ISSN 2077–9879

Eurasian Mathematical Journal

2018, Volume 9, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

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- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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KUSSAINOVA LEILI KABIDENOVNA

(to the 70th birthday)

On May 3, 2018 was the 70th birthday of Leili Kabidenovna Kussainova, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department of Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), Professor (2006), Honorary worker of Education of the Republic of Kazakhstan (2005).

L.K. Kussainova was born in the city of Karaganda. In 1972 she graduated from the Novosibirsk State University (Russian Federation) and then completed her postgraduate studies at the Institute of Mathematics (Almaty). L.K. Kussainova's scientific supervisors were distinguished Kazakh mathematicians T.I. Amanov and M. Otelbayev.

Scientific works of L.K. Kussainova are devoted to investigation of the widths of embeddings of the weighted Sobolev spaces, to embeddings and interpolations of weighted Sobolev spaces with weights of general type.

She has solved the problem of three-weighted embedding of isotropic and anisotropic Sobolev spaces in Lebesgue spaces, the problem of exact description of the Lions-Petre interpolation spaces for a pair of weighted Sobolev spaces.

To solve these problems L.K. Kussainova obtained nontrivial modifications of theorems on Besicovitch-Guzman covers. The first relates to covers by multidimensional parallelepipeds, whereas the second relates to double covers by cubes. These modifications have allowed to obtain the description of the interpolation spaces in the weighted case. Furthermore, by using the double covering theorem the exact descriptions of the multipliers were obtained for a pair of Sobolev spaces of general type.

The maximal operators on a basis of cubes with adjustable side length, which were introduced by L.K. Kussainova, have allowed her to solve the problem of two-sided distribution estimate of widths of the embedding of two-weighted Sobolev spaces with weights of general type in weighted Lebesgue spaces.

Under her supervision 6 theses have been defended: 4 candidates of sciences theses and 2 PhD theses.

The Editorial Board of the Eurasian Mathematical Journal congratulates Leili Kabidenovna Kussainova on the occasion of her 70th birthday and wishes her good health and new achievements in mathematics and mathematical education.

The awarding ceremony
of the Certificate of the Emerging Sources Citation of Index database

In 2016 the Eurasian Mathematical Journal has been included in the Emerging Sources Citation of Index (ESCI) of the "Clarivate Analytics" (formerly "Thomson Reuters") Web of Science. In 2018 the second journal of the L.N. Gumilyov Eurasian National University, namely the Eurasian Journal of Mathematical and Computer Applications was also included in ESCI.

The ESCI was launched in late 2015 as a new database within "Clarivate Analytics". Around 3,000 journals were selected for coverage at launch, spanning the full range of subject areas.

The selection process for ESCI is the first step in applying to the Science Citation Index. All journals submitted for evaluation to the core Web of Science databases will now initially be evaluated for the ESCI, and if successful, indexed in the ESCI while undergoing the more in-depth editorial review. Timing for ESCI evaluation will follow "Clarivate Analytics" priorities for expanding database coverage, rather than the date that journals were submitted for evaluation.

Journals indexed in the ESCI will not receive Impact Factors; however, the citations from the ESCI will now be included in the citation counts for the Journal Citation Reports, therefore contributing to the Impact Factors of other journals. If a journal is indexed in the ESCI it will be discoverable via the Web of Science with an identical indexing process to any other indexed journal, with full citation counts, author information and other enrichment. Articles in ESCI indexed journals will be included in an author's H-Index calculation, and also any analysis conducted on Web of Science data or related products such as InCites. Indexing in the ESCI will improve the visibility of a journal, provides a mark of quality and is good for authors.

To commemorate this important achievement of mathematicians of the L.N. Gumilyov Eurasian National University on June 14, 2018, by the initiative of the "Clarivate Analytics", the awarding ceremony of the Certificate of Emerging Sources Citation Index database of "Clarivate Analytics" to the editorial boards of the Eurasian Mathematical Journal and the Eurasian Journal of Mathematical and Computer Applications was held at the L.N. Gumilyov Eurasian National University. The programme of this ceremony is attached.

Astana June 14, 2018

Venue: L.N. Gumilyov Eurasian National University
Astana, Satpayev street 2, Room 259

14:30- 15:00	Visit to the Museum of the history of Education, Museum of L.N. Gumilyov, Museum of writing
15:00-15:10	*Opening speech of moderator* **A. Moldazhanova** – the First Vice-Rector, Vice-Rector for Academic Works of L.N. Gumilyov Eurasian National University
15:10-15:20	**Oleg Utkin** - Managing Director of Clarivate Analytics in Russia and the CIS
15:20-15:30	*Certification award ceremony of the Eurasian Mathematical Journal, the Eurasian Journal of Mathematical and Computer Applications in international database*
15:30-15:45	**Kordan Ospanov** – Deputy Editor-in-Chief of the Eurasian Mathematical Journal. *History and perspectives of development of the scientific journal Eurasian Mathematical Journal*
15:45-16:00	**Kazizat Iskakov** – Deputy Editor-in-Chief of the Eurasian Journal of Mathematical and Computer Applications. *History and perspectives of development of the scientific journal Eurasian Journal of Mathematical and Computer Applications.*
16:00-16:10	*Closing Ceremony* *Memory photo*
16:10-16:30	*Coffee break for visitors*
16:40-17:20	**Lyaziza Mukasheva** - Official representative of Clarivate Analytics in the Central Asian region *Seminar for editors of scientific journals Scientific library of L.N. Gumilyov Eurasian National University room 104*

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 9, Number 2 (2018), 68 – 81

MODEL-THEORETIC PROPERTIES OF THE #-COMPANION OF A JONSSON SET

A.R. Yeshkeyev, M.T. Kasymetova, N.K. Shamatayeva

Communicated by J.A. Tussupov

Key words: perfect Jonsson theory, Jonsson set, Jonsson fragment, companion, categoricity, lattice of perfect fragment.

AMS Mathematics Subject Classification: 03C60, 03C68, 03C10.

Abstract. The work is devoted to the model theory. The subject of this article is connected with the study of incomplete inductive theories. In particular, model-theoretic properties of Jonsson theories are considered, which are subclasses of inductive theories. This paper considers a fragment of a certain Jonsson subset of a semantic model of a fixed Jonsson theory, and, as a class of models, all models of the given fragment are considered, namely, the paper considers the model-theoretic properties of countable and uncountable categoricity and the properties of elimination of quantifiers of the given fragment's #-companions. Also the properties of #-companions of the fragment's existential formulas are investigated.

