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KUSSAINOVA LEILI KABIDENOVNA

(to the 70th birthday)

On May 3, 2018 was the 70th birthday of Leili Kabidenovna Kussainova, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department of Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), Professor (2006), Honorary worker of Education of the Republic of Kazakhstan (2005).

L.K. Kussainova was born in the city of Karaganda. In 1972 she graduated from the Novosibirsk State University (Russian Federation) and then completed her postgraduate studies at the Institute of Mathematics (Almaty). L.K. Kussainova's scientific supervisors were distinguished Kazakh mathematicians T.I. Amanov and M. Otelbayev.

Scientific works of L.K. Kussainova are devoted to investigation of the widths of embeddings of the weighted Sobolev spaces, to embeddings and interpolations of weighted Sobolev spaces with weights of general type.

She has solved the problem of three-weighted embedding of isotropic and anisotropic Sobolev spaces in Lebesgue spaces, the problem of exact description of the Lions-Petre interpolation spaces for a pair of weighted Sobolev spaces.

To solve these problems L.K. Kussainova obtained nontrivial modifications of theorems on Besicovitch-Guzman covers. The first relates to covers by multidimensional parallelepipeds, whereas the second relates to double covers by cubes. These modifications have allowed to obtain the description of the interpolation spaces in the weighted case. Furthermore, by using the double covering theorem the exact descriptions of the multipliers were obtained for a pair of Sobolev spaces of general type.

The maximal operators on a basis of cubes with adjustable side length, which were introduced by L.K. Kussainova, have allowed her to solve the problem of two-sided distribution estimate of widths of the embedding of two-weighted Sobolev spaces with weights of general type in weighted Lebesgue spaces.

Under her supervision 6 theses have been defended: 4 candidates of sciences theses and 2 PhD theses.

The Editorial Board of the Eurasian Mathematical Journal congratulates Leili Kabidenovna Kussainova on the occasion of her 70th birthday and wishes her good health and new achievements in mathematics and mathematical education.

The awarding ceremony
of the Certificate of the Emerging Sources Citation of Index database

In 2016 the Eurasian Mathematical Journal has been included in the Emerging Sources Citation of Index (ESCI) of the "Clarivate Analytics" (formerly "Thomson Reuters") Web of Science. In 2018 the second journal of the L.N. Gumilyov Eurasian National University, namely the Eurasian Journal of Mathematical and Computer Applications was also included in ESCI.

The ESCI was launched in late 2015 as a new database within "Clarivate Analytics". Around 3,000 journals were selected for coverage at launch, spanning the full range of subject areas.

The selection process for ESCI is the first step in applying to the Science Citation Index. All journals submitted for evaluation to the core Web of Science databases will now initially be evaluated for the ESCI, and if successful, indexed in the ESCI while undergoing the more in-depth editorial review. Timing for ESCI evaluation will follow "Clarivate Analytics" priorities for expanding database coverage, rather than the date that journals were submitted for evaluation.

Journals indexed in the ESCI will not receive Impact Factors; however, the citations from the ESCI will now be included in the citation counts for the Journal Citation Reports, therefore contributing to the Impact Factors of other journals. If a journal is indexed in the ESCI it will be discoverable via the Web of Science with an identical indexing process to any other indexed journal, with full citation counts, author information and other enrichment. Articles in ESCI indexed journals will be included in an author's H-Index calculation, and also any analysis conducted on Web of Science data or related products such as InCites. Indexing in the ESCI will improve the visibility of a journal, provides a mark of quality and is good for authors.

To commemorate this important achievement of mathematicians of the L.N. Gumilyov Eurasian National University on June 14, 2018, by the initiative of the "Clarivate Analytics", the awarding ceremony of the Certificate of Emerging Sources Citation Index database of "Clarivate Analytics" to the editorial boards of the Eurasian Mathematical Journal and the Eurasian Journal of Mathematical and Computer Applications was held at the L.N. Gumilyov Eurasian National University. The programme of this ceremony is attached.

