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## Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

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Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

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- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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**KUSSAINOVA LEILI KABIDENOVNA**

(to the 70th birthday)

On May 3, 2018 was the 70th birthday of Leili Kabidenovna Kussainova, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department of Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), Professor (2006), Honorary worker of Education of the Republic of Kazakhstan (2005).

L.K. Kussainova was born in the city of Karaganda. In 1972 she graduated from the Novosibirsk State University (Russian Federation) and then completed her postgraduate studies at the Institute of Mathematics (Almaty). L.K. Kussainova's scientific supervisors were distinguished Kazakh mathematicians T.I. Amanov and M. Otelbayev.

Scientific works of L.K. Kussainova are devoted to investigation of the widths of embeddings of the weighted Sobolev spaces, to embeddings and interpolations of weighted Sobolev spaces with weights of general type.

She has solved the problem of three-weighted embedding of isotropic and anisotropic Sobolev spaces in Lebesgue spaces, the problem of exact description of the Lions-Petre interpolation spaces for a pair of weighted Sobolev spaces.

To solve these problems L.K. Kussainova obtained nontrivial modifications of theorems on Besicovitch-Guzman covers. The first relates to covers by multidimensional parallelepipeds, whereas the second relates to double covers by cubes. These modifications have allowed to obtain the description of the interpolation spaces in the weighted case. Furthermore, by using the double covering theorem the exact descriptions of the multipliers were obtained for a pair of Sobolev spaces of general type.

The maximal operators on a basis of cubes with adjustable side length, which were introduced by L.K. Kussainova, have allowed her to solve the problem of two-sided distribution estimate of widths of the embedding of two-weighted Sobolev spaces with weights of general type in weighted Lebesgue spaces.

Under her supervision 6 theses have been defended: 4 candidates of sciences theses and 2 PhD theses.

The Editorial Board of the Eurasian Mathematical Journal congratulates Leili Kabidenovna Kussainova on the occasion of her 70th birthday and wishes her good health and new achievements in mathematics and mathematical education.

**The awarding ceremony**
**of the Certificate of the Emerging Sources Citation of Index database**

In 2016 the Eurasian Mathematical Journal has been included in the Emerging Sources Citation of Index (ESCI) of the "Clarivate Analytics" (formerly "Thomson Reuters") Web of Science. In 2018 the second journal of the L.N. Gumilyov Eurasian National University, namely the Eurasian Journal of Mathematical and Computer Applications was also included in ESCI.

The ESCI was launched in late 2015 as a new database within "Clarivate Analytics". Around 3,000 journals were selected for coverage at launch, spanning the full range of subject areas.

The selection process for ESCI is the first step in applying to the Science Citation Index. All journals submitted for evaluation to the core Web of Science databases will now initially be evaluated for the ESCI, and if successful, indexed in the ESCI while undergoing the more in-depth editorial review. Timing for ESCI evaluation will follow "Clarivate Analytics" priorities for expanding database coverage, rather than the date that journals were submitted for evaluation.

Journals indexed in the ESCI will not receive Impact Factors; however, the citations from the ESCI will now be included in the citation counts for the Journal Citation Reports, therefore contributing to the Impact Factors of other journals. If a journal is indexed in the ESCI it will be discoverable via the Web of Science with an identical indexing process to any other indexed journal, with full citation counts, author information and other enrichment. Articles in ESCI indexed journals will be included in an author's H-Index calculation, and also any analysis conducted on Web of Science data or related products such as InCites. Indexing in the ESCI will improve the visibility of a journal, provides a mark of quality and is good for authors.

To commemorate this important achievement of mathematicians of the L.N. Gumilyov Eurasian National University on June 14, 2018, by the initiative of the "Clarivate Analytics", the awarding ceremony of the Certificate of Emerging Sources Citation Index database of "Clarivate Analytics" to the editorial boards of the Eurasian Mathematical Journal and the Eurasian Journal of Mathematical and Computer Applications was held at the L.N. Gumilyov Eurasian National University. The programme of this ceremony is attached.

Astana June 14, 2018

**Venue: L.N. Gumilyov Eurasian National University**
**Astana, Satpayev street 2, Room 259**

| | |
|---|---|
| **14:30- 15:00** | Visit to the Museum of the history of Education, Museum of L.N. Gumilyov, Museum of writing |
| **15:00-15:10** | *Opening speech of moderator* |
| | **A. Moldazhanova** – the First Vice-Rector, Vice-Rector for Academic Works of L.N. Gumilyov Eurasian National University |
| **15:10-15:20** | **Oleg Utkin** - Managing Director of Clarivate Analytics in Russia and the CIS |
| **15:20-15:30** | *Certification award ceremony of the Eurasian Mathematical Journal, the Eurasian Journal of Mathematical and Computer Applications in international database* |
| **15:30-15:45** | **Kordan Ospanov** – Deputy Editor-in-Chief of the Eurasian Mathematical Journal. *History and perspectives of development of the scientific journal Eurasian Mathematical Journal* |
| **15:45-16:00** | **Kazizat Iskakov** – Deputy Editor-in-Chief of the Eurasian Journal of Mathematical and Computer Applications. *History and perspectives of development of the scientific journal Eurasian Journal of Mathematical and Computer Applications.* |
| **16:00-16:10** | *Closing Ceremony* |
| | *Memory photo* |
| **16:10-16:30** | *Coffee break for visitors* |
| **16:40-17:20** | **Lyaziza Mukasheva** - Official representative of Clarivate Analytics in the Central Asian region *Seminar for editors of scientific journals Scientific library of L.N. Gumilyov Eurasian National University room 104* |

EURASIAN MATHEMATICAL JOURNAL
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# ON FUNDAMENTAL SOLUTIONS OF A CLASS OF WEAK HYPERBOLIK OPERATORS

**V.N. Margaryan, H.G. Ghazaryan**

Communicated by V.I. Burenkov

**Key words:** hyperbolic with weight operator (polynomial), multianisotropic Jevre space, Newton polyhedron, fundamental solution.

**AMS Mathematics Subject Classification:** 12E10.

**Abstract.** We consider a certain class of polyhedrons $\Re \subset \mathbb{E}^n$, multi-anisotropic Jevre spaces $G^{\Re}(\mathbb{E}^n)$, their subspaces $G_0^{\Re}(\mathbb{E}^n)$, consisting of all functions $f \in G^{\Re}(\mathbb{E}^n)$ with compact support, and their duals $(G_0^{\Re}(\mathbb{E}^n))^*$. We introduce the notion of a linear differential operator $P(D)$, $h_{\Re}-$hyperbolic with respect to a vector $N \in \mathbb{E}^n$, where $h_{\Re}$ is a weight function generated by the polyhedron $\Re$. The existence is shown of a fundamental solution $E$ of the operator $P(D)$ belonging to $(G_0^{\Re}(\mathbb{E}^n))^*$ with $\operatorname{supp} E \subset \overline{\Omega_N}$, where $\Omega_N := \{x \in \mathbb{E}^n, (x,N) > 0\}$. It is also shown that for any right-hand side $f \in G^{\Re}(\mathbb{E}^n)$ with the support in a cone contained in $\overline{\Omega_N}$ and with the vertex at the origin of $\mathbb{E}^n$, the equation $P(D)u = f$ has a solution belonging to $G^{\Re}(\mathbb{E}^n)$.

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## 1 Formulation of the problem and preliminary facts

Let $P$ be a differential operator with constant coefficients. A distribution $E \in \mathcal{D}'(\mathbb{E}^n)$ is called **a fundamental solution** of the operator $P$, if $P(D)E = \delta^0$, where $\delta^0$ is the Dirac measure concentrated at the origin.

Fundamental solutions play an important role in the investigation of smoothness of solution of differential equations. Their importance is explained by the fact (anyway in the classical case) that the operator of convolution with a fundamental solution $E$ of the operator $P$ is both left and right inverse to the operator $P$. Namely $E * [P(D)u] = u$ and $P(D)(E * f) = f$ for any $u \in \mathcal{E}'(\mathbb{E}^n)$ and $f \in \mathcal{E}'(\mathbb{E}^n)$, where $\mathcal{E}'(\mathbb{X})$ is the set of all distributions with compact support in $\mathbb{X}$.

Thus, many properties of the solution $u = E * f$ of the equation $P(D)u = f$, in particular its smoothness, are determined not only by the properties of the right-hand side $f$, but also by the properties of a fundamental solution.

For a wide class of operators (such as the Laplace operator, wave or heat operators, and others) fundamental solutions were constructed by Cauchy, Fredholm and other classics. In the middle of the last century, by joint efforts of Ehrenpreis, Malgrange and Hörmander, the existence was prowed of a fundamental solution in the space of distributions of finite order for any differential operator with constant coefficients (see [10] or [5], I, Theorem 7.3.10]). Moreover, they proved (see [5], II, Theorem 10.3.1]) that if $E \in S'$ is a fundamental solution of the operator $P$ and $u = E * f$ is a solution to the equation $P(D)u = f \in \mathcal{B}_{p,k}(\mathbb{E}^n)$, then $u \in \mathcal{B}^{loc}_{p,k\tilde{P}}(\mathbb{E}^n)$, where $k \in \mathcal{K}$ is a tempered weight function (see [5], II, Definition 10.1.1), $1 \leq p \leq \infty$, $\tilde{P}$ is

L. Hörmander's function of the operator $P$ (see below, or [5], II, Example 10.1.3]), $\mathcal{B}_{p,k}(\mathbb{E}^n)$ is a Banach space with the norm

$$||u||_{p,\,k} = [(2\,\pi)^{-n} \int |k(\xi)\,\hat{u}(\xi)|^p \,d\xi]^{1/p}.$$

It turns out that the solution $u = E * f$ of the equation $P(D)u = f$ has the best possible local properties described in terms of the $\mathcal{B}^{loc}_{p,k}-$spaces. However, it should be noted that this theorem, which has a universal character, does not distinguish between the smoothness of solutions of equations of various types (such as elliptic, hyperbolic, hypoelliptic and others), whereas it is well known that solutions (for example) of hyperbolic and hypoelliptic differential equations may have quite different smoothness properties.