DOI: https://doi.org/10.32523/2077-9879-2018-9-2-68-81

1 Introduction

This article considers such model-theoretic properties as countable and uncountable categoricity and the properties of #-companions' elimination of quantifiers of the perfect fragment of a certain Jonsson subset of a semantic model of a fixed Jonsson theory. The interest in studying the model-theoretic properties of #-companion goes back to the problems that were formulated when studying inductive theories. This class of theories was well studied in the literature [1] and the main inspirer of the study of this subject was one of the founders of the theory of models Abraham Robinson. The results given in this paper are related to the description of the model-theoretic properties of some, in general, incomplete theories classes, that are subclasses of inductive theories, namely Jonsson theories.

Jonsson theories determine a wide class of known examples of algebras, for example: groups, abelian groups, fields of fixed characteristic, Boolean algebras, linear orders. It should be noted that the most interesting description of the model-theoretic properties of syntactic and semantic character of Jonsson theories and their classes of models was obtained in the case of a perfect Jonsson theory.

It is also important to note that the main research in the field of model theory is carried out within the framework of studying complete theories and their classes of models. Isomorphic embeddings and homomorphisms are considered as morphisms between models, while elementary embeddings are considered in the study of complete theories. In connection with the fact that Jonsson theories, generally speaking, are not complete, it is reasonable to require certain completeness. As a rule, this is a requirement of existential completeness. Considering the fact that the apparatus of the classical models theory is developed for complete theories, we can note the relevance and interest in the transfer or refinement of the model theory's classical results in the framework of the study of Jonsson theories and, accordingly, the development of a technical apparatus for achieving of these objectives.

The concept of a Jonsson set was introduced by one of the authors of this article in [2] and some of its properties were studied in [3-7]. The idea of a Jonsson set goes back to the concept of a linear subspace basis. By the way, the theory of linear spaces over a fixed field is an example of a perfect Jonsson theory.

In this paper we focus our attention on the properties of the #-companion of a Jonsson set's fragment of a perfect fixed Jonsson theory T.

It is known that Jonsson theories, as a subclass of inductive theories, represent a part in which there are certain methods for studying incomplete theories, namely the method of transferring elementary properties of Jonsson theory's center to a Jonsson theory itself. In this paper, this method was first applied to the investigation of the #-companion of a Jonsson subset fragment of a fixed Jonsson theory's semantic model.

As a rule, when studying Jonson's theories, the main objects of research are the following two objects:

(i) a Jonsson theory,

(ii) the class of its existential closed models.

In this paper we are in a more general situation, namely, instead of the theory, we consider the fragment of some Jonsson subset of the fixed Jonsson theory's semantic model, and, as a class of models, all models of this fragment are considered.

It is known that perfect Jonsson theories are quite convenient for model-theoretic studies. In the case of perfectness, we can say that with above-mentioned semantic method, we can give a fairly complete description of the class of models of the center of such a theory [8], for example, such a description was obtained for abelian groups [9].

The concept of the theory's existential primeness of is a natural additional condition when studying Jonsson theory properties. By virtue of the inductivity of theory, the class of existentially closed models of a Jonsson theory is always not empty, but the class of algebraically prime models of considered Jonsson theory can also be empty. The concept of a theory's strong convexity was introduced in A. Robinson's book [10] and later developed in various directions; in this connection we should note the work [11]. A strongly convex theory by definition has a core model, which in its turn is algebraically prime, but it is not necessary is existentially closed. The most spectacular example showing that there are many strongly convex theories is an example of the group theory. This example is characterized by the fact that this is an example of an imperfect Jonsson theory. In the case of the abelian group theory, we have an example of a perfect strongly convex Jonsson theory. The perfectness of a Jonsson theory and some fragment in this theory do not depend on each other, that is, there is a perfect Jonsson theory, with an imperfect fragment in this theory, and, conversely, there is an imperfect Jonsson theory with a perfect fragment in this theory. In this case, a strongly convex perfect Jonsson theory is existentially prime.

Thus, we can conclude that the natural restriction of the class of Jonsson theories to the class of strongly convex perfect Jonsson theories is justified, but separately one can consider a class of existentially prime perfect Jonsson theories.

2 Preliminaries

This section presents the necessary information for understanding the main results of Sections 3 and 4.

Let L be a countable first-order language.

Definition 1. [1, p. 158]. Let T be an arbitrary theory, then a #-companion of a theory, $T^{\#}$ is a theory of the same signature if:

(i) $(T^{\#})_{\forall} = T_{\forall}$;

(ii) if $T_{\forall} = T'_{\forall}$, then $T^{\#} = (T')^{\#}$;

(iii) $T_{\forall\exists} \subseteq T^{\#}$.

The natural interpretations of the companion $T^{\#}$ are T^{*} , T^{f} , T^{M} , T^{l}, where T^{*} is the center of Jonsson theory T, T^{f} is the forcing companion of Jonsson theory T, T^{M} is the model companion of the theory T, $T^{l} = Th(E_T)$, where E_T is the class of existentially closed models of the theory T.

Recall that if E_T is a class of T-existentially closed models of an inductive theory T, then it is always nonempty [8, p. 227].

Definition 2. [1, p. 156]. Let T be an arbitrary theory of a language L. We say that T' is a model companion of T, if

(i) $T'_{\forall} = T_{\forall}$ (T and T' are mutually model-consistent, i.e., any model of T is embedded in the model of T and vice versa),

(ii) T' is model complete.

It is clear that the model companion is a $T^{\#}$-companion of T.

Remark 1. Saying that $T'_{\forall} = T_{\forall}$, we mean exactly that any model of T is a substructure of model of T' and any model of T' is a substructure of model of T.

It is easy to see that T' is a model companion of T if and only if T' is a model companion of $T_{\forall}$.

Proposition 2.1. [1, p. 159]. Let T be an arbitrary theory.

(i) T has a model companion if and only if the class of existentially closed models of $T_{\forall}$ is an elementary class,

(ii) A model companion T, if it exists, is unique and is equal to the theory of existentially closed models of $T_{\forall}$.

A. Robinson also defined the concepts of a finite forcing and a forcing #-companion in model theory [10]. In [10] is shown that a theory in a countable language that admits the joint embedding property has a forcing #-companion, which is a complete theory.

The following theorem shows that Jonsson theory always has a forcing companion, which is a complete theory.

Theorem 2.1. [1, p. 162].

(i) $T^{\#}$ is #-companion of T.

(ii) A theory T is complete if and only if T has the joint embedding property.

Thus, the existence simultaneously of all interpretations of #-companion is connected with the existence of model companion.

