{\beta_n}^\infty w^\gamma vg\right|$$
$$\le \int_c^{\alpha_n}|fg| + \frac{1}{n^2} + \int_{b_n}^\infty |fg| + \frac{1}{n^\gamma}\|vg\|_{L^1(I)}.$$

Thus, $\lim_{n\to\infty}\int_I f_n g = \int_I fg$.

Now we put $F_n(x) := w(x)\int_c^x f_n$, $x \in I$. Then $F_n \in AC_{\mathrm{loc}}(I)$, $\operatorname{supp} F_n$ is a compact in I and $f_n = vF_n + \frac{1}{w}DF_n$ $\mathcal{L}^1$-almost everywhere on I. Using [5, Corollary 5.40], we get

$$\int_I f_n g = \int_I vgF_n + \int_I \frac{1}{w} gDF_n = \int_I vgF_n - \int_I F_n\, d\lambda_{\tilde{g}}.$$

Consequently,

$$\left|\int_I f_n g\right| \le \left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde{g}}\|\right)\sup_{x\in I} w(x)\left|\int_c^x f_n\right|$$

and (3.3) follows by passing to the limit as $n\to\infty$. □

Now we can formulate the criterion of an element $g \in \mathfrak{M}(I)$ belonging to the space $(\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$ and get a two-sided estimate on the norm of the element of the "weak" space in case $v > 0$ $\mathcal{L}^1$-almost everywhere on I.

Theorem 3.1. *Let w satisfy condition* (1.4), $g \in \mathfrak{M}(I)$. *The following statements are equivalent:*
(i) $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$;
(ii) $vg \in L^1(I)$, $g \in L^\infty(I)$, *and* $\chi_{(b,d)}g = 0$ *for some* $b \in I$, $\frac{g}{w}$ *has an* $BPV(I)$ *representative. Moreover, if* $v > 0$ $\mathcal{L}^1$*-almost everywhere on* I*, then*

$$\|g\|_{(\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}} \approx \left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde{g}}\|\right),$$

where $\tilde{g}$ *is an* $BPV(I)$ *representative of* $\frac{g}{w}$.

Proof. (*ii*) $\Rightarrow$ (*i*). For $f \in \mathrm{Ch}_{\infty,w}(I)$ we have $f \in L^1_{\mathrm{loc}}([c,d))$ and therefore

$$\int_I |fg| \le \|g\|_{L^\infty(I)} \int_c^b |f| < \infty.$$

Using Lemma 3.3 for $\tilde g \in \frac{g}{w} \cap BPV(I)$ we get the estimate

$$\|g\|_{(\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}} \le 2\left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde g}\|\right) < \infty.$$

(*i*) $\Rightarrow$ (*ii*). We denote $E := \{x \in I : g(x) \ne 0\}$. Suppose that $\mathcal{L}^1((t,d)\cap E) > 0$ for any $t \in I$. Then there exists $\{[a_k,b_k]\}_1^\infty$ such that $b_k < a_{k+1}$ and $\int_{a_k}^{b_k} |g| > 0$. We choose $\theta_k \in (0,\infty)$ so that the inequality $\theta_k \int_{a_k}^{b_k} |g| \ge 1$ holds. By Lemma 3.1 there exists $f_k \in \mathfrak{M}(I)$ with the properties: $\|f_k\|_{\mathrm{Ch}_{\infty,w}(I)} < 2^{-k}$, $\operatorname{supp} f_k \subset [a_k,b_k]$ and $|f_k| = \theta_k$ on (a_k,b_k). Then for the function $f := \sum_{k=1}^\infty f_k$ we have $\|f\|_{\mathrm{Ch}_{\infty,w}(I)} \le 1$ and

$$\int_I |fg| \ge \sum_{k=1}^\infty \theta_k \int_{a_k}^{b_k} |g| \ge \sum_{k=1}^\infty 1 = \infty.$$

This contradicts $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$. Thus, there exists point $b \in I$ such that $g\chi_{(b,d)} = 0$.