Therefore, the efforts of many mathematicians have been focused on studying the smoothness properties of solutions (including fundamental solutions) for different types of equations. Conditions were found under which the Cauchy problem for a hyperbolic (in that sense or other) operator $P$ for the equation $P(D)u = f$ has an infinitely differentiable solution. It turned out that the weak hyperbolicity condition with respect to the vector $N = (0, ..., 0, 1)$ is necessary, and the condition of hyperbolicity by Görding (hence by Petrovsky) is sufficient for this (see, for instance, [4], [6], [8], [17]).

With regard to the Jevre classes (see [3] , or [5], I, 8.4 ), it turned out that the $s-$ hyperbolicity condition with respect to the vector $N = (0, ..., 0, 1)$ is sufficient for the existence of solutions belonging to the Jevre space $G^s(\Omega_N)$ of the Cauchy problem of equation $P(D)u = 0$ with the appropriate initial conditions, where $\Omega_N := \{x \in \mathbb{E}^n, (x, N) > 0\}$ (see [11]), and $h_{\Re}-$ hyperbolicity condition is sufficient for the existence of solutions of the Cauchy problem belonging to the multianisotropic Jevre space $G^{\delta\,\tilde{\Re}}(\mathbb{E}^n)$, where $\delta > 0$, and polyhedron $\tilde{\Re}$ is defined in a special way via the polyhedron the $\Re$ (see [1], also [10], [2] ).

In the present paper we prove that if $0 \neq N \in \mathbb{E}^n$ is an arbitrary vector, $\Re \subset \mathcal{B}_n$ is an arbitrary polyhedron, $f \in G^{\Re}_0(\Omega_N)$ and $P(D)$ is an operator $h_{\Re}-$ weakly hyperbolic with respect to the vector $N$ operator, then the operator $P(D)$ has a fundamental solution $E \in (G^{\Re}_0(\mathbb{E}^n))^*$ with the support in $\bar{\Omega}_N := \{x \in \mathbb{E}^n, (x, N) \geq 0\}$, and the equation $P(D)u = f$ has a solution belonging to $G^{\Re}(\mathbb{E}^n)$ such that $\operatorname{supp} u \subset \bar{\Omega}_N$.

Let $\mathbb{E}^n$ and $\mathbb{R}^n$ be $n$-dimensional Euclidian spaces of points (vectors) respectively $x = (x_1, ..., x_n)$ and $\xi = (\xi_1, ..., \xi_n)$, $\mathbb{R}^{n,+} := \{\xi \in \mathbb{R}^n, \xi_j \geq 0,\ j = 1, ..., n\}$, $\mathbb{R}^{n,\,0} := \{\xi \in \mathbb{R}^n, \xi_1...\xi_n \neq 0\}$, $\mathbb{C}^n = \mathbb{R}^n \times i\mathbb{R}^n$, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}^n_0 = \mathbb{N}_0 \times ... \times \mathbb{N}_0$ is the set of all $n$- dimensional multi-indices, i.e. the set of all points with non-negative integer coordinates: $\mathbb{N}^n_0 := \{\alpha = (\alpha_1, ..., \alpha_n) :\ \alpha_i \in \mathbb{N}_0\ (i = 1, ..., n)\}$.

For $\xi \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n_0$ and $\nu \in \mathbb{R}^{n,+}$ we put $|\xi| = \sqrt{\xi_1^2 + ... + \xi_n^2}$, $|\xi|^\nu = |\xi_1|^{\nu_1}...|\xi_n|^{\nu_n}$, $|\,\alpha\,| = \alpha_1 + ... + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1}...\xi_n^{\alpha_n}$, $D^\alpha = D_1^{\alpha_1}...D_n^{\alpha_n}$, where $D_j = \partial/\partial\xi_j$ or $D_j = \frac{1}{i}\,\partial/\partial x_j$ $(j = 1, ...n)$.

Let $\mathcal{A} = \{a^1, ..., a^M\}$, be a finite set of points in $R^{n,\,+}$. By the **Newton polyhedron** of the set $\mathcal{A}$ we mean the minimal convex polyhedron $\Re(\mathcal{A})$ in $R^{n,\,+}$ containing all points of $\mathcal{A}$.

A polyhedron $\Re$ with vertices in $\mathbb{R}^{n,\,+}$ is said to be complite if $\Re$ has a vertex at the origin of $\mathbb{R}^n$ and one vertex (distinct from the origin) on each coordinate axis $R^{n,+}$. A complite polyhedron $\Re$ is called **completely regular** if all coordinates of the outward normals of its noncoordinate $(n-1)-$dimensional faces (the set of which we denote by $\Lambda(\Re)$) are positive (see [15] or [13]). We assume that the vectors $\lambda \in \Lambda(\Re)$ are normalized so that $\max_{\nu\in\Re}(\lambda, \nu) = 1$.

Let $\Re$ be a completely regular polyhedron. By $\Re^0$ we denote the set of vertices of the polyhedron $\Re$ and put

$$h_{\Re}(\xi) := \sum_{\nu\in\Re^0} |\xi|^\nu.$$

Let $\alpha \in \mathbb{N}_0^n$, and $\Omega \subset \mathbb{E}^n$. We denote

$$r_{\Re}(\alpha) := \max_{\lambda \in \Lambda(\Re)} (\lambda, \alpha),$$

and by $G^{\Re}(\Omega)$ we denote the set of functions $\varphi \in C^\infty(\Omega)$ such that for any compact $K \subset\subset \Omega$ and $\delta > 0$.

$$||\varphi, K||_{\Re,\,\delta} := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} \delta^{-r_{\Re}(\alpha)} \, [r_{\Re}(\alpha)]^{-r_{\Re}(\alpha)} \, |D^\alpha \varphi(x)| < \infty. \tag{1.1}$$

It is obvious that for $\delta_1 > \delta_2$

$$||\varphi, K||_{\Re,\,\delta_1} \le ||\varphi, K||_{\Re,\,\delta_2} \;\; \forall \varphi \in G^{\Re}(\Omega), \; K \subset\subset \Omega.$$

It is easy to verify that $G^{\Re}(\Omega)$ is a Frechet space with the topology generated by a countable number of seminorms $||\cdot, K_s||_{\Re,\,\delta_s}$, where $\delta_s \searrow 0$ and $K_s \nearrow \Omega$ as $s \to \infty$.

Moreover, (see, for instance, [12 ]), if $\Re_1 \subset \Re_2$ are completely regular polyhedrons then $G^{\Re_2}(\Omega)$ is embedded in $G^{\Re_1}(\Omega)$ (notation: $G^{\Re_2}(\Omega) \hookrightarrow G^{\Re_1}(\Omega)$) and if $\Re = \{\nu : \nu \in \mathbb{R}^{n,\,+}, (\lambda, \nu) \le 1\}$ for a vector $\lambda \in \mathbb{R}^{n,\,+} \cap \mathbb{R}^{n,\,0}$, then $G^{\Re}$ coincides with the classical anisotropic Jevre space $G^\lambda$.

For further purposes, we introduce some additional notation, related to a completely regular polyhedron $\Re$ :

- we put $\Re(0) = \emptyset$, and for $j \in \mathbb{N}$ we denote $\Re(j) := \{\nu \in \mathbb{R}^{n,+}, \, \nu/j \in \Re\}$,

- let $\lambda \in \Lambda(\Re)$, and $\lambda^0 = \lambda^0(\Re) := (\min \lambda_1, ..., \min \lambda_n)$. We also put $\Re^\star = \{\nu \in \mathbb{R}^{n,+}, (\lambda^0, \nu) \le 1\}$ which we call the polyhedron conjugate to the polyhedron $\Re$ .

Note that for any $n-$ dimensional completely regular polyhedron $\Re \subset \mathbb{R}^{n,\,+}$ the polyhedron $\Re^\star$ is an $n-$dimensional polyhedron in $\mathbb{R}^{n,\,+}$ with vertex at the origin of $\mathbb{R}^n$, $\Re \subset \Re^\star$, wherein $\Re = \Re^\star$ if and only if the set $\Lambda(\Re)$ consists of a single vector (for example, for $n = 2$ when $\Re$ is a right triangle with a vertex at the origin of $\mathbb{R}^2$.)