Definition 3. [1, p. 80]. A theory T is called a Jonsson theory if:

(i) the theory T has infinite models;

(ii) the theory T is inductive;

(iii) the theory T has the joint embedding property (JEP)

(iv) the theory T has the amalgam property (AP).

Definition 4. [8, p. 169]. The center of a Jonsson theory T is an elementary theory T^{*} of the semantic model C of T, i.e. $T^{*} = Th(C)$.

Definition 5. [8, p. 344]. A Jonsson theory T is called perfect if every semantic model of T is a saturated model of T^*.

Lemma 2.1. [8, p. 157]. The semantic model M of a Jonsson theory T is existentially closed in the theory T .

The following theorem is a criterion of the concept of perfectness.

Theorem 2.2. ***(Criterion of Perfectness)*** [8, p. 158]. Let T be an arbitrary Jonsson theory, then the following conditions are equivalent:
(i) the theory T is perfect;
(ii) T^* is a model companion of the theory T.

Corollary 2.1. In a perfect Jonsson theory, the #-companion coincides with the center of the given theory.

Definition 6. [12, p. 289]. A model of theory is called algebraically prime if it is isomorphically embedded in any model of the considered theory.

Definition 7. [11, p. 157].The model of signature of given theory (hereinafter referred to as the structure) is called a core model if it is isomorphic to the unique substructure of each model of this theory.

Definition 8. [20, p. 105]. Let M be a structure and N is an extension of M . We say that M is existentially closed in N if for any $\overline{a}$ from M and for any quantifier-free formula $f(\overline{x}, \overline{y})$ of language M, if $N \vdash (\exists \overline{y}) f(\overline{a}, \overline{y})$ then $M \vdash (\exists \overline{y}) f(\overline{a}, \overline{y})$.

Definition 9. [8, p. 310]. A theory is called existentially prime if its both classes E_T and AP are exist and has a non-empty intersection.

Definition 10. [11, p. 157]. A theory T is called convex if for any its model $\mathfrak{A}$ and for any collection $\{\mathfrak{B}_i | i \in I\}$ of substructures of $\mathfrak{A}$ which are models of T, the intersection $\bigcap_{i \in I} \mathfrak{B}_i$ is a model of T, provided it is non-empty. If in addition such an intersection is never empty, then T is called strongly convex.

If a theory is strongly convex, then the intersection of all models is contained in some of its model.

Convex theories are theories with the following important algebraic property: every non-empty subset of a model of T generates a unique substructure which is a model of T (namely the intersection of all models of T contained in the given model that contains this set) .

If T is strongly convex, then the intersection of all models of T contained in the given model of T, which is also a model of T (since each model of T contains a core model and a core model does not contain any model of T).

If T satisfies JEP and is strongly convex, then the core model of T is unique up to isomorphism. This model is isomorphic to exactly one substructure of each model of T, and it is uniquely determined as the largest structure with this property. When we talk about the concept of structure, we mean a model of language.

Let a strongly convex Jonsson perfect theory T be complete for existential sentences in the language L.

Definition 11. We say that a set is A-definable if it is definable by some formula. A set A is called a Jonsson theory T if it satisfies the following properties:
(i) A is a definable subset of M;
(ii) $dcl(A)$ is a carrier of some existentially closed submodel M, where $dcl(A)$ is the set of all A-definable elements $a \in A$ such that for some formula $\varphi(x) \in L(A)$, it follows that $\varphi(M) = \{a\}$
.

Let A be a Jonsson set in T and N be an existentially closed submodel of a semantic model M of the considered Jonsson theory T, where $dcl(A) = N$.

We denote by $Th_{\forall\exists}(N)$ the sets of all $\forall\exists$- sentences of the language, which true in the model N.

Definition 12. We say that all $\forall\exists$-consequences of an arbitrary theory create a Jonsson fragment of this theory, if the deductive closure of these $\forall\exists$-consequences is a Jonsson theory.

3 Categoricity of fragments

In this section we give results on a countable and uncountable categoricity of fixed fragments.

We denote by $Fr(A)$ a perfect strongly convex Jonsson fragment of a Jonsson set A .

The following facts characterize the properties of such fragments.

Lemma 3.1. *$Fr(A)$ is a Jonsson theory.*

The proof can be extracted from [8].

Let $Fr^{\#}(A)$ be the $\#$-companion of a fragment $Fr(A)$.

Theorem 3.1. *Let A be a Jonsson set. Then the following conditions are equivalent:*

(i) $Fr(A)$ is perfect;

(ii) $Fr^{\#}(A)$ is $\forall\exists$-axiomatizable.

The proof can be extracted from [8].

Theorem 3.2. *Let A be Jonsson set. Then the following conditions are equivalent:*

(i) $Fr(A)$ is perfect;

(ii) $Fr(A)$ has a model companion.

The proof follows from the criterion of perfectness.

Lemma 3.2. *If $Fr^{\#}(A)$ is the $\#$-companion of a Jonsson fragment $Fr(A)$ and $Fr^{M}(A)$ is the model companion of $Fr(A)$, then $Fr^{\#}(A) = Fr^{M}(A)$.*

The proof follows from Corollary 3.1.

It is easy to verify the validity of the following statement.

Lemma 3.3. *Let A_1 and A_2 be Jonsson sets. Then the following conditions are equivalent:*

(i) $Fr(A_1)$ and $Fr(A_2)$ are mutually model consistent;

(ii) $Fr^{\#}(A_1) = Fr^{\#}(A_2)$.

Proof. If $Fr(A_1)$ and $Fr(A_2)$ are mutually model consistent, then $Fr(A_{1\forall}) = Fr(A_{2\forall})$. Therefore, $Fr^{\#}(A_1) = Fr^{\#}(A_2)$ (by the definition of a $\#$-companion). Conversely, if the $\#$-companions of fragments of Jonsson sets A_1 and A_2 coincide, then $Fr(A_{1\forall}) = Fr(A_{2\forall})$. But according to part (i) of the definition of a $\#$-companion $Fr(A_1) = (Fr^{\#}(A_1))_{\forall}$ and $Fr(A_2) = (Fr^{\#}(A_2))_{\forall}$. Consequently $Fr(A_{1\forall}) = Fr(A_{2\forall})$. Hence, $Fr(A_1)$ and $Fr(A_2)$ are mutually model consistent. □

It is well known that the concepts of model completeness and completeness of theory do not coincide. But there is Lindstrom's theorem [1, p. 54], which connects these concepts. The following theorem has a relation to Lindstrom's theorem on model completeness.

