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When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

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Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

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- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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KUSSAINOVA LEILI KABIDENOVNA

(to the 70th birthday)

On May 3, 2018 was the 70th birthday of Leili Kabidenovna Kussainova, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department of Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), Professor (2006), Honorary worker of Education of the Republic of Kazakhstan (2005).

L.K. Kussainova was born in the city of Karaganda. In 1972 she graduated from the Novosibirsk State University (Russian Federation) and then completed her postgraduate studies at the Institute of Mathematics (Almaty). L.K. Kussainova's scientific supervisors were distinguished Kazakh mathematicians T.I. Amanov and M. Otelbayev.

Scientific works of L.K. Kussainova are devoted to investigation of the widths of embeddings of the weighted Sobolev spaces, to embeddings and interpolations of weighted Sobolev spaces with weights of general type.

She has solved the problem of three-weighted embedding of isotropic and anisotropic Sobolev spaces in Lebesgue spaces, the problem of exact description of the Lions-Petre interpolation spaces for a pair of weighted Sobolev spaces.

To solve these problems L.K. Kussainova obtained nontrivial modifications of theorems on Besicovitch-Guzman covers. The first relates to covers by multidimensional parallelepipeds, whereas the second relates to double covers by cubes. These modifications have allowed to obtain the description of the interpolation spaces in the weighted case. Furthermore, by using the double covering theorem the exact descriptions of the multipliers were obtained for a pair of Sobolev spaces of general type.

The maximal operators on a basis of cubes with adjustable side length, which were introduced by L.K. Kussainova, have allowed her to solve the problem of two-sided distribution estimate of widths of the embedding of two-weighted Sobolev spaces with weights of general type in weighted Lebesgue spaces.

Under her supervision 6 theses have been defended: 4 candidates of sciences theses and 2 PhD theses.

The Editorial Board of the Eurasian Mathematical Journal congratulates Leili Kabidenovna Kussainova on the occasion of her 70th birthday and wishes her good health and new achievements in mathematics and mathematical education.

The awarding ceremony
of the Certificate of the Emerging Sources Citation of Index database

In 2016 the Eurasian Mathematical Journal has been included in the Emerging Sources Citation of Index (ESCI) of the "Clarivate Analytics" (formerly "Thomson Reuters") Web of Science. In 2018 the second journal of the L.N. Gumilyov Eurasian National University, namely the Eurasian Journal of Mathematical and Computer Applications was also included in ESCI.

The ESCI was launched in late 2015 as a new database within "Clarivate Analytics". Around 3,000 journals were selected for coverage at launch, spanning the full range of subject areas.

The selection process for ESCI is the first step in applying to the Science Citation Index. All journals submitted for evaluation to the core Web of Science databases will now initially be evaluated for the ESCI, and if successful, indexed in the ESCI while undergoing the more in-depth editorial review. Timing for ESCI evaluation will follow "Clarivate Analytics" priorities for expanding database coverage, rather than the date that journals were submitted for evaluation.

Journals indexed in the ESCI will not receive Impact Factors; however, the citations from the ESCI will now be included in the citation counts for the Journal Citation Reports, therefore contributing to the Impact Factors of other journals. If a journal is indexed in the ESCI it will be discoverable via the Web of Science with an identical indexing process to any other indexed journal, with full citation counts, author information and other enrichment. Articles in ESCI indexed journals will be included in an author's H-Index calculation, and also any analysis conducted on Web of Science data or related products such as InCites. Indexing in the ESCI will improve the visibility of a journal, provides a mark of quality and is good for authors.

To commemorate this important achievement of mathematicians of the L.N. Gumilyov Eurasian National University on June 14, 2018, by the initiative of the "Clarivate Analytics", the awarding ceremony of the Certificate of Emerging Sources Citation Index database of "Clarivate Analytics" to the editorial boards of the Eurasian Mathematical Journal and the Eurasian Journal of Mathematical and Computer Applications was held at the L.N. Gumilyov Eurasian National University. The programme of this ceremony is attached.