**Lemma 1.1** *Let a completely regular polyhedron $\Re \subset \mathbb{R}^{n,+}$ and a natural number $m$ be fixed. Then the initial topology of $G^{\Re}(\Omega)$ coincides with the topology generated by the seminorms*

$$||\varphi, K||^{(m)}_{\Re,\,\delta} := \sup_{j \ge m} \max_{\alpha \in \Re(j) \setminus \Re(j-m)} \sup_{x \in K} \delta^{-(j-m)} \, (j-m)^{-(j-m)} \, |D^\alpha \varphi(x)| \tag{1.2}$$

*for all compacts $K \subset\subset \Omega$ and numbers $\delta > 0$.* Namely, for any $K \subset\subset \Omega$, $\delta > 0$ and $\varphi \in G^{\Re}(\Omega)$ the following inequality holds

$$\frac{\delta^m}{1+\delta^m} \, ||\varphi, K||_{\Re,\,\delta} \le ||\varphi, K||^{(m)}_{\Re,\,\delta} \le (2m)^m \, [1 + (\frac{\delta}{2})^m] \, ||\varphi, K||_{\Re,\,\delta/2}. \tag{1.3}$$

*Proof.* For an arbitrary compact $K \subset\subset \Omega$, any number $\delta > 0$, any function $\varphi \in G^{\Re}(\Omega)$ and any multi-index $\alpha \in \mathbb{N}_0^n$ we have

$$|D^\alpha \varphi(x)| \le ||\varphi, K||_{\Re,\,\delta} \; \delta^{r_{\Re}(\alpha)} \; [r_{\Re}(\alpha)]^{r_{\Re}(\alpha)} \;\; \forall x \in K.$$

Since $0 \le j - r_{\Re}(\alpha) \le m$ for any $j \ge m$ and $\alpha \in \Re(j) \setminus \Re(j-m)$, it follows that $\delta^{r_{\Re}(\alpha)} \le \delta^{j-m} \, (1 + \delta^m)$. On the other hand, since it is obvious that for any $k > 1$ $[(r_{\Re}(\alpha))^{r_{\Re}(\alpha)}/(j-m)^{j-m}] \le [(k\,m)/(k-1)]^m$, for $k = 2$ we have $[(r_{\Re}(\alpha))^{r_{\Re}(\alpha)}/(j-m)^{j-m}] \le (2m)^m \, 2^{j-m}$. Finally for all $j \ge m$, $\alpha \in \Re(j) \setminus \Re(j-m)$ and $x \in K$ we get

$$|D^\alpha \varphi(x)| \le (2m)^m \, (1 + \delta^m) \, ||\varphi, K||_{\Re,\,\delta} \, (2\delta)^{j-m} \, (j-m)^{j-m},$$

from which immediately follows the right-hand side inequality in (1.3).

Since it is obvious that the set $\bigcup_{j\geq m}[(\Re(j)\setminus\Re(j-m))\cap\mathbb{N}_0^n]$ coincides with the set $\mathbb{N}_0^n$, it follows that for any compact $K\subset\subset\Omega$, any number $\delta>0$ and any function $\varphi\in G^{\Re}(\Omega)$ we have (see. (1.1))

$$||\varphi,K||_{\Re,\,\delta}=\sup_{j\geq m}\max_{\alpha\in\Re(j)\setminus\Re(j-m)}\sup_{x\in K}\delta^{-r_{\Re}(\alpha)}\,[r_{\Re}(\alpha)]^{-r_{\Re}(\alpha)}\,|D^{\alpha}\varphi(x)|$$

$$=\sup_{j\geq m}\max_{\alpha\in\Re(j)\setminus\Re(j-m)}\sup_{x\in K}\{[\delta^{-(j-m)}\,(j-m)^{(j-m)}\,|D^{\alpha}\varphi(x)|\,]\cdot$$

$$\cdot[\delta^{j-m-r_{\Re}(\alpha)}\,(j-m)^{(j-m)}/(r_{\Re}(\alpha))^{r_{\Re}(\alpha)}]\}.$$

Since $-m\leq j-m-r_{\Re}(\alpha)\leq 0$ for any $j\geq m$, and $\alpha\in\Re(j)\setminus\Re(j-m)$ we have

$$(j-m)^{(j-m)}/(r_{\Re}(\alpha))^{r_{\Re}(\alpha)}\leq 1,\ \ \delta^{j-m-r_{\Re}(\alpha)}\leq 1+\delta^{-m},$$

this gives

$$||\varphi,K||_{\Re,\,\delta}\leq(1+\delta^{-m})\,||\varphi,K||^{(m)}_{\Re,\,\delta},\ \ \forall K\subset\subset\Omega,\,\delta>0,\,\varphi\in G^{\Re}(\Omega).$$

Thus the left-hand side inequality in (1.3) and therefore Lemma 1.1 are proved. □

Below, we shall use the following property of completely regular polyhedrons.

**Proposition 1.1** *Let $\Re$ be a completely regular polyhedron and $\rho=\rho(\Re):=\left(\max_{\lambda\in\Lambda(\Re)}\max_{1\leq j\leq n}\lambda_j\right)^{-1}$. Then*

*1) there exists a numbers $m\in\mathbb{N}_0$ and $c=c(\Re,m)>0$ such that for all $j\geq m$*

$$c^{-(j-m)}\,h_{\Re}^{j-m}(\xi)\leq\sum_{\alpha\in\Re(j)\setminus\Re(j-m)}|\xi^{\alpha}|\leq c^{j}\,h_{\Re}^{j}(\xi)\ \ \forall\xi\in\mathbb{R}^n,\ |\xi|\geq 1, \tag{1.4}$$

*2) $r_{\Re^*}(\alpha)\leq r_{\Re}(\alpha)$ for any $\alpha\in\mathbb{N}_0^n$,*

*3) for all $\alpha,\beta\in\mathbb{N}_0^n$, $\alpha\leq\beta$,*

$$\rho\,r_{\Re}(\alpha)\leq r_{\Re^*}(\alpha)\leq r_{\Re}(\alpha-\beta)+r_{\Re^*}(\beta)\leq r_{\Re}(\alpha),$$

*4) for any $\alpha\in\mathbb{N}_0^n$ there exists a number $c=c(\alpha)>0$ such that*

$$||D^{\alpha}\varphi,K||_{\Re,\delta}\leq c\,[\max\{\delta^{\rho},\delta\}]^{r_{\Re}(\alpha)}\,||\varphi,K||_{\Re,\delta/2}\ \ \forall K\subset\subset\Omega,\,\delta>0,\,\varphi\in G^{\Re}(\Omega).$$

*Proof.* The first three statements are obvious. Let us prove the fourth one. Let $K\subset\subset\Omega$, $\delta>0$, $\varphi\in G^{\Re}(\Omega)$, then

$$||D^{\alpha}\varphi,K||_{\Re,\delta}=\sup_{\beta\in\mathbb{N}_0^n}\sup_{x\in K}\delta^{-r_{\Re}(\beta)}\,[r_{\Re}(\beta)]^{-r_{\Re}(\beta)}\,|D^{\alpha+\beta}\varphi(x)|$$

$$=\sup_{\beta\in\mathbb{N}_0^n}\sup_{x\in K}\{[(\delta/2)^{-r_{\Re}(\alpha+\beta)}\,(r_{\Re}(\alpha+\beta))^{-r_{\Re}(\alpha+\beta)}\,|D^{\alpha+\beta}\varphi(x)|]$$

$$\cdot[\delta^{r_{\Re}(\alpha+\beta)}\,(r_{\Re}(\alpha+\beta)/2)^{r_{\Re}(\alpha+\beta)}]/[\delta^{r_{\Re}(\beta)}\,(r_{\Re}(\beta))^{r_{\Re}(\beta)}]\}\ \ \forall x\in K.$$

Since by virtue of statement 3) for any $0\neq\beta\in\mathbb{N}_0^n$ $\rho\,r_{\Re}(\alpha)\leq r_{\Re}(\alpha+\beta)-r_{\Re^*(\beta)}\leq r_{\Re}(\alpha)$ and $[(r_{\Re}(\alpha+\beta)/2)^{r_{\Re}(\alpha+\beta)}]/[(r_{\Re}(\beta))^{r_{\Re}(\beta)}]\leq c$, for some $c>0$ depending only on $\alpha$ this gives us a direct proof of statement 4). □

**Lemma 1.2** *Let $\Re\subset\mathbb{R}^{n,+}$ be a completely regular polyhedron. Then for any $\delta>0$ there exist positive numbers $\delta_1=\delta_1(\delta,\Re)$ and $c=c(\delta,\Re)$ such that for all $K\subset\subset\Omega$, $\psi\in G^{\Re^*}(\Omega)$ and $\varphi\in G^{\Re}(\Omega)$*

$$||(\psi\,\varphi), K||_{\Re,\,\delta_1} \le c\,||\varphi, K||_{\Re,\,\delta}\,||\psi, K||_{\Re^*,\,\delta}. \tag{1.5}$$

*Proof.* Applying the Leibnitz' formula, we get for any $\delta > 0$, $K \subset\subset \Omega$, $\psi \in G^{\Re^*}(\Omega)$, $\varphi \in G^{\Re}(\Omega)$ and $\alpha \in \mathbb{N}_0^n$

$$|D^\alpha(\psi\,\varphi)(x)| \le \sum_{\beta\le\alpha} C_\alpha^\beta\,|D^{\alpha-\beta}\varphi(x)|\,|D^\beta\psi(x)|$$

$$\le ||\varphi, K||_{\Re,\,\delta}\,||\psi, K||_{\Re,\,\delta}\sum_{\beta\le\alpha} C_\alpha^\beta\,\delta^{r_\Re(\alpha-\beta)}\,[r_\Re(\alpha-\beta)]^{r_\Re(\alpha-\beta)}$$

$$\cdot\delta^{r_{\Re^*}(\beta)}\,[r_{\Re^*}(\beta)]^{r_{\Re^*}(\beta)}\ \ \forall x \in K.$$

From this and by virtue of Proposition 1.1 we have for any $\alpha \in \Re(j)\setminus\Re(j-1)$ $(j = 1, 2, ...)$ and for all $x \in K$

$$|D^\alpha(\psi\,\varphi)(x)| \le ||\varphi, K||_{\Re,\,\delta}\,||\psi, K||_{\Re^*,\,\delta}\,[max\{\delta, \delta^\rho\}]^{r_\Re(\alpha)}$$

$$\cdot\sum_{l=1}^{j}\ \sum_{\beta\in\Re^*(l)\setminus\Re^*(l-1),\,\beta\le\alpha} C_\alpha^\beta\,[r_\Re(\alpha-\beta)]^{r_\Re(\alpha-\beta)}\cdot\,[r_{\Re^*}(\beta)]^{r_{\Re^*}(\beta)} \le [max\{\delta, \delta^\rho\}]^{r_\Re(\alpha)}$$

$$\cdot\,||\varphi, K||_{\Re,\,\delta}\,||\psi, K||_{\Re^*,\,\delta}\sum_{l=1}^{j} l^l\,(j-l+1)^{j-l+1}\sum_{\beta\in\Re^*(l)\setminus\Re^*(l-1),\,\beta\le\alpha} C_\alpha^\beta.$$