Astana June 14, 2018

Venue: L.N. Gumilyov Eurasian National University
Astana, Satpayev street 2, Room 259

14:30- 15:00	Visit to the Museum of the history of Education, Museum of L.N. Gumilyov, Museum of writing
15:00-15:10	*Opening speech of moderator* **A. Moldazhanova** – the First Vice-Rector, Vice-Rector for Academic Works of L.N. Gumilyov Eurasian National University
15:10-15:20	**Oleg Utkin** - Managing Director of Clarivate Analytics in Russia and the CIS
15:20-15:30	*Certification award ceremony of the Eurasian Mathematical Journal, the Eurasian Journal of Mathematical and Computer Applications in international database*
15:30-15:45	**Kordan Ospanov** – Deputy Editor-in-Chief of the Eurasian Mathematical Journal. *History and perspectives of development of the scientific journal Eurasian Mathematical Journal*
15:45-16:00	**Kazizat Iskakov** – Deputy Editor-in-Chief of the Eurasian Journal of Mathematical and Computer Applications. *History and perspectives of development of the scientific journal Eurasian Journal of Mathematical and Computer Applications.*
16:00-16:10	*Closing Ceremony* *Memory photo*
16:10-16:30	*Coffee break for visitors*
16:40-17:20	**Lyaziza Mukasheva** - Official representative of Clarivate Analytics in the Central Asian region *Seminar for editors of scientific journals Scientific library of L.N. Gumilyov Eurasian National University room 104*

EURASIAN MATHEMATICAL JOURNAL
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DISCRETENESS AND ESTIMATES OF SPECTRUM OF A FIRST ORDER DIFFERENCE OPERATOR

K.N. Ospanov

Communicated by I.N. Parasidis

Key words: difference operator, coercive estimate, compactness of the resolvent, singular numbers.

AMS Mathematics Subject Classification: 39A70, 47B39.

Abstract. We investigated a minimal closed in the space l_2 first order nonsymmetric difference operator L. The matrix of zero order coefficients of L may be an unbounded operator. The study of L is motivated by applications to stochastic processes and stochastic differential equations. We obtained compactness conditions and exact with respect to the order two-sided estimates for s-numbers of the resolvent of L. Note that these estimates for s-numbers do not depend on the oscillations of the coefficients of L, in contrast to the case of a differential operator.

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1 Introduction

In this work we concider the following first order difference operator

$$L_0 y = -\Delta y + Qy,$$

where $y = \{y_j\}_{j=-\infty}^{+\infty}$, $\Delta y = \{\Delta y_j\}_{j=-\infty}^{+\infty} = \{y_{j+1} - y_j\}_{j=-\infty}^{+\infty}$, and $Q = (q_{ij})_{i,j=-\infty}^{+\infty}$ is a real matrix. Let Φ be the class of all compactly supported sequences (i.e. $\Phi = \{y = \{y_j\}_{j=-\infty}^{+\infty} : \exists n \in \mathbb{N} \ \forall |j| \geqslant n \ y_j = 0\}$). We assume that the domain $D(L_0)$ is Φ. Under certain conditions on the matrix Q (see Theorem 2.2) L_0 is closable in the Hilbert space l_2 of all sequences $u = \{u_j\}_{j=-\infty}^{+\infty}$ with finite norm $\|u\|_2 = \left(\sum_{j=-\infty}^{+\infty} |u_j|^2 \right)^{1/2}$. We denote the closure of L_0 by L.

The study of L is motivated by applications to stochastic processes and stochastic differential equations, see [2, 3] and the references therein. It is well-known that differential operators with continuous coefficients defined on non-compact regions can be reduced to L with a banded matrix Q. The operator L can be used to study regularity properties of the second order differemce equation with fast-growing first order coefficients (see [1] for the one-dimensional case).

The operator L is not symmetric, and the matrix Q can be an unbounded operator on l_2. Properties of linear difference and differential operators with unbounded coefficients differ markedly from those known in the case of bounded coefficients. For instance, elements of the domain $D(L)$ of L need not belong to the difference Sobolev space. Even though $D(L)$ is a subset of the Sobolev space, it may happen that the spectrum of L is not discrete.