Now we assume that $g \notin L^\infty(I)$. Then there exists $h \in L^1((c,b))$ such that $\int_c^b |hg| = \infty$. Let $a_1 := b$ and $a_k \downarrow c$ as $k \to \infty$. By Lemma 3.1 there exists $f_k \in \mathfrak{M}(I)$ with properties: $\operatorname{supp} f_k \subset [a_{k+1},a_k]$, $\|f_k\|_{\mathrm{Ch}_{\infty,w}(I)} < 2^{-k}$ and $|f_k| = |h|$ on (a_{k+1},a_k). Then for the function $f := \sum_{k=1}^\infty f_k$ we have $\|f\|_{\mathrm{Ch}_{\infty,w}(I)} \le 1$ and

$$\int_I |fg| \ge \int_c^b |hg| = \infty.$$

This contradicts the relation $g \in (\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}$, that is $g \in L^\infty(I)$.

By Lemma 3.2 we have $vg \in L^1(I)$, $\frac{g}{w} \cap BPV(I) \ne \emptyset$. If $v > 0$ $\mathcal{L}^1$-almost everywhere on I the statement 1 of Lemma 3.2 implies the estimate $3\|g\|_{(\mathrm{Ch}_{\infty,w}(I))'_{\mathrm{w}}} \ge \left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde g}\|\right)$ for $\tilde g \in \frac{g}{w} \cap BPV(I)$. □

Using the results for the weighted Cesàro space, we can also characterize the space $(\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}$ in the case of $v > 0$ $\mathcal{L}^1$-almost everywhere on I.

Theorem 3.2. *Let w satisfy condition* (1.4), *$v > 0$ $\mathcal{L}^1$-almost everywhere on I and $g \in \mathfrak{M}(I)$. The following statements are equivalent:*

(*i*) *$g \in (\mathrm{Cs}_{\infty,w}(I), \|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}$;*

(*ii*) *$vg \in L^1(I)$, $\frac{g}{w}$ has an $BPV(I)$ representative and $\int_I v(t)\|g\|_{L^\infty([t,d))}\,dt < \infty$.*

Moreover, $\|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}} \approx \left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde g}\|\right)$, where $\tilde g$ is an $BPV(I)$ representative of $\frac{g}{w}$.

Proof. First, $\int_I v(t)\|g\|_{L^\infty([t,d))}\,dt < \infty$ is equivalent to $g \in (\mathrm{Cs}_{\infty,w}(I))'_{\mathrm{w}}$ by [1, Remark 4.3], [15, Theorem 4].

(*ii*) $\Rightarrow$ (*i*). Since $\int_I v(t)\|g\|_{L^\infty([t,d))}\,dt < \infty$ then $g \in L^\infty_{\mathrm{loc}}(I)$. Moreover, for $f \in \mathrm{Cs}_{\infty,w}(I)$ we have $\int_I |fg| < \infty$, and the estimate

$$\|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}} \le 2\left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde g}\|\right) < \infty$$

follows from Lemma 3.3.

(*i*) $\Rightarrow$ (*ii*). By Lemma 3.2 we have $vg \in L^1(I)$, $\frac{g}{w} \cap BPV(I) \ne \emptyset$ and the estimate $3\|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}} \ge \left(\|vg\|_{L^1(I)} + \|\lambda_{\tilde g}\|\right)$ holds for $\tilde g \in \frac{g}{w} \cap BPV(I)$. Further, since $\|g\|_{(\mathrm{Cs}_{\infty,w}(I))'_{\mathrm{w}}} \le \|g\|_{(\mathrm{Cs}_{\infty,w}(I),\|\cdot\|_{\mathrm{Ch}_{\infty,w}(I)})'_{\mathrm{w}}}$, we obtain $\int_I v(t)\|g\|_{L^\infty([t,d))}\,dt < \infty$. □

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