On the other hand, obviously, there exists a constant $\kappa_1 = \kappa_1(\Re) > 0$ such that

$$\sum_{l=1}^{j}\frac{l^l\,(j-l+1)^{j-l+1}}{[r_\Re(\alpha)]^{r_\Re(\alpha)}}\sum_{\beta\in\Re^*(l)\setminus\Re^*(l-1),\,\beta\le\alpha} C_\alpha^\beta \le \kappa_1^{r_\Re(\alpha)+1}.$$

Substituting here $\delta_1 := \kappa_1\,max\{\delta, \delta^\rho\}$, we get for all $x \in K$ and $\alpha \in \Re(j)\setminus\Re(j-1)$ $(j = 1, 2, ...)$

$$|D^\alpha(\psi\,\varphi)(x)| \le \kappa_1\,\delta_1^{r_\Re(\alpha)}\,[r_\Re(\alpha)]^{r_\Re(\alpha)}\,||\varphi, K||_{\Re,\,\delta}\,||\psi, K||_{\Re^*,\,\delta}.$$

Hence we obtain inequality (1.5) with the constant $c = \kappa_1$. □

By this lemma and Proposition 1.1 the following statement follows directly.

**Corollary 1.1** *Let $\Re \subset \mathbb{R}^{n,+}$ be a completely regular polyhedron and $\alpha \in \mathbb{N}_0^n$. Then for any $\delta > 0$ there exist positive numbers $\delta_1 = \delta_1(\delta, \Re)$ and $c = c(\delta, \Re, \alpha)$ such that for all $K \subset\subset \Omega$, $\psi \in G^{\Re^*}(\Omega)$ and $\varphi \in G^{\Re}(\Omega)$*

$$||D^\alpha(\psi\,\varphi), K||_{\Re,\,\delta_1} \le c\,||\varphi, K||_{\Re,\,\delta}\,||\psi, K||_{\Re^*,\,\delta}. \tag{1.6}$$

Let for the polyhedron $\Re \subset \mathbb{R}^{n,+}$ the vector $\lambda^0 = \lambda^0(\Re)$ be defined as above. Denote by $\mathcal{B}_n$ the set of all completely regular polyhedrons $\Re \subset \mathbb{R}^{n,+}$ for which $\min\{\lambda_1^0(\Re), ..., \lambda_n^0(\Re)\} > 1$.

Let $\Re \in \mathcal{B}_n$. It is known (see, for instance, [5], I, Theorem 1.4.2), that for any domain $\Omega \subset \mathbb{E}^n$ the set $G^{\Re^\star}(\Omega)$ contains a non-zero function belonging to $C_0^\infty(\Omega)$.

Since $\Re \subset \Re^\star$, hence $G^{\Re}(\Omega) \supset G^{\Re^\star}(\Omega)$, and the set $G^{\Re}(\Omega)$ also contains a non-zero function belonging to $C_0^\infty(\Omega)$.

For a compact $K \subset\subset \Omega$, and a number $\delta > 0$ we denote by $G_0^{\Re}(K)$ the set $G_0^{\Re}(K) := \{\varphi \in G^{\Re}(\Omega), \operatorname{supp}\varphi \subset K\}$ with the topology generated by the seminorms $||\cdot, K||_{\Re,\delta}$ and put $G_0^{\Re}(\Omega) := \bigcup_{K\subset\subset\Omega} G_0^{\Re}(K)$.

It is easy to verify that in $G_0^{\Re}(\Omega)$ one can define convergence as follows: we say that a sequence $\{\varphi_s\}$ converges to zero as $s\to\infty$, in $G_0^{\Re}(\Omega)$, if 1) there exists a compact $K\subset\subset\Omega$, such that $\operatorname{supp}\varphi_s\subset K$, $s=1,2,...$ and 2) $||\varphi_s,K||_{\Re,\delta}\to 0$ as $s\to\infty$ for any $\delta>0$.

**Corollary 1.2** *Let* $\Re\subset\mathcal{B}_n$, *then* $G^{\Re}(\Omega)\cdot G_0^{\Re^\star}(\Omega):=\{g=\varphi\cdot\psi:\varphi\in G^{\Re}(\Omega),\psi\in G_0^{\Re^\star}(\Omega)\}$ $\hookrightarrow G_0^{\Re}(\Omega)$.

Follows by Lemma 1.2 and inequality (1.6).

## 2 Some properties of the basic and the dual spaces

For a polyhedron $\Re\in\mathcal{B}_n$ and a domain $\Omega\subset\mathbb{E}^n$ by $(G_0^{\Re}(\Omega))^*$ (respectively by $(G^{\Re}(\Omega))^*$ ) we denote the set of all linear continuous functionals defined on $G_0^{\Re}(\Omega)$ (respectively on $G^{\Re}(\Omega)$).

Applying the criterion for the boundedness of a linear functional in coutably - normed spaces (see, for example, [9], Chapter 4, Section 1-1, Theorem 1 and Section 1-4) we obtain that a linear functional $f$ defined on $G_0^{\Re}(\Omega)$ belongs to $(G_0^{\Re}(\Omega))^*$ if and only if for any compact $K\subset\subset\Omega$ there exist some numbers $\delta>0$ and $c>0$ such that

$$|f(\varphi)|\le c||\varphi,K||_{\Re,\delta}\quad\forall\varphi\in G^{\Re}(\Omega),\ \operatorname{supp}\varphi\subset K. \tag{2.1}$$

Respectively, a linear functional $f$ defined on $G^{\Re}(\Omega)$ belongs to $(G^{\Re}(\Omega))^*$, if there exist a compact $K\subset\subset\Omega$ and some positive numbers $\delta$ and $c$ such that

$$|f(\varphi)|\le c||\varphi,K||_{\Re,\delta}\quad\forall\varphi\in G^{\Re}(\Omega). \tag{2.1$'$}$$

It follows by Lemma 1.1 that for any $f\in(G_0^{\Re}(\Omega))^*$ (respectively $f\in(G^{\Re}(\Omega))^*$ ) and $\alpha\in\mathbb{N}_0^n$ the expression $(-1)^{|\alpha|}f(D^\alpha\varphi)$ : $\varphi\in G_0^{\Re}(\Omega)$ (respectively $\varphi\in G^{\Re}(\Omega)$ ) generates a funcional belonging to $(G_0^{\Re}(\Omega))^*$ (respectively belonging to $(G^{\Re}(\Omega))^*$ ). This funcional will be denoteed by $D^\alpha f$.

For a compact set $K$ and $\varphi\in G^{\Re}(\mathbb{E}^n)$ we denote by $\bigcup_{x\in K}\operatorname{supp}\varphi(x-y)$ the set of all those $y\in\mathbb{E}^n$, for which there exists a point $x\in K$ such that $\varphi(x-y)\ne0$. Let $f\in(G_0^{\Re}(\mathbb{E}^n))^*$ we set

$$\mathcal{D}_K(f,\varphi):=[\operatorname{supp}f]\cap[\overline{\bigcup_{x\in K}\operatorname{supp}\varphi(x-\cdot)}].$$

Let $f\in(G_0^{\Re}(\mathbb{E}^n))^*$ and $\varphi\in G^{\Re}(\mathbb{E}^n)$ be such that for any compact set $K$ the set $\mathcal{D}_K(f,\varphi)$ is also compact and $\psi\in G_0^{\Re^\star}(\mathbb{E}^n)$, $\psi(x)=1$ in a neibourhood of the set $\mathcal{D}_K(f,\varphi)$. We define (see Corollary 1.2) the following convolution of functions

$$(f*\varphi)_\psi(x)=f_y[\psi(y)\,\varphi(x-y)]\ \ x,y\in\mathbb{E}^n.$$

Since the expression $f_y[\psi(y)\,\varphi(x-y)]$ does not depend on the choice of the function $\psi$, in the sequel, in our notation $(f*\varphi)_\psi(x)$, we omit the symbol $\psi$, denoting it simply by $(f*\varphi)(x)$.

It is easy to verify that in this case for any compact set $K$ the set $\mathcal{D}_K(f,\varphi)$ is also compact: $f\in(G_0^{\Re}(\mathbb{E}^n))^*$, $\varphi\in G^{\Re}(\mathbb{E}^n)$, besides $\operatorname{supp}f\cup\operatorname{supp}\varphi\subset\{x\in E^n,(x,N)\ge0\}$ for a vector $0\ne N\in\mathbb{R}^n$. Moreover, either $\operatorname{supp}f$ or *supp* $\varphi$ lies in the cone $\{x\in E^n,(x,N)\ge\varepsilon\,|x|\}$ for some number $\varepsilon>0$.