In present work we prove the discreteness of the spectrum of L (or compactness of the inverse L^{-1} in l_2). For the compact operator L^{-1}, we give two-sided estimates of the singular values $s_k(L^{-1})$ (eigenvalues of $\sqrt{L^{-1}(L^{-1})^*}$ numbered in nonincreasing order). Here $(L^{-1})^*$ is the adjoint operator to L^{-1}, and $k = 1, 2, \ldots$.

Note that the above mentioned estimates of $s_k(L^{-1})$ are exact with respect to the order and do not depend on oscillations of the sequences $\{q_{ij}\}_{j=-\infty}^{+\infty}$ $(i = 1, 2, \dots)$ and $\{q_{ij}\}_{i=-\infty}^{+\infty}$ $(j = 1, 2, \dots)$. We recall that (see [4] and also [5]), in general, certain oscillation conditions on coefficients are necessary for exact order estimates of singular values of the resolvents of differential operators. The last fact reinforces the motivation for the present study.

We also proved a uniform estimate for $y \in D(L)$ with unbounded weight and for the difference Δy. These estimates show that the solution to the following equation

$$Ly = f, \tag{1.1}$$

with $f \in l_2$ is stable. They also can be used to find an approximate solution to (1.1).

Two-sided estimates for the smallest eigenvalue of a symmetric matrix, which do not depend on the size of the matrix, were proved in [6]. Spectral results for the second order difference operator with tridiagonal matrix are obtained in [1]. For the compactness conditions of singular differential operators we refer to [7, 8, 9, 10, 11, 12] and references therein.

To prove our main results, we will use an a priori estimate for $y \in D(L)$ and theorems on boundedness and compactness of the embedding operator of the weighted difference Sobolev space in l_2, which were obtained in [13], see also [14].

2 Preliminaries

First we give a statement about the separability of a nonlinear operator in the class Φ of compactly supported sequences. We consider the following map

$$Ay = -\Delta y + B(y)$$

in Φ, where $y = \{y_j\}_{j=-\infty}^{+\infty}$, $\Delta y = \{\Delta y_j\}_{j=-\infty}^{+\infty} = \{y_{j+1} - y_j\}_{j=-\infty}^{+\infty}$ and $B(y)$ are real sequences.

Lemma 2.1. *Let $(\cdot\,,\,\cdot)$ be the scalar product in l_2. If*

$$(B(w), w) \geqslant C\|w\|_2^2, \quad \forall w \in \Phi, C > 0, \tag{2.1}$$

then

$$\|\Delta w\|_2 + \|B(w)\|_2 \leqslant \left(2\sqrt{\frac{2}{C}} + 1\right) \|Aw\|_2 \tag{2.2}$$

holds for any $w \in \Phi$,

Proof. We consider the functional (Az, z), where $z = \{z_j\}_{j=-\infty}^{+\infty} \in \Phi$. Since z is compactly supported by (2.1) we obtain

$$(Az, z) \geqslant -\sum_{j=-\infty}^{+\infty} (z_{j+1} - z_j)z_j + C\sum_{j=-\infty}^{+\infty} z_j^2. \tag{2.3}$$

But

$$\sum_{j=-\infty}^{+\infty} (z_{j+1} - z_j)z_j = -\sum_{j=-\infty}^{+\infty} (\Delta z_j)^2 + \sum_{j=-\infty}^{+\infty} (z_{j+1} - z_j)z_{j+1}$$

$$= -\sum_{j=-\infty}^{+\infty} (\Delta z_j)^2 - \sum_{j=-\infty}^{+\infty} (z_{j+1} - z_j)z_j.$$

Hence

$$-\sum_{j=-\infty}^{+\infty}(z_{j+1}-z_j)z_j=\frac{1}{2}\sum_{j=-\infty}^{+\infty}(\Delta z_j)^2.$$

Then by (2.3), we have

$$\|\Delta z\|_2\leqslant\sqrt{\frac{2}{C}}\|Az\|_2,\ \|z\|_2\leqslant\frac{1}{C}\|Az\|_2. \tag{2.4}$$