Astana June 14, 2018

Venue: L.N. Gumilyov Eurasian National University
Astana, Satpayev street 2, Room 259

14:30- 15:00	Visit to the Museum of the history of Education, Museum of L.N. Gumilyov, Museum of writing
15:00-15:10	*Opening speech of moderator* **A. Moldazhanova** – the First Vice-Rector, Vice-Rector for Academic Works of L.N. Gumilyov Eurasian National University
15:10-15:20	**Oleg Utkin** - Managing Director of Clarivate Analytics in Russia and the CIS
15:20-15:30	*Certification award ceremony of the Eurasian Mathematical Journal, the Eurasian Journal of Mathematical and Computer Applications in international database*
15:30-15:45	**Kordan Ospanov** – Deputy Editor-in-Chief of the Eurasian Mathematical Journal. *History and perspectives of development of the scientific journal Eurasian Mathematical Journal*
15:45-16:00	**Kazizat Iskakov** – Deputy Editor-in-Chief of the Eurasian Journal of Mathematical and Computer Applications. *History and perspectives of development of the scientific journal Eurasian Journal of Mathematical and Computer Applications.*
16:00-16:10	*Closing Ceremony* *Memory photo*
16:10-16:30	*Coffee break for visitors*
16:40-17:20	**Lyaziza Mukasheva** - Official representative of Clarivate Analytics in the Central Asian region *Seminar for editors of scientific journals Scientific library of L.N. Gumilyov Eurasian National University room 104*

Short communications

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MARCINKIEWICZ-TYPE INTERPOLATION THEOREM AND ESTIMATES FOR CONVOLUTIONS FOR MORREY-TYPE SPACES

V.I. Burenkov, D.K. Chigambayeva, E.D. Nursultanov

Communicated by V.S. Guliyev

Key words: Morrey spaces, generalized Morrey-type spaces, interpolation theorem, convolution operator.

AMS Mathematics Subject Classification: 42B35, 46E30, 46B70, 47B38, 47G10.

Abstract. We introduce a class of Morrey-type spaces $M^{\lambda}_{p,q,\Omega}$, which includes the classical Morrey spaces. We discuss their properties and we prove a Marcinkiewicz-type interpolation theorem. This theorem is then applied to obtaining a Young-O'Neil-type inequality for the convolution operator in Morrey-type spaces.

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1 Introduction

Let $0 < p \leqslant \infty$ and $0 \leqslant \lambda \leqslant \frac{n}{p}$. The Morrey spaces M^{λ}_{p} are the spaces of all functions $f \in L^{loc}_{p}(\mathbb{R}^n)$ such that

$$||f||_{M^{\lambda}_{p}} \equiv ||f||_{M^{\lambda}_{p}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \sup_{t>0} t^{-\lambda} \|f\|_{L_p(B_t(x))} < \infty,$$

where $B_t(x)$ is the open ball of radius $t > 0$ with center at the point $x \in \mathbb{R}^n$ (see [17]). If $\lambda = 0$, then $M^{0}_{p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = \frac{n}{p}$, then $M^{n/p}_{p}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > \frac{n}{p}$, then $M^{\lambda}_{p} = \Theta$, where Θ is the set of all functions that are equivalent to zero on $\mathbb{R}^n$.

These spaces were introduced by Morrey in 1938 and arose in connection with some problems of the theory of partial differential equations and theory of variations. There is a number of books and survey papers on the Morrey and Morrey-type spaces and classical operators of real analysis in the Morrey-type spaces, for example, see [4], [5], [15], [22], [23], [1].