**Theorem 2.1** *Let* $\Re\in\mathcal{B}_n$, $f\in(G_0^{\Re}(\mathbb{E}^n))^*$ $\varphi\in G^{\Re}(\mathbb{E}^n)$ *are such that for any compact* $K$ *the set* $\mathcal{D}_K(f,\varphi)$ *is also compact. Then*

1)$D^\alpha(f*\varphi)(x)=[(D^\alpha f)*\varphi](x)=[f*D^\alpha\varphi](x)\ \ \forall x\in\mathbb{E}^n$,

2)$(f*\varphi)\in G^{\Re}(\mathbb{E}^n)$,

3)$\operatorname{supp}(f*\varphi)\subset\overline{\operatorname{supp}f+\operatorname{supp}\varphi}$,

4) *let* $\varphi_s \in (G_0^{\Re}(\mathbb{E}^n))^*$ $(s = 1, 2, ...)$, $\varphi_s \to 0$ *as* $s \to \infty$ *in the topology of* $G^{\Re}(\mathbb{E}^n)$ *and for any compact* $K_0$ *there is a compact* $K_1$ *such that* $\mathcal{D}_{K_0}(f, \varphi_s) \subset K_1$ $(s = 1, 2, ...)$, *then* $(f * \varphi_s)(x) \to 0$ *as* $s \to \infty$ *in the topology of* $G^{\Re}(\mathbb{E}^n)$.

*Proof.* Let $e^k$ be the unit vector in the direction $x_k$ $(k = 1, 2, ..., n)$.

Since $\frac{1}{i\,h}[\varphi(x + h\,e^k - \cdot) - \varphi(x - \cdot)] \to [D_k \varphi(x - \cdot)]$ as $h \to 0$ in the topology of $G^{\Re}(\mathbb{E}^n)$, and $\mathcal{D}_K(f, \varphi(x + h\,e^k - \cdot) - \varphi(x - \cdot)) \subset \mathcal{D}_K(f, \varphi) + \{\vartheta\, e^k, |\vartheta| \le h\} =: K'$ is a compact set, by virtue of Lagrange's formula we have for any $\delta > 0$

$$||\frac{1}{i\,h}[\varphi(x + h\,e^k - \cdot) - \varphi(x - \cdot)] - D_k \varphi(x - \cdot),\, K||_{\Re,\,\delta} \le |h|\, ||D_k^2 \varphi(x - \cdot),\, K'||_{\Re,\,\delta}.$$

For the same reason, by the definition of the convolution we get

$$\lim_{h \to 0} \{\frac{1}{h}\, [(f * \varphi)(x + h\,e^k) - (f * D_k \varphi)(x)]\} = 0.$$

Hence, according to Proposition 1.1 (see point 4)) and applying the estimate (2.1) we obtain $D_k(f * \varphi)(x) = (f * D_k \varphi)(x)$.

The equality $D_k(f * \varphi)(x) = ((D_k f) * \varphi)(x)$ follows immediately from the definition of the convolution operation and the definition of $D_k f$ .

Repeating these arguments the required number of times, we obtain the proof of the first part of the theorem.

Let us prove the second part of the theorem. Let a compact $K \subset \mathbb{E}^n$ be fixed. By the conditions of the theorem, $\mathcal{D}_K(f, \varphi)$ is compact. Let $\chi \in G_0^{\Re^*}(E^n)$ be a function that is equal to unity in some neibourhood of $\mathcal{D}_K(f, \varphi)$ and $K_0 := supp\chi$. Therefore by virtue of inequality (2.1) and according to the first part of the theorem, which has already been proved, we have with some positive numbers $\delta_0$ and $c_0$ and for all $\alpha \in \mathbb{N}_0^n$ and $x \in K$

$$|D^\alpha (f * \varphi)(x)| = |\, [f * D^\alpha \varphi](x)| = |f_y[\chi(y)\, (D^\alpha \varphi)(x - y)]|$$

$$\le c_0\, ||\chi(\cdot)\, (D^\alpha \varphi)(x - \cdot),\, K_0||_{\Re,\,\delta_0}.$$

Hence, by virtue of Lemma 1.2 and according to the condition $\chi \in G_0^{\Re^*}(E^n)$ we obtain with some positive constants $\delta_1, c_1, c_2$ for all $\alpha \in \mathbb{N}_0^n$ and $x \in K$

$$|D^\alpha (f * \varphi)(x)| \le c_1\, ||(D^\alpha \varphi)(x - \cdot), K_0||_{\Re,\,\delta_1} \cdot ||\chi,\, K_0||_{\Re^*,\,\delta_1}$$

$$\le c_2\, ||D^\alpha \varphi,\, K + K_0||_{\Re,\,\delta_1},$$

where, as usual $K + K_0 := \{(x + y) : x \in K, y \in K_0\}$.

It is obvious that

a) there is a number $\delta_2 > 0$ such that for all positive numbers $\delta$, $\delta_1$ and for all $l, j \in \mathbb{N}$

$$\delta^{-(l-1)}\, (l-1)^{-(l-1)}\, \delta_1^{-(j-1)}\, (j-1)^{-(j-1)}\, \delta_2^{j+l-2}\, (j+l-2)^{j+l-2} \le 1,$$

b) if $\alpha \in \Re(l) \setminus \Re(l-1)$, $\beta \in \Re(j) \setminus \Re(j-1)$, then $\alpha + \beta \in \Re(j+l) \setminus \Re(j+l-2)$.

Therefore, from this and Lemma 1.1 (see (1.3)) we obtain with some positive numbers $c_3, c_4$

$$|D^\alpha (f * \varphi)(x)| \le c_3\, \delta^{l-1}\, (l-1)^{(l-1)}\, [\,\delta^{-(l-1)}]\, (l-1)^{-(l-1)}\, ||D^\alpha \varphi,\, K + K_0||_{\Re^{(1)},\,\delta_1}$$

$$= c_3\, \delta^{l-1}\, (l-1)^{(l-1)} \sup_{j \in \mathbb{N}, \beta \in \Re(j) \setminus \Re(j-1)}\, \sup_{z \in K + K_0} \{\, \delta^{-(l-1)}\, (l-1)^{-(l-1)}\, \delta_1^{-(j-1)} (j-1)^{-(j-1)}$$

$$\cdot (j-1)^{-(j-1)}\, \delta_2^{j+l-2} (j+l-2)^{j+l-2}\, [\delta_2^{-(j+l-2)}\, (j+l-2)^{-(j+l-2)}\, |(D^{\alpha+\beta} \varphi)(z)|]\}$$

$$\leq c_3\,\delta^{l-1}\,(l-1)^{(l-1)}\,||\varphi, K+K_0||^{(2)}_{\Re,\,\delta_2}. \qquad (2.2)$$

Hence and by virtue of Lemma 1.1 we obtain the proof of the second part of the theorem.

The proof of the third statement is obtained by the following simple considerations: if $supp f\cap supp\varphi(x^0-\cdot)\neq\emptyset$ for some point $x^0\in\mathbb{E}^n$, then there exists a point $y\in supp f$ such that $supp(f*\varphi)\subset\overline{supp f+supp\varphi}$.

Since the fourth statement follows directly from estimate (2.2), the theorem is completely proved. □

For a compact set $K\subset\mathbb{E}^n$ and a point $\eta\in\mathbb{R}^n$ we denote by $H_K(\eta):=\sup\limits_{x\in K}(x,\eta)$. We need the following proposition to prove the main result (Theorem 3.1).

**Theorem 2.2** *Let $K_0\subset\mathbb{E}^n$ be a convex compact, $\Re\in\mathcal{B}_n$ and $\zeta=\xi+i\eta\in\mathbb{C}^n$. The entire analytic function $\phi(\zeta)$ is the Fourer - Laplace transformation of*

*1) a function belonging to $G_0^{\Re}(\mathbb{E}^n)$ with the support in $K_0$ if and only if for any $\vartheta>0$ there exists a number $c=c(\vartheta)>0$ such that*

$$|\phi(\zeta)|\leq c\cdot\,exp\,[H_{K_0}(\eta)-\vartheta\,h_{\Re}(\zeta)], \qquad (2.3)$$

*2) an element of $(G^{\Re}(\mathbb{E}^n))^*$ with the support in $K_0$ if and only if there exists a number $\vartheta_0>0$ such that for any $\varepsilon>0$ and for a number $c=c(\varepsilon)>0$*

$$|\phi(\zeta)|\leq c\,exp\,[H_{K_0}(\eta)+\varepsilon\,|\eta|+\vartheta_0\,h_{\Re}(\zeta)]. \qquad (2.4)$$

*Proof.* Let $\varphi\in G_0^{\Re}(\mathbb{E}^n)$. It is obvious that $\hat{\varphi}(\zeta):=\int\varphi(x)\,e^{-(x,\,\zeta)}dx$ is an entire analytic function. Let us prove that the function $\phi(\zeta):=\hat{\varphi}(\zeta)$ satisfies relation (2.3).