Theorem 3.3. *Let A be Jonsson set and $Fr(A)$ be a perfect Jonsson theory.*
The following conditions are equivalent:
(i) $Fr(A)$ is complete;
(ii) $Fr(A)$ is model complete.

Proof. In the beginning, we note that, from the perfectness of an existentially prime strongly convex Jonsson theory the perfectness of a fragment follows.

(i) $\Rightarrow$ (ii) Let $Fr(A)$ be a complete fragment of a Jonsson set A. Let $Fr^*(A)$ be a central completion of the fragment $Fr(A)$.

As is known $Fr(A) \subseteq Fr^*(A)$, but by the assumption $Fr(A)$ is a complete fragment, hence the theory $Fr(A) = Fr^*(A)$. Then $Fr^*(A)$ is a central completion of a fragment of a Jonsson set of the existentially prime strongly convex Jonsson theory. Then, by the criterion of the perfectness of $Fr(A)$, the fragment $Fr(A)$ is a perfect fragment of a Jonsson set A. But then the center of this fragment is equal to its model companion, and the latter is a model complete theory.

(ii)$\Rightarrow$(i). The proof uses JEP of fragment $Fr(A)$ and the model completeness of $Fr^{\#}(A)$.

We assume the opposite, i.e. $Fr(A)$ is incomplete, then there is a sentence φ of the fragment $Fr(A)$, such that φ is not deducible in $Fr(A)$ and $\neg\varphi$ is not deducible in $Fr(A)$. Hence, $Fr(A) \bigcup \varphi$ is an inconsistent set of sentences and $Fr(A) \bigcup \neg\varphi$ is an inconsistent set of sentences. Therefore, there is a model $A_1 \in Mod\ (Fr(A) \bigcup \varphi)$ and there is a model $A_2 \in Mod\ (Fr(A) \bigcup \neg\varphi)$. Therefore, $A_1 \models \varphi$ and $A_2 \models \neg\varphi$. Then, $A_1 \in Mod\ (Fr(A)$,$A_2 \in Mod\ (Fr(A)$. Since $Fr(A)$ admits JEP, then there exist such isomorphic embeddings $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$, where $B \models Fr(A)$, but $Fr(A)$ is model-complete, hence f_1, f_2 are elementary embeddings. Then $B \models \varphi \wedge \neg\varphi$. We obtained a contradiction. Therefore, $Fr(A)$ is complete. □

Corollary 3.1. Mutually model consistent complete Jonsson fragments of Jonsson subsets of a semantic model of an existentially prime strongly convex Jonsson theory are logically equivalent to each other.

It is clear that the core model of theory is minimal in this sense.

Futher we consider the model-theoretic properties of a Jonsson fragment $Fr(A)$ of a Jonsson subset A of a semantic model of some existentially prime strongly convex first-order countable language's Jonsson theory T.

Definition 13. [12, p. 292]. A formula $\varphi(x_1, x_2, ..., x_n)$ is a $\triangle$-formula with respect to the theory T if there are existential formulas $\psi_1(\overline{x})$ and $\psi_2(\overline{x})$ such that $T \models (\varphi \leftrightarrow \psi_1)$ and $T \models (\varphi \leftrightarrow \psi_2)$.

Note that existential formulas are sometimes called the $\exists$-formulas.

Thus, $\triangle$-formulas are formulas invariant under embeddings between models of T. Together with the $\exists$-formulas they constitute the main classes which are used for defining relations in algebraically prime models.

Definition 14. [12, p. 293].

(i) $(A, a_0, x_1, ..., a_{n-1}) \Rrightarrow_\Gamma (B, b_0, b_1, ..., b_{n-1})$ means that for every formula $\varphi(x_0, x_1, ..., x_{n-1})$ in , if $A \models \varphi(\overline{a})$, then $B \models \varphi(\overline{b})$.

(ii) $(A, \overline{a}) \equiv_\Gamma (B, \overline{b})$ means that $(A, \overline{a}) \Rrightarrow_\Gamma (B, \overline{b})$ and $(B, \overline{b}) \equiv_\Gamma (A, \overline{a})$.

As classes we will consider either $\triangle$ or $\exists$.

Definition 15. [12, p. 293]. A formula $\varphi(x_0, x_1, ..., x_n)$ is complete for Γ-formulas if φ is consistent with T and for every formula $\psi(x_0, x_1, ..., x_n)$ in Γ having no more free variables than φ either $T \models \forall\overline{x}(\varphi \to \psi)$ or $T \models \forall\overline{x}(\varphi \to \neg\psi)$.

Definition 16. [12, p. 293]. A model $\mathfrak{A}$ is a Γ_1, Γ_2 -atomic model of T if $\mathfrak{A}$ is a model of T and for every n, every n-tuple of elements of A satisfies in $\mathfrak{A}$ some formula in Γ_1 which is complete for Γ_2-formulas.

Let $Fr(A)$ be a perfect strongly convex fragment be complete for existential sentences. Then $Fr(A)$ is an existentially prime theory and, in particular, it has a core model, which due to the fact [12, p. 308], that this core model is $(\triangle, \exists)$ - atomic.

R_1 means that for every $\exists$-formula consistent with T, there is $\theta(\overline{x}) \in \triangle$ consistent with T, which implies in T the $\varphi\overline{x}$-formula.

We define the concepts and results connected with them, which are necessary for the proof of Theorem 3.6.

Theorem 3.4. ***(Saracino)*** [13]. If L is a countable language and T is a complete ω -categorical theory, then T has an ω-categorical model companion.

Definition 17. [12, p. 304].

(i) $\mathfrak{A}$ is called a $\sum$*nice*-algebraically prime model of T, if $\mathfrak{A}$ is a countable model of T and for every model $\mathfrak{B}$ of T, every $n \in \omega$ and for all $a_0, a_1, ..., a_{n-1} \in A, b_0, b_1, ..., b_{n-1} \in B$, if $(A, a_0, a_1, ..., a_{n-1}) \Rightarrow_\exists (B, b_0, b_1, ..., b_{n-1})$, then for every $a_n \in A$ there is some $b_n \in B$ such that $(A, a_0, a_1, ..., a_n) \Rightarrow_\exists (B, b_0, b_1, ..., b_n)$.

(ii) $\mathfrak{A}$ is called a $\sum^*$*nice*-algebraically prime model of T, if $\mathfrak{A}$ is a countable model and for every model $\mathfrak{B}$ of T, every $n \in \omega$ and for all $a_0, a_1, ..., a_{n-1} \in A, b_0, b_1, ..., b_{n-1} \in B$, if $(A, a_0, a_1, ..., a_{n-1}) \equiv_\exists (B, b_0, b_1, ..., b_{n-1})$, then for every $a_n \in A$ there is some $b_n \in B$ such that $(A, a_0, a_1, ..., a_n) \equiv_\exists (B, b_0, b_1, ..., b_n)$.