This paper is devoted to the interpolation properties of the Morrey-type spaces. Some results for the classical Morrey spaces were obtained in Stampacchia [25], Campanato and Murthy [12], Peetre [22]. In particular in [22] it is proved that

$$(M^{\lambda_0}_{p}, M^{\lambda_1}_{p})_{\theta,\infty} \subset M^{\lambda}_{p},$$

where $1 \leqslant p < \infty$, $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, $0 < \theta < 1$. In Ruiz and Vega [24], Blasco, Ruiz and Vega [2] it is proved that this inclusion is strict.

In [16] a more detailed investigation of the interpolation problem for the Morrey spaces was carried out. In particular, it was proved that the inclusion

$$(M^{\lambda_0}_{p_0}, M^{\lambda_1}_{p_1})_{\theta,\infty} \subset M^{\lambda}_{p},$$

where $1 \leqslant p_0, p_1 < \infty$, $0 < \lambda_0 < \frac{n}{p_0}$, $0 < \lambda_1 < \frac{n}{p_1}$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, holds if and only if $p_0 = p_1$.

The case of the local Morrey-type spaces $LM^{\lambda}_{p,q}$ was considered in [6], [7], [10]. Let $0 < p, q \leqslant \infty$, $\lambda > 0$ if $q < \infty, \lambda \geqslant 0$ if $q = \infty$. A function $f \in LM^{\lambda}_{p,q}$ if $f \in L^{loc}_p(\mathbb{R}^n)$ and

$$||f||_{LM^{\lambda}_{p,q}} = \left(\int_0^{\infty} \left(t^{-\lambda} \|f\|_{L_p(B_t(0))} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$

In [10] it was proved, in particular, that the local Morrey-type spaces $LM^{\lambda}_{p,q}$ form an interpolation scale when p is fixed, i.e.

$$(LM^{\lambda_0}_{p,q_0}, LM^{\lambda_1}_{p,q_1})_{\theta,q} = LM^{\lambda}_{p,q}\,,$$

where $0 < p, q_0, q_1, q \leqslant \infty$, $0 < \theta < 1$, $\lambda_0 > 0$ if $q_0 < \infty$, $\lambda_0 \geqslant 0$ if $q_0 = \infty$, $\lambda_1 > 0$ if $q_1 < \infty$, $\lambda_1 \geqslant 0$ if $q_1 = \infty$, $\lambda_0 \neq \lambda_1$ and $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$.

Further generalization of interpolation properties for the general local Morrey-type spaces $LM^{\lambda}_{p,q}(G,\mu)$ was discussed in [6], namely, it was proved that

$$(LM^{\lambda_0}_{p,q_0}(G,\mu), LM^{\lambda_1}_{p,q_1}(G,\mu))_{\theta,q} = LM^{\lambda}_{p,q}(G,\mu)\,,$$

under the same assumptions on the numerical parameters.

In this paper, we introduce the generalized Morrey-type spaces $M^{\lambda}_{p,q,\Omega}$, which coincide with the classical Morrey spaces M^{λ}_p in the case $q = \infty$ and $\Omega = \mathbb{R}^n$. According to the above results it follows that the classical interpolation theorems for this scale of the spaces do not take place. Nevertheless, we prove a certain analogue of an interpolation theorem. Compared with the classical interpolation theorems the assumptions of the theorem are formulated in terms of the local Morrey-type spaces $LM^{\lambda}_{p,q,x}$ and the statement in terms of the generalized Morrey-type spaces $M^{\lambda}_{p,q,\Omega}$. We say that this theorem is an interpolation theorem of Marcinkiewicz-type, because it allows us to obtain from, in a certain sense "weak estimates" for quasi-additive operators in terms of the local Morrey-type spaces, "strong estimates" in terms of the generalized Morrey-type spaces.

Also we study estimates for the norm of the convolution operator

$$(Tf)(x) = (k * f)(x) = \int_{\mathbb{R}^n} k(x-y) f(y) dy$$

in the spaces $M^{\lambda}_{p,q}(\Omega)$, where $k(\cdot)$ is a locally integrable function on $\mathbb{R}^n$.