Therefore

$$\|B(z)\|_2\leqslant\left(\sqrt{\frac{2}{C}}+1\right)\|Az\|_2. \tag{2.5}$$

By (2.4) and (2.5), we get (2.2). □

If $w=\{w_j\}_{j=-\infty}^{+\infty}\in\Phi$, and Q satisfies the following condition:

$$(Qw,w)\geqslant C_0\|w\|_2^2,\quad C_0>0, \tag{2.6}$$

then (2.2), (2.4) and (2.5) imply that

$$\|-\Delta w\|_2+\|Qw\|_2+\|w\|_2\leqslant\left(\frac{1}{C_0}+2\sqrt{\frac{2}{C_0}}+1\right)\|L_0w\|_2. \tag{2.7}$$

It is easy to see that this estimate holds for any $w\in D(L)$.

Using the closedness of L, we can prove that under condition (2.6) the operator L is continuously invertible.

Thus, we proved the following theorem.

Theorem 2.1. *Let $Q=(q_{ij})_{i,j=-\infty}^{+\infty}$ satisfy condition* (2.6)*. Then the operator L is continuously invertible and for any $w\in D(L)$ estimate* (2.7) *holds.*

Inequality (2.7) shows that the gomain $D(L)$ is a subset of the weighed difference Sobolev space W with norm

$$\|w\|_W=\left\{\sum_{i=-\infty}^{+\infty}\left[(\Delta w_i)^2+\left(\sum_{j=-\infty}^{+\infty}q_{ij}w_j\right)^2\right]\right\}^{1/2}.$$

If inequality (2.7) holds with some constant, then L is called a separable operator [4, 14].

3 Main results

Theorem 3.1. *Let $Q=(q_{ij})_{i,j=-\infty}^{+\infty}$ satisfy condition* (2.6) *and, for some $0<C_1\leqslant C_2<\infty$, the following inequalities:*

$$C_1\left(\sum_{j=-\infty}^{+\infty}q_{jj}^2v_j^2\right)^{1/2}\leqslant\|Qv\|_2\leqslant C_2\left(\sum_{j=-\infty}^{+\infty}q_{jj}^2v_j^2\right)^{1/2},$$

for any $v=\{v_j\}_{j=-\infty}^{+\infty}\in\Phi$. Then the inverse L^{-1} to operator L is compact in l_2 if and only if

$$\lim_{|i|\to+\infty}|q_{ii}|=+\infty. \tag{3.1}$$

Proof. By Theorem 2.1 L^{-1} is bounded from l_2 to the weighted Sobolev space W of the sequences $z=\{z_j\}_{j=-\infty}^{+\infty}$ with finite norm

$$\|z\|_W=\left\{\sum_{j=-\infty}^{+\infty}\left[(\Delta z_j)^2+C_1^2q_{jj}^2z_j^2\right]\right\}^{1/2}.$$

It is known (see [13] and [14] (Chapter 9, Theorem 3)) that W is compactly embedded in l_2 if and only if equality (3.1) holds. □

The nth singular (approximation) number of the compact operator T acting in l_2 is the number

$$s_n(T)=\inf_{K\in\{\mathbf{L}_n\}}\|T-K\|_{l_2\to l_2},\quad n=0,\ 1,\ 2,\ \dots,$$

where $\{\mathbf{L}_n\}$ is a collection of all operators of rank $\leqslant n$ acting in l_2. It is well known that $s_n(T)$ is equal to nth eigenvalue of the self-adjoint positive operator $\sqrt{T^*T}$. Let $p\in[1,+\infty)$. If $\sum\limits_{n=0}^{+\infty}[s_n(T)]^p<\infty$, then it is said that the operator T belongs to the Schatten class σ_p. σ_p with the norm

$$\|T\|_{\sigma_p}=\left(\sum_{n=0}^{+\infty}[s_n(T)]^p\right)^{1/p}$$

is a Banach space. If $T\in\sigma_p$ for some $p\in[1,+\infty)$, then it is said that T is an operator of finite type.