Remark 2. For a $\exists$-complete perfect strongly convex fragment, from [12, p. 308] it follows that, since this fragment has a core model, it is a core model. And accordingly, from $(\triangle, \exists)$-atomicity follows $(\sum, \sum)$-atomicity [12, p. 320].

Theorem 3.5. *Let $Fr(A)$ be $\exists$-complete perfect strongly convex Jonsson fragment of Jonsson set A and $\mathfrak{A}$ is a countable model of the theory $Fr(A)$.*

Then (i)⇒(ii) and (ii)⇒(iii), where:

(i) $\mathfrak{A}$ is $(\sum, \sum)$-atomic,

(ii) $\mathfrak{A}$ is $\sum^*$ −nice,

(iii) $\mathfrak{A}$ is existentially closed and $\sum$ −nice.

Proof follows by Remark 1 and [12, p. 305].

Theorem 3.6. [12, p. 302]. Let T be a complete theory for existential sentences. Then any two countable $(\sum, \sum)$-atomic models of T are isomorphic.

Theorem 3.7. *Let $Fr(A)$ be a $\forall\exists$- complete perfect strongly convex Jonsson fragment of a Jonsson set A. Then the following conditions are equivalent:*

(i) $Fr^{\#}(A)$ is ω -categorical;

(ii) $Fr(A)$ is ω-categorical.

Proof. (i)⇒(ii) Let $Fr^{\#}(A)$ be ω-categorical. By part (ii) of Theorem 2.1 $Fr^{\#}(A)$ is a complete theory. Then, by Theorem 3.4, $Fr^{\#}(A)$ has an ω-categorical $\#$-companion $Fr^{\#'}(A)$. By virtue of model consistency $Fr(A)$ and $Fr^{\#}(A)$, $Fr^{\#}(A)$ and $Fr^{\#'}(A)$, $Fr^{\#'}(A)$ is model consistent with $Fr(A)$, therefore, $Fr^{\#'}(A)$ is a $\#$-companion of $Fr(A)$, in particular, $Fr^{\#'}(A)$ is model-complete. By virtue of model completeness of $Fr^{\#'}(A)$, any formula in a language $Fr^{\#'}(A)$ is equivalent to some $\exists$-formula. Then, by virtue of Robinson's theorem on the uniqueness of a model companion, and by the criterion of perfectness of a Jonsson theory, it follows, that

$Fr^{\#}(A) = Fr^{\#'}(A)$. Since $Fr^{\#'}(A)$ is ω-categorical, its unique countable model is countably saturated and belongs to $Mod\ Fr(A)$, since $Mod\ Fr^{\#}(A) \subseteq Mod\ Fr(A)$. By virtue of the criterion of perfectness of Jonsson theory, follows that $Mod\ Fr^{\#}(A) = E_{Fr(A)}$ and in $E_{Fr(A)}$ there is a unique, up to an isomorphism, countable model M, which is (L, L)-atomic (in the sense of Definition 17), where L is a whole language. Hence M is a $(\sum_1, \sum_1)$-atomic model of $Fr^{\#}(A)$, due to its model completeness ($Fr^{\#}(A) = Fr^{\#'}(A)$). By virtue of $\exists$-completeness of $Fr(A)$, M is the a $(\sum_1, \sum_1$-atomic model of $Fr(A)$. Then, by Theorem 3.5, the model M is a $\sum^* -nice$-model. Let $B \in Mod\ Fr(A), card\ B = \omega$. In view of $\exists$-completeness of $Fr(A)$, we have that $M \equiv_{\exists} B$ (the basis of induction), then by the definition of $\sum^* -niceness$ we obtain inductively that $(M, a)_{a \in M} \equiv_{\exists} (M_1, f(a))_{a \in M}$. Let an f-mapping be such that $f(a) = a$ for any $a \in M$. Then $(M, a)_{a \in M} \equiv_{\exists} (B, f(a))_{a \in M}$. Hence $M \preceq_{\exists} B$. Therefore B is a $(\sum_1, \sum_1)$-atomic model. Then by Theorem 3.6 we have that $B \cong M$. Since a model B is arbitrary, the fragment $Fr(A)$ is ω-categorical.

(ii)⇒(i) Let a fragment $Fr(A)$ be ω-categorical. Suppose that the fragment $Fr^{\#}(A)$ is not ω-categorical, then there are nonisomorphic countable models M and B of the fragment $Fr^{\#}(A)$. But since $Fr(A) \subseteq Fr^{\#}(A)$, then $Mod\ Fr(A) \subseteq Mod\ Fr^{\#}(A)$. Consequently, M and B belong to $Mod\ Fr(A)$. We obtained a contradiction with the ω-categoricity of $Fr(A)$. □

Now we define the concepts and the results associated with them, necessary for proving Theorem 3.10.

Definition 18. [12, p. 307]. We say that a theory T admits R_1, if for any existential formula $\varphi(\overline{x})$ consistent with T there is a formula $\psi(\overline{x}) \in \triangle$ consistent with T such that $T \models \psi \to \varphi$.

Definition 19. [12, p. 301]. A countable model of T is called a countably algebraically universal model if all countable models of the given theory are isomorphically embedded in it.

Theorem 3.8. [12, p. 309]. Let T be a $\forall\exists$- theory complete for existential sentences, and assume that T satisfies R_1. Then the following conditions are equivalent:

(i) T has an algebraically prime model,
(ii) T has a $(\exists, \triangle)$ -atomic model,
(iii) T has a $(\triangle, \exists)$ -atomic model,
(iv) T has a $\triangle - nice$ algebraically prime model,
(v) T has exactly one algebraically prime model.

With the help of the following concept, M. Morley (see [14, p. 266]) established a criterion of ω_1-categoricity of a complete theory.

Definition 20. [12, p. 278]. A model $\mathfrak{A}$ is called a proper prime elementary extension of $\mathfrak{B}$ if $\mathfrak{A} \supsetneqq \mathfrak{B}$ and for any model $\mathfrak{C}$ such that $\mathfrak{C} \supsetneqq \mathfrak{B}$ follows that $\mathfrak{A} \prec \mathfrak{C}$.

Theorem 3.9. ***(Morley)*** [14, p. 267]. A complete theory T is ω_1 - categorical if and only if any countable model has a proper prime elementary extension.