Let us recall the well-known Young's inequality. Let $1 \leqslant p, q, r \leqslant \infty$ and $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. Then $k * f \in L_q(\mathbb{R}^n)$ for all $f \in L_p(\mathbb{R}^n)$, and the following inequality

$$\|k * f\|_{L_q} \leqslant \|f\|_{L_p} \|k\|_{L_r}$$

holds. For $1 < p, q, r < \infty$, O'Neil [21] has proved the strong Young's inequality

$$\|k * f\|_{L_q} \leqslant c \|f\|_{L_p} \|k\|_{L_{r,\infty}},$$

where $c > 0$ depends only on the parameters p, q, r and $\|k\|_{L_{r,\infty}}$ denotes the norm in the Lorentz spaces.

Further results and generalizations of Young-O'Neil's inequality were obtained in Blozinski [3], Kerman [13], Kostyuchenko and Nursultanov [14], Stepanov [26], Nursultanov and Tikhonov [20], Burenkov and Tararykova [11], and others. In this paper we continue the study Young-O'Neil's inequality in the case of Morrey spaces. In particular, by applying the Marcinkiewicz-type interpolation theorem, we prove a Young-O'Neil-type inequality for the classical Morrey spaces.

2 Spaces $M^\lambda_{p,q,\Omega}$

Let $\Omega \subset \mathbb{R}^n$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $0 \leqslant \lambda \leqslant \frac{n}{p}$. We consider the generalized Morrey-type spaces $M^\lambda_{p,q,\Omega}$ that are defined for $q < \infty$ as the spaces of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ such that

$$||f||_{M^\lambda_{p,q,\Omega}} = \left(\int\limits_0^\infty \left(t^{-\lambda} \sup_{x\in\Omega} \|f\|_{L_p(B_t(x))}\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,$$

and for $q = \infty$,

$$||f||_{M^\lambda_{p,\infty,\Omega}} = \sup_{x\in\Omega} \sup_{t>0} t^{-\lambda} \|f\|_{L_p(B_t(x))} < \infty.$$

Note that the introduced spaces coincide with the classical Morrey spaces in the case $q = \infty$ and $\Omega = \mathbb{R}^n$, i.e.

$$M^\lambda_{p,\infty,\mathbb{R}^n} = M^\lambda_p.$$

However, these spaces differ from the global Morrey-type spaces $GM^\lambda_{p,q,\Omega}$, which are defined as the spaces of all functions f Lebesgue measurable on $\mathbb{R}^n$ with finite quasi-norm

$$\|f\|_{GM^\lambda_{p,q,\Omega}} = \sup_{x\in\Omega} \left(\int\limits_0^\infty \left(t^{-\lambda} \|f\|_{L_p(B_t(x))}\right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

if $q < \infty$ and usual modification if $q = \infty$. For $\Omega = \mathbb{R}^n$ they were introduced by Burenkov and Guliyev [8, 9].

Clearly

$$M^\lambda_{p,q,\Omega} \subset GM^\lambda_{p,q,\Omega}$$

and

$$\|f\|_{GM^\lambda_{p,q,\Omega}} \leqslant \|f\|_{M^\lambda_{p,q,\Omega}}.$$

If $\Omega = \{x\}$ is a singleton, then

$$M^\lambda_{p,q,\Omega} = GM^\lambda_{p,q,\Omega} \equiv LM^\lambda_{p,q,x},$$

where $LM^\lambda_{p,q,x}$ are the local Morrey-type spaces [8, 9].

Note also that the generalized Morrey-type spaces are close to the net spaces $N_{p,q}(M)$ introduced by Nursultanov [18, 19].