First we note that for any $\alpha\in\mathbb{N}_0^n$ and $\zeta\in\mathbb{C}^n$

$$|\zeta^\alpha|\,|\phi(\zeta)|\leq|K_0|\,e^{H_{K_0}(\eta)}\,\sup_{x\in K_0}|D^\alpha\varphi(x)|, \qquad (2.5)$$

where $|K_0|:=measK_0$. Let a number $m\in\mathbb{N}$ be chosen so that inequality (1.4) holds for all $j\geq m$. Then from (2.5) we have for all $\delta>0$, $j\geq m$ and $\zeta\in\mathbb{C}^n$

$$\sum_{\alpha\in\Re(j)\setminus\Re(j-m)}|\zeta^\alpha||\phi(\zeta)|\leq|K_0|\,\delta^{j-m}\,(j-m)^{j-m}\,e^{H_{K_0}(\eta)}$$

$$\cdot\sum_{\alpha\in\Re(j)\setminus\Re(j-m)}\delta^{-(j-m)}\,(j-m)^{-(j-m)}\,\sup_{x\in K_0}|D^\alpha\varphi(x)|$$

$$\leq|K_0|\,\delta^{j-m}\,(j-m)^{j-m}\,||\varphi,K_0||^{(m)}_{\Re,\delta}\,e^{H_{K_0}(\eta)}\,[\sum_{\alpha\in\Re(j)\setminus\Re(j-m)}1].$$

Since with a constant $\kappa=\kappa(\Re,m)>0$ the inequality

$$card[\,(\Re(j)\setminus\Re(j-m))\cap\mathbb{N}_0^n\,]\leq\kappa^{j-m+1},\ \ j=m,m+1,...,$$

holds, in vertue of point 1) of Proposition 1.1 with a constant $\kappa_1=\kappa_1(\Re)>0$ we have for all $\zeta=\xi+i\,\eta\in\mathbb{C}^n, |\zeta|\geq1,$ and $j=m,m+1,...$

$$|\phi(\zeta)|\leq\kappa\,(\delta\,\kappa_1\,(j-m)/h_{\Re}(\zeta)\,)^{j-m}\,||\varphi,K_0||^{(m)}_{\Re,\delta}\,e^{H_{K_0}(\eta)}.$$

Denoting by $j_0=m+[h_{\Re}(\zeta)/\kappa_1\,\delta\,e]$, where $[a]$ is the integer part of $a$, we get for all $\zeta\in\mathbb{C}^n, |\zeta|\geq1$

$$|\phi(\zeta)|\leq\kappa\,e^{-(j_0-m)}\,||\varphi,K_0||^{(m)}_{\Re,\delta}\,e^{H_{K_0}(\eta)}.$$

Since $j_0 - m \geq h_{\Re}(\zeta)/(\kappa_1\,\delta\,e) - 1$, hence we obtain

$$|\phi(\zeta)| \leq \kappa\,e\,||\varphi, K||^{(m)}_{\Re,\delta}\;\; exp\,\{H_{K_0}(\eta) - \frac{h_{\Re}(\zeta)}{\kappa_1\,\delta\,e}\}.$$

Let $\vartheta := 1/(\kappa_1\,\delta\,e)$. In virtue of $h_{\Re}(\zeta) \leq c_1 < \infty$ for $|\zeta| \leq 1$, from this we have with a constant $c_2 > 0$ for all $\zeta \in \mathbb{C}^n$

$$|\phi(\zeta)| \leq c_2\,||\varphi, K_0||^{(m)}_{\Re,\delta}\;\; exp\,\{H_{K_0}(\eta) - \vartheta\,h_{\Re}(\zeta)\}. \tag{2.6}$$

So, assuming $c = c_2\,||\varphi, K_0||^{(m)}_{\Re,\delta}$, we get inequality (2.3). Thus, the necessity of the first statement is proved.

We proceed with the proof of the sufficiency of the first statement. Let $\phi$ be an entire analytic function, satisfying (2.3). We shall prove that there exists a function $\varphi \in G^{\Re}_0(\mathbb{E}^n):\ \operatorname{supp}\varphi \subset K_0$ such that $\phi(\zeta) = \hat{\varphi}(\zeta)$.

First we note that for an entire analytic function $\phi$ , satisfying condition (2.3), the integral $\int \phi(\xi + i\,\eta)\,e^{i\,(x,\,\xi+i\,\eta)}d\xi$ converges for any point $\eta \in \mathbb{R}^n$, and the integral does not depend on $\eta$.

We set

$$\varphi(x) := (2\pi)^{-n}\int \phi(\xi)\,e^{i\,(x,\xi)}d\xi \quad ( = (2\pi)^{-n}\int \phi(\xi + i\,\eta)\,e^{i\,(x,\,\xi+i\,\eta)}\,d\xi\ ).$$

We will show that $\varphi \in G^{\Re}(\mathbb{E}^n)$. It follows immediately from estimate (2.3) that $\varphi \in C^{\infty}(\mathbb{E}^n)$, and there is a number $c > 0$ such that for any $\vartheta > 0$ and for all $\alpha \in \mathbb{N}^n_0$, $x \in \mathbb{E}^n$

$$|D^{\alpha}\varphi(x)| \leq (2\pi)^{-n}\int |\phi(\xi)|\,|\xi^{\alpha}|d\xi \leq c\sup_{\xi\in\mathbb{R}^n}|\xi^{\alpha}|\,e^{-\vartheta\,h_{\Re}(\xi)}.$$

This together with (1.4) implies that for some positive constants $\kappa_2 = \kappa_2(\Re)$ and $c_4$ and all $\alpha \in \Re(j)\setminus\Re(j-1)$ $(j = 1, 2, ...)$

$$|D^{\alpha}\varphi(x)| \leq \sup_{\xi\in\mathbb{R}^n}[\kappa_2\,h_{\Re}(\xi)]^j\,e^{-\vartheta\,h_{\Re}(\xi)} \leq c_4\,\kappa_2^j\,(\frac{j}{\vartheta})^j\,e^{-j} = c_4\,(\frac{\kappa_2}{e\,\vartheta})^j\,j^j,\ \ x \in \mathbb{E}^n.$$

Since $j-1 \leq r_{\Re}(\alpha) \leq j$ for any $\alpha \in \Re(j)\setminus\Re(j-1)$, when $\vartheta = \kappa_2/(e\,\delta)$ for any compact $K \in \mathbb{E}^n$ and any number $\delta > 0$ we have $||\varphi, K||_{\Re,\delta} \leq c_5$ with a contant $c_5 > 0$. Applying Lemma 1.1, we obtain that $\varphi \in G^{\Re}(\mathbb{E}^n)$.

Let us prove that $\operatorname{supp}\varphi \subset K_0$. Let $x^0 \notin K_0$. Since $K_0$ is a convex set and $x^0 \notin K_0$, there exists a point $\eta^0 \in \mathbb{R}^n$ and a number $a > 0$ such that $(x^0, \eta^0) - H_{K_0}(\eta^0) \geq 2a$. We show that $\varphi(x) = 0$ for $\{x : |x - x^0| < a\}$. Since for any $\vartheta > 0$ $(x, \vartheta\,\eta^0) - H_{K_0}(\vartheta\,\eta^0) \geq \vartheta\,a$ if $|x - x^0| < a$, for some a constant $c_6 > 0$ we have for such $x$

$$|\varphi(x)| = (2\pi)^{-n}\,|\int \phi(\xi + i\,\vartheta\,\eta^0)\,e^{i(x,\xi+i\,\vartheta\,\eta^0)}d\xi| \leq c_6\,e^{H_{K_0}(\vartheta\,\eta^0)-(x,\vartheta\,\eta^0)} \leq c_6\,e^{-a\,\vartheta}.$$

Hence, in view of the arbitrariness of the number $\vartheta > 0$ and the point $x^0 \notin K_0$ we obtain first, that $\varphi(x) = 0$ for $x:\ |x - x_0| < a$ and, secondly, that $\operatorname{supp}\varphi \subset K_0$. Thus the first part of the theorem is proved.

Now we will prove the second part of the theorem. Let $f \in (G^{\Re}(\mathbb{E}^n))^*$, $\operatorname{supp} f \subset K_0$. We show that its Fourer - Laplace transformation $F(\zeta) := \hat{f}(\zeta)$ is an entire analytic function satisfying inequality (2.4).

We choose a function $\chi \in G_0^{\Re^\star}(\mathbb{E}^n)$ such that $\operatorname{supp}\chi \subset K_0(\varepsilon) := K_0 + \{x \in \mathbb{E}^n; |x| \le \varepsilon\}$ for some $\varepsilon > 0$ and $\chi(x) = 1$ for $x \in K_0(\varepsilon/2)$. Since $e^{-i(x,\zeta)} \in G^{\Re}(\mathbb{E}^n)$ for any point $\zeta \in \mathbb{C}^n$, by virtue of (2.1) we get for some positive constants $\delta_0$ and $c_7$

$$|F(\zeta)| = |f_x(\chi(x)\, e^{-i(x,\zeta)})| \le c_7\, ||\chi(\cdot)\, e^{-i(\cdot,\zeta)}, K_0(\varepsilon)||_{\Re,\delta_0},\ \ \zeta \in \mathbb{C}^n.$$

By virtue of Lemma 1.2, from this we have, for some positive constants $\delta_1$ and $c_8$

$$|F(\zeta)| \le c_8\, ||e^{-i(\cdot,\zeta)}, K_0(\varepsilon)||_{\Re,\delta_1},\ \ \zeta \in \mathbb{C}^n.$$

By carrying out calculations analogous to those carried out in the proof of the first part of the theorem, we immediately obtain inequality (2.4) for some constant $\vartheta_0 = \vartheta_0(\delta_1) > 0$. Thus, the necessity of the second statement is proved.

Now we will show that $F$ is an entire analytic function. Since for any point $\zeta \in \mathbb{C}^n$ $\sum\limits_{j=0}^{k} \frac{[-i\,(\cdot,\zeta)]^j}{j!} \to e^{-i\,(\cdot,\zeta)}$ for $k \to \infty$ in the topology of $G^{\Re}(\mathbb{E}^n)$ and $f \in (G^{\Re}(\mathbb{E}^n))^*$, it follows that

$$f_x(\chi(x) \sum_{j=0}^{k} \frac{[-i\,(x,\zeta)]^j}{j!}) \to f_x(\chi(x)\, e^{-i(x,\zeta)}) = F(\zeta)$$

as $k \to \infty$. This proves that $F$ is an entire analytic function.

Now we will prove the converse assertion: let $F$ be an entire analytic function, satisfying inequality (2.4). We shall show that there exists an element $f \in (G^{\Re}(\mathbb{E}^n))^*$, with support in $K_0$ such that $\hat{f}(\zeta) = F(\zeta)$.