In the framework of study of Jonsson theories, we give an analogue of Definition 20.

Definition 21. [8, p. 188]. Let $\mathfrak{A}, \mathfrak{B} \in E_T$ and $\mathfrak{A} \subsetneqq \mathfrak{B}$. Then $\mathfrak{B}$ is called an algebraically prime model extension of $\mathfrak{A}$ in E_T if for any model $\mathfrak{C} \in E_T$ from the fact that $\mathfrak{A}$ is isomorphically imbedded in $\mathfrak{C}$ follows that $\mathfrak{B}$ is isomorphically imbedded in $\mathfrak{C}$.

The following result is a refinement of Theorem 3.9 in the framework of study of Jonsson theories' fragments.

Theorem 3.10. Let $Fr(A)$ be Jonsson fragment, which is an existentially prime perfect Jonsson theory complete for an existential sentences of a Jonsson universal theory, for which R_1 is satisfied. Then the following are equivalent:

(i) the theory $Fr^{\#}(A)$ is ω_1-categorical,

(ii) any countable model in $E_{Fr(A)}$ has an algebraically prime model extension in $E_{Fr(A)}$.

Proof. (i)⇒(ii) Let $Fr^{\#}(A)$ be ω_1-categorical, then it is perfect by virtue of Morley's theorem on uncountable categoricity. Then, by virtue of criterion of perfectness of a Jonsson theory, we have what $Fr^{\#}(A)$ is a model complete theory and $Mod\ Fr^{\#}(A) = E_{Fr(A)}$. If $Fr^{\#}(A)$ is a model complete, then any isomorphic embedding is elementary. Since $Fr^{\#}(A)$ is a complete theory, then, applying Theorem 2.2, we obtain the required statement.

(ii)⇒(i) By applying Lemma 3.11.2 [8] to the semantic model $\mathfrak{C}$ of an existential prime perfect $Fr(A)$ Jonsson theory (it exists, i.e. $Fr(A)$-Jonsson theory), we get that the model is ω-universal. In general, its cardinality is more than countable. Therefore we consider its countable elementary submodel $\mathfrak{D}$. Since $\mathfrak{C}$ is existentially closed (Lemma 3.11.3. [8]), its elementary submodel $\mathfrak{D}$ is also existentially closed. Hence we have that it is countably algebraically universal. Since the fragment by the assumption is existentially prime, then $Fr(A)$ has an algebraically prime model $\mathfrak{A}_0$. We define by induction $\mathfrak{A}_{\delta+1}$, which is an algebraically prime model extension of model $\mathfrak{A}_\delta$ and $\mathfrak{A}_\lambda = \cup\{\mathfrak{A}_\delta | \delta < \lambda\}$. Then let $\mathfrak{A} = \cup\{\mathfrak{A}_\delta | \delta < \omega_1\}$. Suppose that $\mathfrak{B} \models Fr(A)$ and *card* $\mathfrak{B} = \omega_1$. In order to show that $\mathfrak{B} \approx \mathfrak{A}$, we decompose $\mathfrak{B}$ into a chain $\{\mathfrak{B}_\delta \mid \delta < \omega_1\}$ of countable models. In view of the existential prime Jonsson theory $Fr(A)$, this is possible. We define a function $g : \omega_1 \to \omega_1$ and a chain $\{f_\delta : \mathfrak{A}_{g\delta} \to \mathfrak{B}_\delta \mid 0 < \delta < \omega_1\}$ of isomorphisms by induction on δ:

(i) $g0 = 0$ and $f_0 : \mathfrak{A}_o \to \mathfrak{B}_o$.

(ii) $g\lambda = \bigcup\{g\delta | \delta < \lambda\}$ and $f_\lambda = \bigcup\{f_\delta | \delta < \lambda\}$.

(iii) $f_{\delta+1}$ is equal to the union of the chain $\{f_\delta^\gamma | \gamma \leq \rho\}$, which is determined by induction on γ.

(iv) $f_{\delta+1}^0 = f_\delta$, $f_{\delta+1}^\lambda = \bigcup\{f_{\delta+1}^\lambda | \gamma < \lambda\}$.

(v) Suppose that $f_1^\gamma : \mathfrak{A}_{g\delta+\gamma} \to \mathfrak{B}_{\delta+1}$. If $f_\delta^{\gamma+1}$ is a mapping onto, then $\rho = \gamma$. Otherwise, by virtue of the algebraic primeness of $\mathfrak{A}_{g\delta+\gamma+1}$ we can continue $f_\delta^{\gamma+1}$ to $f_{\delta+1}^{\gamma+1} : \mathfrak{A}_{g\delta+\gamma+1} \to \mathfrak{B}_{\delta+1}$.

(vi) $g(\delta + 1) = g\delta + \rho$.

It is clear that $f = \bigcup\{f_\delta | \delta < \omega_1\}$ isomorphically maps $\mathfrak{A}$ on $\mathfrak{B}$. Now it remains to apply Theorem 3.8. Since $\mathfrak{B}$ is an arbitrary model of $Fr(A)$, and $\mathfrak{A}$ is exactly one algebraic prime and existentially closed model by virtue of the assumption and construction, it follows that, $E_{Fr(A)}$ of uncountable cardinality, it has exactly one model, hence the semantic model of an existential prime Jonsson theory T is saturated, i.e. an existential prime Jonsson theory $Fr(A)$ is perfect. This implies that $Mod\ Fr^{\#}(A) = E_{Fr(A)}$. Consequently, $Fr^{\#}(A)$ is ω_1-categorical. □

4 Properties of #-companions of lattices of existential formulas of a fragment

In order to obtain the results of this section, we are largely indebted to F. Weisspfening's paper [15], in which he studies the properties of lattices of existential formulas for inductive theories.

In this section we will use and refer to the definitions of concepts and the results from [15] and other sources [8], [16-17], we will use their Jonsson analogues to describe the properties of lattices of perfect fragments of a fixed Jonsson theory and their relationship with the #-companion of this fragment.

We recall the main definitions of paper [15]. Let a theory T be fixed.

Definition 22. [15, p. 843]. Let $\varphi^T, \psi^T \in E_n(T)$ and $\varphi^T \cap \psi^T = 0$. Then ψ^T is called a complement of φ^T, if $\varphi \cup \psi = 1$; ψ^T is a pseudo-complement of φ^T, if for all $\mu^T \in E_n(T)$ $\varphi^T \cap \mu^T = 0 \Rightarrow \mu^T \leq \psi^T$. ψ^T is called weakly complemented φ^T, if for all $\mu^T \in E_n(T)$ $(\varphi^T \cup \mu^T) \cap \mu^T = 0 \Rightarrow \mu^T = 0$.