We denote by $N(\lambda,T)$ the number of those $s_n(T)$ which exceed $\lambda>0$:

$$N(\lambda,T)=\sum_{\{n:\,s_n(T)>\lambda\}}1.$$

Assume that

$$q_n^*=max\left\{k\geqslant 0:(k+1)^{-1}\geqslant\sum_{j=n-k}^{n+k}q_{jj}^2\right\}\quad(n\in\mathbb{Z}),$$

and

$$A_n=\begin{cases}|q_{nn}|^{-1}, & |q_{nn}|>1\\ q_n^*, & |q_{nn}|\leqslant 1.\end{cases}$$

Theorem 3.2. *Let Q satisfy the assumptions of Theorem 3.1. Then*

$$\frac{1}{24}\sum_{\{n:\sqrt{3}\,C_1^{-1}A_n\geqslant\lambda\}}1\leqslant N\left(\lambda,L^{-1}\right)\leqslant 4\sum_{\{n:4\sqrt{3}\,C_2^{-1}A_n\geqslant\lambda\}}1. \tag{3.2}$$

Proof. $N(\lambda,L^{-1})$ coincides with the number of approximation numbers of the imbedding operator $E:W\to l_2$ which exceed $\lambda>0$. Therefore, by Theorems 5 and 6 of [14] (Chapter 9), we obtain (3.2). □

We denote by $\{B_n\}_{n=0}^{+\infty}$ the nonincreasing rearrangement of the sequence $\{A_n\}_{n=-\infty}^{+\infty}$. Using Corollary 1 of [14] (Chapter 9) and Theorem 3.2, we obtain the following statement.

Theorem 3.3. *Let the matrix Q satisfy the conditions of Theorem* (3.1). *Then the following estimates hold:*

$$\frac{1}{3\sqrt{3}C_1}B_{24n}\leqslant s_n\left(L^{-1}\right)\leqslant\frac{4\sqrt{3}}{C_2}B_{[n/4]},\quad n=0,\ 1,\ 2,\ \dots.$$

Since $\{B_n\}_{n=0}^{+\infty}$ is a nonincreasing sequence, Theorem 3.3 implies the following statement.

Theorem 3.4. *Let* $1 \leqslant p < +\infty$, *and the matrix* Q *satisfy the assumptions of Theorem* (3.1). *Then* $L^{-1} \in \sigma_p$ *if and only if* $\sum\limits_{n=0}^{+\infty} B_n^p < +\infty$. *Moreover, for some* $0 < C_3 \leqslant C_4 < \infty$,

$$C_3 \left(\sum_{n=0}^{+\infty} B_n^p \right)^{1/p} \leqslant \|L^{-1}\|_{\sigma_p} \leqslant C_4 \left(\sum_{n=0}^{+\infty} B_n^p \right)^{1/p}.$$

Example 1. We consider the following operator

$$Ly = -\Delta y + \widetilde{Q} y,$$

where $\widetilde{Q} = (\widetilde{q}_{ij})_{i,j=-\infty}^{+\infty}$, and

$$\widetilde{q}_{ij} = \begin{cases} 10 + |i|, & i = j \\ \frac{1}{4(1+3i^2)(1+4j^2)}, & i \neq j. \end{cases}$$

It is easy to show that, for some $0 < C_5 \leqslant C_6 < \infty$,

$$C_5 \sum_{j=-\infty}^{+\infty} (1 + |j|)^2 y_j^2 \leqslant \left\| \widetilde{Q} y \right\|^2 \leqslant C_6 \sum_{j=-\infty}^{+\infty} (1 + |j|)^2 y_j^2.$$

Therefore, by Theorem 3.1, the inverse L^{-1} to L exists and it is a compact operator in the space l_2. By Theorem 3.3, for $s_n(L^{-1})$ the following inequalities hold: for some $0 < C_7 \leqslant C_8 < \infty$,

$$\frac{C_7}{n+1} \leqslant s_n(L^{-1}) \leqslant \frac{C_8}{n+1}, \quad n = 0,\, 1,\, 2,\, \dots .$$

By Theorem 3.4, $L^{-1} \in \sigma_p$ if and only if $p > 1$. Moreover, L^{-1} is a Hilbert - Schmidt operator.

Acknowledgments

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