3 Interpolation theorem

Theorem 3.1. *Let* $\Omega \subset \mathbb{R}^n$, $0 < \alpha_0, \alpha_1, \beta_0, \beta_1 < \infty$, $\alpha_0 \neq \alpha_1$, $\beta_0 \neq \beta_1$, $0 < p, q \leqslant \infty$, $0 < \sigma \leqslant \tau \leqslant \infty$, $0 < \theta < 1$ *and*

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \ \beta = (1-\theta)\beta_0 + \theta\beta_1.$$

Let T be a quasi-additive operator [1] *given on* $LM^{\beta_0}_{q,\sigma,x} + LM^{\beta_1}_{q,\sigma,x}$, $x \in \Omega$. *Suppose that for some* $M_1, M_2 > 0$ *the following inequalities hold*

$$\|Tf\|_{LM^{\alpha_i}_{p,\infty,x}} \leqslant M_i \|f\|_{LM^{\beta_i}_{q,\sigma,x}}, \quad x \in \Omega, \ f \in LM^{\beta_i}_{q,\sigma,x}, \ i = 0, 1, \tag{3.1}$$

[1] that is for some $A > 0$ and for almost all $y \in \mathbb{R}^n$ $|T(f+g)(y)| \leqslant A(|(Tf)(y)| + |(Tg)(y)|)$ for all functions $f, g \in LM^{\beta_0}_{q,\sigma,x} + LM^{\beta_1}_{q,\sigma,x}$ and $x \in \Omega$.

then

$$\|Tf\|_{M^{\alpha}_{p,\tau,\Omega}} \leqslant cAM_0^{1-\theta}M_1^{\theta}\|f\|_{M^{\beta}_{q,\tau}(\Omega)} \tag{3.2}$$

for all functions $f \in M^{\beta}_{q,\tau,\Omega}$, *where* $c > 0$ *depends only on* $\alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma, \theta$.

Corollary 3.1. *Assume that the assumptions of Theorem* 3.1 *on the numerical parameters are satisfied. Then there exists* $c > 0$, *depending only on* $\alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma, \theta$, *such that if* $x \in \mathbb{R}^n$, T *is a quasi-additive operator on* $LM^{\beta_0}_{q,\sigma,x} + LM^{\beta_1}_{q,\sigma,x}$, *and for some* $M_1, M_2 > 0$ *the following inequalities hold*

$$\|Tf\|_{LM^{\alpha_i}_{p,\infty,x}} \leqslant M_i\|f\|_{LM^{\beta_i}_{q,\sigma,x}}$$

for all functions $f \in LM^{\beta_i}_{q,\sigma,x}$, $i = 0, 1$, *then*

$$\|Tf\|_{LM^{\alpha}_{p,\tau,x}} \leqslant cAM_0^{1-\theta}M_1^{\theta}\|f\|_{LM^{\beta}_{q,\tau,x}}$$

for all functions $f \in LM^{\beta}_{q,\tau,x}$.

Corollary 3.2. *Assume that the assumptions of Theorem* 3.1 *on the numerical parameters are satisfied. Then there exists* $c > 0$, *depending only on* $\alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \sigma, \theta$, *such that if* $\Omega \subset \mathbb{R}^n$, T *is a quasi-additive operator given on* $LM^{\beta_0}_{q,\sigma,x} + LM^{\beta_1}_{q,\sigma,x}$, $x \in \Omega$, *and for some* $M_1, M_2 > 0$ *inequalities* (3.1) *are satisfied, then*

$$\|Tf\|_{GM^{\alpha}_{p,\tau,\Omega}} \leqslant cAM_0^{1-\theta}M_1^{\theta}\|f\|_{GM^{\beta}_{q,\tau,\Omega}}$$

for all functions $f \in GM^{\beta}_{q,\tau,\Omega}$.

4 O'Neil-type inequality for convolutions in Morrey-type spaces

Let, for $0 < p, \theta \leqslant \infty$, $L_{p,\theta}(\mathbb{R}^n)$ be the Lorentz space of functions defined on $\mathbb{R}^n$, and, for a measurable set $D \subset \mathbb{R}^n$, $L_{p,\theta}(D)$ be the set of all functions f measurable on D for which extensions of f by 0 outside D belong to $L_{p,\theta}(\mathbb{R}^n)$.