From (2.4) and from the first part of the theorem, which has already been proved, it follows that for any $\varphi \in G_0^{\Re}(\mathbb{E}^n)$ and $\eta \in \mathbb{R}^n$ the integral $\int F(\xi + i\,\eta)\, \hat{\varphi}(-\xi - i\,\eta)\, d\xi$ converges. Since $F$ and $\hat{\varphi}$ are entire analytic functions, this integral does not depend on $\eta \in \mathbb{R}^n$. Denote

$$f(\varphi) := (2\pi)^{-n} \int F(\xi + i\,\eta)\, \hat{\varphi}(\xi + i\,\eta)\, d\xi.$$

Since by virtue of (2.4), Remark 2.1 (see inequality (2.6)) and Lemma 1.1 for any compact set $K$ and number $\delta_2 > 0$ there exists a number $c_9 > 0$ such that for all $\varphi \in G_0^{\Re}(\mathbb{E}^n)$ with support in $K$

$$|f(\varphi)| \le (2\pi)^{-n} \int \hat{\varphi}(\zeta)\, e^{\vartheta_0\, h_{\Re}(\xi)}\, d\xi \le c_9.||\varphi,\, K||_{\Re,\delta_2},$$

This means (see (2.1)) that $f \in (G_0^{\Re}(\mathbb{E}^n))^*$.

We show that $\operatorname{supp} f \subset K_0$. Let $x^0 \notin K(\varepsilon)$. Then there exist a point $\eta^0 \in \mathbb{R}^n$ and a number $a > 0$ such that $(x^0, \eta^0) - H_{K(\varepsilon)}(\eta^0) \ge 2a$. Let a function $\varphi \in G_0^{\Re}(\mathbb{E}^n)$ satisfy the condition $supp\varphi \subset \{x \in \mathbb{E}^n, |x - x_0| < a\}$. Then by virtue of (2.3) and (2.4) for $\vartheta > \vartheta_0$ we have for some positive constants $c_{10}$ and $c_{11}$

$$|f(\varphi)| \le (2\,\pi)^{-n} \int |\Phi(\xi + i\,t\,\eta^0)|\, |\hat{\varphi}(-\xi - i\,t\,\eta^0)|\, d\xi$$

$$\le c_{10}\, exp[H_{K_0}(t\,\eta^0) + \varepsilon\, t\, |\eta^0| - (x,\, t\,\eta^0)] \int |e^{(\vartheta_0 - \vartheta)\, h_{\Re}(\xi + i\,t\,\eta^0)}|\, d\xi$$

$$\le c_{10}\, exp[H_{K_0}(t\,\eta^0) + \varepsilon\, t\, |\eta^0| - (x,\, t\,\eta^0)] \int e^{(\vartheta_0 - \vartheta)\, h_{\Re}(\xi)}|\, d\xi \le c_{11}\, e^{-a\,t} \to 0$$

as $t \to \infty$.

Since the point $x^0 \notin K(\varepsilon)$ and the number $\varepsilon > 0$ are arbitrary, we obtain that $\operatorname{supp} f \subset K_0$. □

## 3 Main result

For a linear differential operator with constant coefficients $P(D) = \sum\limits_{\alpha}\gamma_\alpha D^\alpha$, where the sum goes over a finite set of multi-indices $(P) := \{\alpha \in \mathbb{N}_0^n, \gamma_\alpha \neq 0\}$, by $P(\xi) := \sum\limits_{\alpha}\gamma_\alpha \xi^\alpha$ we denote the characteristic polynomial (the complete symbol) of the operator $P(D)$, by $m = m(P) := \max\limits_{\alpha\in(P)} |\alpha|$ we denote its order and by $P_m(\xi) := \sum\limits_{|\alpha|=m} \gamma_\alpha \xi^\alpha$ its main ($m-$homogenous) part.

We represent the polynomial $P$ as a sum of $j-$homogeneous polynomials $(j = 0, 1, ..., m)$

$$P(\xi) = \sum_{j=0}^{m} P_j(\xi) = \sum_{j=0}^{m} \sum_{|\alpha|=j} \gamma_\alpha \xi^\alpha. \tag{3.1}$$

**Definition 1** Let a polynomial $P$ be represented in form (3.1), $0 \neq N = (N_1, ..., N_n) \in E^n$ and $P_m(N) \neq 0$. The polynomial $P$ is called

1) **hyperbolic** (by Gȯrding) with respect to a vector $N$ (see [4] or [5], II, Definition 12.3.3), if there exists a number $\tau_0 > 0$ such that $P(\xi + i\tau N) \neq 0$ for all $\xi \in \mathbb{R}^n$, and $\tau \in \mathbb{R}^1 : |\tau| \geq \tau_0$;

2) **strongly hyperbolic** (by Petrowsky) with respect to a vector $N$ (see [16] ), if all zeros of the polynomial $P_m(\xi + \tau N)$, are real and simple;

2') **weakly hyperbolic** if among those zeros of this polynomial there are multiple zeros (see, for instance, [6], [7], [8]);

3) $s-$**hyperbolic** $(s > 1)$ with respect to a vector $N$ (see [11]), if there exists a number $c > 0$ such that $P(\xi + i\tau N) \neq 0$ for all $(\xi, \tau) \in \mathbb{R}^{n+1}$ satisfying the condition $|\tau| \geq c\,(1 + |\xi|^{1/s})$;

4) $h_\Re-$**hyperbolic** (for a polyhedron $\Re \in \mathcal{B}_n$) with respect to a vector $N$ if there exists a number $c > 0$ such that $P(\xi + i\tau N) \neq 0$ for all $\xi \in \mathbb{R}^n$, $\tau \in \mathbb{C}$ and $|Re\tau| \geq c\, h_\Re(\xi)$.

For an operator $R(D)$ (polynomial $R(\xi)$) and a number $\tau \in \mathbb{R}^1$ we denote by $\tilde{R}$ the L. Hörmander function

$$\tilde{R}(\xi, \tau) := \sqrt{\sum_{\alpha\in\mathbb{N}_0^n} |R^{(\alpha)}(\xi)|^2\, |\tau|^{2\,|\alpha|}}.$$

**Definition 2** Let $q$ be a non-negative function defined in $\mathbb{R}^n$. We say that a polynomial $P$ is $q$ -stronger than a polynomial $Q$ and write $P \succ^q Q$, or $Q \prec^q P$, if there exists a constant $c > 0$ suh that

$$\tilde{Q}(\xi, \tau) \leq c\, \tilde{P}(\xi, \tau) \;\; \forall (\xi, \tau) \in \mathbb{R}^{n+1} : \, |\tau| \geq q(\xi).$$

It is known (see [13]) that if a polynomial $P$, represented as (3.1) is weakly hyperbolic and $P - P_m \prec^q P_m$, with a non-negative function $q$, then there exist positive numbers $c_0$ and $\kappa_0$ such that for all $\kappa \geq \kappa_0$

$$|P(\xi + i\,\tau\, N)| \geq c_0\, \tilde{P}(\xi, \tau) \;\; \forall (\xi, \tau) \in \mathbb{R}^{n+1} : \, |\tau| \geq \kappa\, q(\xi). \tag{3.2}$$

First we prove the following general proposition

**Lemma 3.1** *Let $\Re \subset \mathbb{R}^{n,+}$ be a completely regular polyhedron, $0 \neq N \in E^n$, $\kappa > 0$ and $q_{\Re,N}(\xi) := \min\limits_{t\in\mathbb{R}^1} h_\Re(\xi - t\,N)$. Then for any polynomial $P$ the following conditions are equivalent:*

1) $P(\xi + i\,\tau\, N) \neq 0 : \, \xi \in \mathbb{R}^n, \; \tau \in \mathbb{C}, \; |Re\tau| \geq \kappa\, h_\Re(\xi)$,

2) $P(\xi + i\,\tau\, N) \neq 0 : \, \xi \in \mathbb{R}^n, \; \tau \in \mathbb{C}, \; |Re\tau| \geq \kappa\, q_{\Re,N}(\xi)$,

3) $P(\xi + i\,\tau\, N) \neq 0 : \, (\xi, \tau) \in \mathbb{R}^{n+1}, \; |\tau| \geq \kappa\, q_{\Re,N}(\xi)$.

*Proof.* Since $h_\Re(\xi) \geq q_{\Re,N}(\xi) \;\; \forall \xi \in \mathbb{R}^n$, from 2) immediately follows 1).

We show that 1) $\Rightarrow$ 2). Let, to the contrary, condition 1) is satisfied, but there exist some points $\xi^0 \in \mathbb{R}^n$, $\tau^0 \in \mathbb{C}$ such that $|Re\tau^0| \geq \kappa\, q_{\Re,N}(\xi^0)$ and $P(\xi^0 + i\,\tau^0\, N) = 0$. Since for any

$\vartheta \in \mathbb{R}^1$ $P(\xi^0 - \vartheta N + i(\tau^0 - i\vartheta)N) = P(\xi^0 + i\tau^0 N) = 0$ and $Re(\tau^0 - i\vartheta) = Re\tau^0$, by virtue of 1) we have $|Re(\tau^0)| = |Re(\tau^0 - i\vartheta)| < \kappa h_{\Re}(\xi - \vartheta N)$. Consequently $|Re\tau^0| < \kappa q_{\Re,N}(\xi^0)$. We have obtained a contradiction, which proves that 1) $\Rightarrow$ 2).