Definition 23. [15, p. 843].

(i) φ^T is called complemented, if φ^T has a complement.
(ii) φ^T is called weakly complemented, if φ^T has a weakly complement.
(iii) φ^T is called pseudo-complemented, if φ^T has a pseudo-complement.
(iv) $E_n(T)$ is called complemented, if every is $\varphi^T \in E_n(T)$ complemented.
(v) $E_n(T)$ is called weakly complemented, if every is $\varphi^T \in E_n(T)$ weakly complemented.
(vi) $E_n(T)$ is called pseudo-complemented, if every is $\varphi^T \in E_n(T)$ pseudo-complemented.

Futher we consider the formulas preserved under extensions of models and submodels.

Definition 24. [16, p. 147]. A formula $\varphi(x_1, ..., x_n)$ is called preserved under extensions of models in the $Mod\,T$, if for any models A and B of the theory T such that $A \subset B$ and, for any $a_1, ..., a_n \in A$, $A \models \varphi[a_1, ..., a_n] \Rightarrow B \models \varphi[a_1, ..., a_n]$.

Definition 25. [16, p. 147]. A formula $\varphi(x_1, ..., x_n)$ is called is preserved under submodels in the $Mod\,T$, if for any models A and B of the theory T such that $A \subset B$ and, for any $a_1, ..., a_n \in A$, $B \models \varphi[a_1, ..., a_n] \Rightarrow A \models \varphi[a_1, ..., a_n]$.

Now we consider the concept of an invariant formula and the relation between the invariance of an existential formula and the complementarity of its class in $E(T)$.

Definition 26. [15, p. 844]. A formula φ is called invariant in $Mod\,T$, if it is simultaneously preserved under extensions of models in $Mod\,T$ and submodels in $Mod\,T$.

Definition 27. [15, p. 843]. A theory T is called positively model complete if it is model complete and every existential formula of a language L is equivalent in T to a positive existential formula.

We introduce the necessary definitions and state known results, which establish the relationship between model completeness, elimination of quantifiers of a theory T and properties of lattices of existential formulas $E_n(Fr(A))$.

We can refine some results from [8] and the classical model theory in the frame of study of fragments of Jonsson theories.

The following results will be used to obtain the main results of this section.

Theorem 4.1. [15, p. 846]. Existential formulas φ is invariant in $Mod\,(Th_{\forall\exists}(E_T))$, where E_T is the class of all existentially closed models of T, if and only if φ^T is weakly complemented in $E(T)$.

Theorem 4.2. [16, p. 145].

(i) A theory T is model complete if and only if every formula is preserved under submodels in $Mod\,(T)$.

(ii) A theory T is model complete if and only if every formula is preserved under extensions of models in $Mod\,(T)$.

Theorem 4.3. [16, p. 145].

(i) Let T' be a model companion of a theory T, where T is a universal theory. Then, T' is a model completion of T if and only if the theory T admits elimination of quantifiers.

(ii) Let T' be a model companion of T. Then, T' is a model completion of T if and only if the theory T has the amalgama property.

Theorem 4.4. [16, p. 145]. A theory T is a submodel complete if and only if T admits the elimination of quantifiers

Theorem 4.5. [15, p. 843]. A theory T is positively model complete if and only if every $\varphi^T \in E_n(T)$ has a positively existential complement.

Theorem 4.6. [15, p. 843]. A theory T has a model companion if and only if $\varphi^T \in E_n(T)$ has a weakly complement.

Theorem 4.7. [15, p. 843]. A theory T has a model companion if and only if $\varphi^T \in E_n(T)$ is a Stone algebra.

Theorem 4.8. [15, p. 843]. A theory T has a model companion if and only if every $\varphi^T \in E_n(T)$ has a weakly quantifier-free complement.

Theorem 4.9. [8, p. 165]. Let T be a Jonsson theory. Then the following conditions are equivalent:

1) T is perfect;
2) T has a model companion.

In [14], [19] a connection was established between the completeness and model completeness of a Jonsson theory.

Theorem 4.10. *Let T be a perfect Jonsson theory. Then the following conditions are equivalent:*

(i) T is complete;
(ii) T is model complete.

In [5] a connection was established between the perfectness of a Jonsson theory and the properties of the lattice $E_n(T)$.

Theorem 4.11. *Let T be complete for $\exists$-sentences of a Jonsson theory. Then the following conditions are equivalent:*

(i) T is perfect;
(ii) T^* is model-complete;
(iii) $E_n(T)$ is a Boolean algebra.

Let us now study the model-theoretic properties of the #-companion of a perfect Jonsson fragment. Let T be a Jonsson theory of a countable language L, A be a Jonsson subset of the semantic model of T, $Fr(A)$ be a fragment of the Jonsson set A. Let $E_n(Fr(A))$ be the distributive lattice of equivalence classes of $\varphi^{Fr(A)} = \{\psi \in E_n(L) | Fr(A) \vdash \varphi \leftrightarrow \psi, \varphi \in E_n(L), E(Fr(A)) = \bigcup_{n<\omega} E_n(Fr(A))\}$.

Now we consider a fragment of a Jonsson set A, which is complete for $\exists$-sentences.

Theorem 4.12. *Let $Fr(A)$ be a perfect fragment of a Jonsson set A, $Fr^{\#}(A)$ be the #-companion. Then*

(i) $Fr^{\#}(A)$ admits elimination of quantifiers if and only if every $\varphi \in E_n(Fr(A))$ has a quantifier-free complement;

(ii) $Fr^{\#}(A)$ is positively model-complete if and only if every $\varphi \in E_n(Fr(A))$ has an existential complement.

Proof. (i). Since $Fr(A)$ is a fragment of a Jonsson set A and M' is a semantic model of $Fr(A)$, since $Fr(A)$ is perfect,we have that $Fr^{\#}(A) = Th(M')$ and the center admits elimination of quantifiers.

Then by Theorem 4.4 $Fr^{\#}(A)$ is submodel complete. Then the theory $Fr^{\#}(A)$ by definition is model complete, and by Theorem 4.11, $E_n(Fr(A))$ is a Boolean algebra, i. e. every $\varphi^{Fr(A)} \in E_n(Fr(A))$ has a complement.

By elimination of quantifiers $Fr^{\#}(A)$, where $Fr^{\#}(A)$ is the completion of Jonsson fragment $Fr(A)$, and every class $\varphi \in E_n(Fr(A))$ has a quantifier-free complement.