Lemma 4.1. *Let* $1 < p, q, r < \infty$, $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$, $f \in L_{r,\infty}(B - D)$, $g \in L_p(D)$, *where* $B, D \subset \mathbb{R}^n$ *are measurable sets, then*

$$\Big\| \int_D f(\cdot - y)g(y)dy \Big\|_{L_q(B)} \leqslant c\|f\|_{L_{r,\infty}(B-D)}||g||_{L_p(D)},$$

where $c > 0$ *depends only on the parameters* n, p, q *and* r.

Next we introduce the following variant of local Morrey-type space, based on using Lorentz spaces. Let $0 < p, \theta, q \leqslant \infty$, $\lambda \in \mathbb{R}$. We say that $f \in V^{\lambda}_{p,\theta,q}$ if f is measurable on $\mathbb{R}^n$ and

$$\|f\|_{V^{\lambda}_{p,\theta,q}} \equiv \|f\|_{V^{\lambda}_{p,\theta,q}(\mathbb{R}^n)} = \left(\int_0^{\infty} \left(t^{-\lambda}\|f\|_{L_{p,\theta}(B_t(0)\setminus B_{\frac{t}{2}}(0))}\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$

Lemma 4.2. *Let* $0 < p, \theta, q \leqslant \infty$, $-\infty < \lambda < \infty$. *Then there exist* $c_1, c_2 > 0$ *such that*

$$c_1\left(\sum_{m\in\mathbb{Z}} \left(2^{-\lambda m}\|f\|_{L_{p,\theta}(D_m)}\right)^q\right)^{1/q} \leqslant \|f\|_{V^{\lambda}_{p,\theta,q}} \leqslant c_2\left(\sum_{m\in\mathbb{Z}} \left(2^{-\lambda m}\|f\|_{L_{p,\theta}(D_m)}\right)^q\right)^{1/q},$$

where $D_m = B_{2^{m+1}}(0) \setminus B_{2^m}(0)$, $m \in \mathbb{Z}$.

Theorem 4.1. *Let $z \in \mathbb{R}^n$, $1 < p < q < \infty$, $0 < \nu \leqslant \lambda < \frac{n}{q}$, $1+\frac{1}{q} = \frac{1}{p}+\frac{1}{r}+\frac{\lambda-\nu}{n}$ and $\frac{1}{s} = \frac{1}{r} - \frac{\nu}{n}$. If $k \in L_{r,\infty}(\mathbb{R}^n) \cap V^{-\nu}_{s,\infty,\infty}(\mathbb{R}^n)$, then the convolution operator*

$$(k * f)(x) = \int\limits_{\mathbb{R}^n} k(x-y)f(y)dy$$

is bounded from $LM^{\nu}_{p,1,z}$ to $LM^{\lambda}_{q,\infty,z}$. Moreover, the following estimate holds

$$\|k * f\|_{LM^{\lambda}_{q,\infty,z}} \leqslant c\big(\|k\|_{L_{r,\infty}(\mathbb{R}^n)} + \|k\|_{V^{-\nu}_{s,\infty,\infty}(\mathbb{R}^n)}\big)\|f\|_{LM^{\nu}_{p,1,z}}$$

for all functions $f \in LM^{\nu}_{p,1,z}$, where $c > 0$ depends only on n, p, q, λ, ν.

Remark 1. Note that the assumption $k \in V^{-\nu}_{s,\infty,\infty}(\mathbb{R}^n)$ in Theorem 4.1 is essential, i.e. the following inequality

$$\|k * f\|_{LM^{\lambda}_{q,\infty,z}} \leqslant c\|k\|_{L_{r,\infty}(\mathbb{R}^n)}\|f\|_{LM^{\nu}_{p,1,z}}$$

does not hold for any $c > 0$ depending only on the parameters n, p, q, r, ν, λ.

As an example, we can consider the following functions

$$f(x) = \begin{cases} 1, & |x| \in [l^{\alpha}, l^{\alpha}+1], \quad l \in \mathbb{Z}_+ \\ 0, & \text{otherwise} \end{cases}$$

and

$$k(x) = \begin{cases} (l+1)^{-\frac{1}{r}}, & |x| \in [l^{\alpha}, l^{\alpha}+1], \quad l \in \mathbb{Z}_+ \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha > \frac{1}{\nu p}$. Then, for any $z \in \mathbb{R}^n$, $\|f\|_{LM^{\nu}_{p,1,z}} < \infty$, $\|k\|_{L_{r,\infty}(\mathbb{R}^n)} < \infty$, but $\|k * f\|_{LM^{\lambda}_{q,\infty,z}} = \infty$.

Let $\Omega \subset \mathbb{R}^n$, $0 < p < \infty$, $0 < q \leqslant \infty$, $0 < \lambda < \infty$, ${}^cB_t(x) = \Omega \setminus B_t(x)$. We define the space ${}^cM^{\lambda}_{p,q,\Omega}$ as the set of all Lebesgue measurable functions f, for which the following quasi-norm is finite: if $q < \infty$, then

$$\|f\|_{{}^cM^{\lambda}_{p,q,\Omega}} = \left(\int\limits_0^{\infty} \left(t^{\lambda} \inf_{x \in \Omega} \|f\|_{L_p({}^cB_t(x))} \right)^q \frac{dt}{t} \right)^{1/q},$$

and if $q = \infty$, then

$$\|f\|_{{}^cM^{\lambda}_{p,\infty,\Omega}} = \sup_{t>0} t^{\lambda} \inf_{x \in \Omega} \|f\|_{L_p({}^cB_t(x))}.$$

Theorem 4.2. *Let $0 < p \leqslant q < h < \infty$, $0 \leqslant \nu \leqslant \lambda < \frac{n}{q}$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} + \frac{\lambda-\nu}{n}$, $0 < \tau \leqslant \infty$, $\frac{\gamma}{n} = \frac{1}{p} - \frac{\nu}{n} - \frac{1}{h} > 0$. Then*

$$\|k * f\|_{M^{\lambda}_{q,\tau,\Omega}} \leqslant c\|k\|_{L_{r,\infty}(\mathbb{R}^n} \left(\|f\|_{M^{\nu}_{p,\tau,\Omega}} + \|f\|_{{}^cM^{\gamma}_{h,\tau,\Omega}} \right),$$

where $c > 0$ depends only on n, p, q, λ, ν, h and τ.

Open problem: can in Theorem 3.1 inequalities (3.1) be replaced by the inequalities

$$\|Tf\|_{M^{\alpha_i}_{p,\infty,\Omega}} \leqslant M_i \|f\|_{M^{\beta_i}_{q,\sigma,\Omega}}, \quad f \in M^{\beta_i}_{q,\sigma,\Omega},\ i = 0, 1?$$

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References

[1] D.R. Adams, *Morrey spaces.* Birkhäuser Mathematics, Lecture notes in applied and numerical harmonic analysis (2015), 124 p.

[2] O. Blasco, A. Ruiz, L. Vega, *Non interpolation in Morrey-Campanato and block spaces.* Ann. Scuola Norm. Super. Pisa 28 (1999), no. 1, 31–40.

[3] A.P. Blozinski, *On a convolution theorem for $L_{p,q}$ spaces.* Trans. Amer. Math. Soc. 164 (1972), 255–265.

[4] V.I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces I.* Eurasian Mathematical Journal 3 (2012), no. 3, 11–32.