Conversely, suppose that every class $\varphi \in E_n(Fr(A))$ has a quantifier-free complement. Then $E_n(Fr(A))$ is a Boolean algebra, then by Theorem 4.11, $Fr^{\#}(A)$ is model-complete, and then, in its turn, by virtue of part (ii) of Theorem 4.2, we have that any formula of the theory $Fr^{\#}(A)$ is equivalent to some existential formula, i.e. the class of all such formulas belongs to $E_n(Fr^{\#}(A))$. By $\exists$-completeness of the theory $Fr(A)$, $E_n(Fr(A)) = E_n(Fr^{\#}(A))$. Consequently, by virtue of that, every $\varphi^{Fr(A)} \in E_n(Fr(A))$ has a quantifier-free complement, $E_n(Fr(A))$ is a Boolean algebra, and every formula in $E_n(Fr^{\#}(A))$ is a without quantifiers formula. Therefore, $Fr^{\#}(A)$ admits elimination of quantifiers.

We now prove part (ii). Let $Fr^{\#}(A)$ be positively model-complete. Then by Definition 22 the theory $Fr^{\#}(A)$ is model-complete and for every existential formula φ there is a positive existential formula ψ such that $Fr^{\#}(A) \vdash \varphi \leftrightarrow \psi$. By Theorem 4.11, $E_n(Fr(A))$ is a Boolean algebra, i. e. every $\varphi^{Fr(A)} \in E_n(Fr(A))$ has an existential complement, and since for every existential formula φ there is a positive existential formula ψ such that $Fr^{\#}(A) \vdash \varphi \leftrightarrow \psi$, we obtain that every $\varphi \in E_n(Fr(A))$ has a positive existential complement. Thus, the necessary condition of part (ii) is proved.

We prove the sufficiency of part (ii). Let every $\varphi^{Fr(A)} \in E_n(Fr(A))$ have a positive existential complement. Then, by Theorem 4.5, $Fr(A)$ is positive model-complete and therefore, by definition, model-complete. Then, by Theorem 4.10 we have that the fragment $Fr(A)$ is complete, and since the theory $Fr^{\#}(A)$ is a central complement of the theory $Fr(A)$, we obtain that $Fr(A) = Fr^{\#}(A)$. Thus, $Fr^{\#}(A)$ is positively model complete. □

In the following theorem necessary and sufficient conditions for the perfectness of a Jonsson theory T are found in terms of lattices of the $E_n(Fr(A)$ existential formulas.

Theorem 4.13. *Let $Fr(A)$ be a perfect fragment of a Jonsson set A. $Fr^{\#}(A)$ be a #-companion. Then the following conditions are equivalent:*

(i) $Fr(A)$ is perfect;

(ii) $E_n(Fr(A))$ is weakly complemented;

(iii) $\varphi \in E_n(Fr(A))$ is a Stone algebra.

Proof. (i) $\Longrightarrow$ (ii). Let a Jonsson theory $Fr(A)$ be perfect. Then $Fr(A)$ is perfect, and by Theorem 4.12 it has a #-companion $Fr^{\#}(A)$. From [8] it is known that $Fr^{f}(A) = Fr^{0}(A)$, where $Fr^{0}(A) = Th_{\forall\exists}(E_{Fr(A)})$ is Kaiser's hull of the fragment $Fr(A)$. Since by the definition of a #-companion $Fr^{\#}(A)$ is model-complete, we have, by part (ii) of Theorem 4.2, that every formula of the language is preserved under submodels in $Mod\ Fr^{\#}(A)$.

Consequently, every existential formula of this language is preserved under submodels in $Mod\ Fr^{\#}(A)$, while at the same time every existential formula of the language is preserved under extensions of models in $Mod\ Fr^{\#}(A)$, and therefore, by Definition 26 this formula is invariant in $Mod\ Fr^{\#}(A)$. Hence, by Theorem 4.1, it follows that every existential formula is weakly complemented. Thus, $E_n(Fr(A))$ is weakly complemented.

(ii) $\Longrightarrow$ (i). If $E_n(Fr(A))$ is weakly complemented, then by Theorem 4.6 the fragment $Fr(A)$ has a model companion. Then by Theorem 4.12 $Fr(A)$ is perfect. So (i) $\Longleftrightarrow$ (ii).

(i) $\Longrightarrow$ (iii). Note that by part (ii) of Theorem 4.5 the model companion of Jonsson theory is its model completion. Then, from the perfectness of fragment, by Theorem 4.7 it follows that $E_n(Fr(A))$ is a Stone algebra.

(iii) $\Longrightarrow$ (i). If $E_n(Fr(A))$ is a Stone algebra, then by Theorem 4.7, the fragment $Fr(A)$ has a #-companion, and, consequently, by Theorem 4.11, the fragment is perfect. □

Theorem 4.14. *Let $Fr(A)$ be a perfect fragment of Jonsson set A, $Fr^{\#}(A)$ be its #-companion. Then the following conditions are equivalent:*

(i) $Fr^{\#}(A)$ is a Jonsson theory;

(ii) every $\varphi \in E_n(Fr(A))$ has a weakly quantifier-free complement.

To prove the necessity we need the following statement.

If the model companion T^M is defined, then a model companion $(T_\forall)^M$ is defined and

$$T^M = (T_\forall)^M \qquad (*)$$

(see [1]).

Proof. (i) $\Longrightarrow$ (ii). Let $Fr^{\#}(A)$ be a Jonsson theory, then from [8] it follows that the fragment $Fr(A)$ is perfect. Then, by Theorem 4.13, the fragment has a #-companion, which equals to the theory $Fr^{\#}(A)$, which, by part (ii) of Theorem 4.3, is a model completion of fragment $Fr(A)$. By virtue of the mutual model consistency of the fragment $Fr(A)$ and theory $Fr_\forall(A)$-all universal consequences of the fragment $Fr(A)$ and (*), the model completion of T is a model completion of $T_\forall$. Then, by Theorem 4.8 every $\varphi^{Fr(A)} \in E_n(Fr(A))$ has a weakly quantifier-free complement.

(ii) $\Longrightarrow$ (i). Every $\varphi^{Fr(A)} \in E_n(Fr(A))$ has a weakly quantifier-free complement. Then every $\varphi^{Fr(A)} \in E_n(Fr(A))$ has a weakly complement, i.e. $E_n(Fr(A))$ is weakly complemented. Then by [8] it follows, that the theory $Fr^{\#}(A)$ is a Jonsson theory. □

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Received: 31.01.2018