[5] V.I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces II.* Eurasian Mathematical Journal 4 (2013), no. 1, 21–45.

[6] V.I. Burenkov, D.K. Chigambayeva, E.D. Nursultanov, *Description of the interpolation spaces for a pair of local Morrey-type spaces and their generalizations.* Trudy Math. Inst. Steklov 284 (2014), 97–128.

[7] V.I. Burenkov, D.K. Darbayeva, E.D. Nursultanov, *Description of interpolation spaces for general local Morrey-type spaces.* Eurasian Mathematical Journal 4 (2013), no. 1, 46–53.

[8] V.I. Burenkov, H.V. Guliyev, *Nessesary and sufficient conditions for the boundedness of the maximal operator in the local Morrey-type spaces.* Doklady Ross. Akad. Nauk. Matematika 391 (2003), no. 5, 591–594 (in Russian). English trans. in Russian Acad. Sci. Dokl. Math. (2003).

[9] V.I. Burenkov, H.V. Guliyev, *Nessesary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces.* Studia Mathematica 163 (2004), no. 2, 157–176.

[10] V.I. Burenkov, E.D. Nursultanov, *Description of interpolation spaces for local Morrey-type spaces.* Trudy Math. Inst. Steklov 269 (2010), 46–56.

[11] V.I. Burenkov, T.V. Tararykova, *An analog of Young's inequality for convolutions of functions for general Morrey-type spaces.* Trudy Math. Inst. Steklov 293 (2016), 107–126.

[12] A. Campanato, M.K.V. Murthy, *Una generalizzazione del teorema di Riesz-Thorin.* Ann. Scuola Norm. Super. Pisa 19 (1965), 87–100.

[13] R.A. Kerman, *Convolution theorems with weights.* Trans. Amer. Math. Soc. 280 (1983), no. 1, 207–219.

[14] A.G. Kostyuchenko, E.D. Nursultanov, *On integral operators in L_p-spaces.* Fundam. Prikl. Mat. 5 (1999), no. 2, 475–491.

[15] P.G. Lemarié-Rieusset, *The role of Morrey spaces in the study of Navier-Stokes and Euler equations.* Eurasian Mathematical Journal 3 (2012), no. 3, 62–93.

[16] P.G. Lemarié-Rieusset, *Multipliers and Morrey spaces.* Potential Analysis 38 (2013), no. 3, 741–752.

[17] C.B. Morrey, *On the solution of quasi-linear elliptic partial differential equations.* Trans. Amer. Math. Soc. 43 (1938), 126–166.

[18] E.D. Nursultanov, *Net spaces and the Fourier transform.* Dokl. Akad. Nauk 361 (1998), no. 5, 597–599.

[19] E.D. Nursultanov, *Net spaces and inequalities of Hardy-Littlewood type.* Sb. Math. 189 (1998), no. 3, 399–419.

[20] E.D. Nursultanov, S. Tikhonov, *Convolution inequalities in Lorentz spaces.* J. Fourier Anal. Appl. 17 (2011), 486-505.

[21] R. O'Neil, *Convolution operators and $L(p,q)$ spaces.* Duke Math. J. 30 (1963), 129–142.

[22] J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces.* J. Func. Anal. 4 (1969), 71–87.

[23] M.A. Ragusa, *Operators in Morrey type spaces and applications.* Eurasian Mathematical Journal 3 (2012), no. 3, 94–109.

[24] A. Ruiz, L. Vega, *Corrigenda to "Unique continuation for Schrödinger operators" and a remark on interpolation on Morrey spaces.* Publ. Mat., Barc. 39 (1995), 405–411.

[25] G. Stampacchia, $\mathcal{L}_{p,\lambda}$*-spaces and interpolation.* Comm. Pure Appl. Math. 17 (1964), 293–306.

[26] V.D. Stepanov, *Some topics in the theory of integral convolution operators.* Dalnauka. Vladivostok (2000), 228 p.

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