It is obvious that 2) $\Rightarrow$ 3). We show that 3) $\Rightarrow$ 2). Let, to the contrary, Condition 3) is satisfied, but there exist some points $\xi^0 \in \mathbb{R}^n$, $\tau^0 \in \mathbb{C}$ : such that $|Re\tau^0| \geq \kappa q_{\Re,N}(\xi^0)$ and $P(\xi^0 + i\tau^0 N) = 0$. Since $0 = P(\xi^0 + i\tau^0 N) = P(\xi^0 - Im\tau^0 N + i\,Re\tau^0 N)$, by virtue of 3) we have $|Re\tau^0| < \kappa q_{\Re,N}(\xi^0 - Im\tau^0 N) = \kappa \min_{t\in\mathbb{R}^1} h_{\Re}(\xi^0 - Im\tau^0 N - tN) = \kappa \min_{\vartheta\in\mathbb{R}^1} h_{\Re}(\xi^0 - \vartheta N) = \kappa q_{\Re,N}(\xi^0)$. We have obtained a contradiction, proving that 3) $\Rightarrow$ 2). □

**Remark 1** Note that for the function $q_{\Re,N}$, which was introduced above, in the classical case, when $N = (1, 0, ..., 0)$, $q_{\Re,N}(\xi) \equiv h_{\Re}(0, \xi_2, ..., \xi_n)$ $\forall \xi \in \mathbb{R}^n$.

The next statement follow immediately from Lemma 3.1.

**Corollary 3.1** *Let $\Re \in \mathcal{B}_n$. A polynomial $P$ is $h_{\Re}-$hyperbolic with respect to a vector $0 \neq N \in \mathbb{E}^n$ if and only there is a member $\kappa > 0$ such that if $P(\xi + i\tau N) \neq 0$ for all $(\xi, \tau) \in \mathbb{R}^{n+1}$, $|\tau| \geq \kappa q_{\Re,N}(\xi)$.*

**Corollary 3.2** *Let $\Re \in \mathcal{B}_n$ and $P$ be a polynomial, weakly hyperbolic with respect to a vector $N$, represented in form (3.1). If $P - P_m \prec^{q_{\Re,N}} P_m$, then the polynomial $P$ is $h_{\Re}-$hyperbolic with respect to the vector $N$.*

*Proof.* Under the assumptions of the corollary, inequality (3.2) holds for all $\kappa \geq \kappa_0$, for some positive constants $c_0$, $\kappa_0$. By Lemma 2.1 this implies that $P(\xi + i\tau N) \neq 0$ for all $(\tau, \xi) \in \mathbb{R}^{n+1}$, for which $|\tau| \geq \kappa q_{\Re,N}(\xi)$, which means that the polynomial $P$ is $h_{\Re}-$hyperbolic with respect to the vector $N$. □

The main results of this paper are the following Theorems 3.1 and 3.2.

**Theorem 3.1** *Let $\Re \in \mathcal{B}_n$, $0 \neq N \in \mathbb{E}^n$ and $P(D)$ be a $h_{\Re}-$ hyperbolic with respect to the vector $N$ operator, represented as (3.1), i.e. (see Corollary 3.1) $P_m(N) \neq 0$ and there is a number $\kappa > 0$ such that $P(\xi + i\tau N) \neq 0$ for all $(\xi, \tau) \in \mathbb{R}^{n+1}$ for which $|\tau| \geq \kappa q_{\Re,N}(\xi)$.*

*Then operator $P(D)$ has a fundamental solution $E \in (G_0^{\Re}(\mathbb{E}^n))^*$ with* $\operatorname{supp} E \subset \bar{\Omega}_N$, where $\Omega_N := \{x \in \mathbb{E}^n, (x, N) > 0\}$.

*Proof.* Let $\xi \in \mathbb{R}^n$ and $\tau_j(\xi)$ $(j = 1, ..., m)$ be the roots of the polynomial $P(\xi + i\tau N)$. Then

$$P(\xi + i\tau N) = i^m P_m(N) \prod_{j=1}^{m} (\tau - \tau_j(\xi)) \;\; \forall (\tau, \xi) \in \mathbb{R}^{n+1}. \tag{3.3}$$

By Lemma 3.1, we have for some constant $\kappa_1 > 0$

$$|Re\tau_j(\xi)| \leq \kappa_1 h_{\Re}(\xi), \;\; \xi \in \mathbb{R}^n, \; j = 1, ..., m. \tag{3.4}$$

Let $t \leq -2\kappa_1$ and $\tau = \tau(t, \xi) := t\,h_{\Re}(\xi)$. Then it follows from (3.3) and (3.4) that

$$|P(\xi + i\tau N)| \geq |P_m(N)| \left[\frac{|t|}{2}\, h_{\Re}(\xi)\right]^m, \;\; \xi \in \mathbb{R}^n.$$

Let $t \leq -2\kappa_1$ and $\sigma(t) := \{\zeta = \xi + i\,t\,h_{\Re}(\xi) : t \leq -2\kappa_1\}$. This implies

$$|P(\zeta)| \geq |P_m(N)| \left[\frac{|t|}{2}\, h_{\Re}(Re\zeta)\right]^m, \;\; \zeta \in \sigma(t). \tag{3.5}$$

By virtue of Theorem 2.2 and estimate (3.5), the integral $\int_{\sigma(t)} \hat{\varphi}(\zeta)/P(\zeta)\,d\zeta$ converges for any $\varphi \in G_0^{\Re}(\mathbb{E}^n)$ and $t < -2\kappa_1$ . Since the function $\hat{\varphi}(\zeta)/P(\zeta)$ is analytic in the domain $\omega := \bigcup_{t<-2\kappa_1} \sigma(t)$, this integral does not depend on $t$ for $t < -2\kappa_1$. Denote

$$\breve{E}(\varphi(\cdot)) := E(\varphi(-\cdot)) := (2\pi)^{-n} \int_{\sigma(t)} \hat{\varphi}(\zeta)/P(\zeta)\, d\zeta,\ t < -2\,\kappa_1. \qquad (3.6)$$

It is obvious that $E$ is a linear functional defined on $G_0^{\Re}(\mathbb{E}^n)$. Let us show that $E \in (G_0^{\Re}(\mathbb{E}^n))^*$.

Since $h_{\Re}(\xi + i\,\eta) \geq h_{\Re}(\xi)$ for all $\xi,\, \eta \in \mathbb{R}^n$, by applying estimate (2.6) and Lemma 1.1, for any $\vartheta > 0$ we obtain the existence of positive numbers $\delta$ and $c$ such that for any convex compact set $K \subset \mathbb{E}^n$

$$|\hat{\varphi}(\zeta)| \leq c\,||\varphi, K||_{\Re,\,\delta}\ \ exp[H_K(t\,h_{\Re}(\xi)\,N) - \vartheta\, h_{\Re}(\xi)]$$
$$= c\,||\varphi, K||_{\Re,\,\delta}\ \ exp[t\,h_{\Re}(\xi)\,H_K^{'}(N) - \vartheta\, h_{\Re}(\xi)]$$
$$\forall \varphi \in G_0^{\Re}(\mathbb{E}^n), \operatorname{supp}\varphi \subset K, \zeta \in \omega, \qquad (3.7)$$

where $\xi = Re\zeta,\ H_K^{'}(N) := \inf_{x \in K}(x, N)$.

Since the number $\vartheta > 0$ $(\vartheta \geq t\,H_K^{'}(N)\ )$ is arbitrary, using estimate (3.5), from here and (3.6) we obtain with a constant $c_1 > 0$

$$|\breve{E}(\varphi)| \leq c_1\,||\varphi, K||_{\Re,\,\delta}\ \ \forall \varphi \in G_0^{\Re}(\mathbb{E}^n), \operatorname{supp}\varphi \subset K,$$

i.e. $E \in (G_0^{\Re}(\mathbb{E}^n))^*$.

Since, by Theorem 2.2, $\hat{\varphi}$ is an entire analytic function for $\varphi \in G_0^{\Re}(\mathbb{E}^n)$, then

$$\breve{E}[P(D)\varphi] = (2\pi)^{-n} \int_{\sigma(t)} \hat{\varphi}(\zeta)\, d\zeta = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\varphi}(\xi\, d\xi = \varphi(0),$$

i.e. $P(D)E = \delta^0$, where $\delta^0$ is the Dirac measure concentrated at the origin.

Thus it is proved that $E \in (G_0^{\Re}(\mathbb{E}^n))^*$ is a fundamental solution. Let us show that $\operatorname{supp} E \subset \bar{\Omega}_N$.

Let $\varphi \in G_0^{\Re}(\mathbb{E}^n)$, $\operatorname{supp}\varphi \subset \Omega_N$ and $K_0$ be the convex hull of $\operatorname{supp}\varphi$. Since $H_{K_0}^{'}(N) > 0$ and $h_{\Re}(\xi) \geq c_2\ \forall \xi \in \mathbb{R}^n$ with a constant $c_2 > 0$, by virtue of estimates (2.3), (3.5) and (3.7) there exist positive numbers $\delta$ and $c_3$ such that

$$|\breve{E}(\varphi)| \leq c_3\,||\varphi, K_0||_{\Re,\,\delta}\ |t|^{-m}\, e^{c_2\,t\,H_{K_0}^{'}(N)} \int_{\sigma(t)} e^{-\vartheta\, h_{\Re}(\xi)} |d\zeta|.$$

Since the right-hand side of this relation tends to zero as $t \to -\infty$, it follows that $\operatorname{supp} E \subset \bar{\Omega}_N$. □

**Theorem 3.2** *Let an operator* $P(D)$ *satisfy the assumptions of Theorem 3.1,* $f \in G^{\Re}(\mathbb{E}^n)$, $\varepsilon > 0$ *and* $\operatorname{supp} f \subset \{x \in \mathbb{E}^n : (x, N) \geq \varepsilon\,|x|\}$. *Then the equation* $P(D)u = f$ *has a solution* $u \in G^{\Re}(\mathbb{E}^n)$ *with* $\operatorname{supp} u \subset \bar{\Omega}_N$.

*Proof.* Since $\mathcal{D}_K(E, f)$ is compact for any compact set $K$, the convolution $(E * f)$ exists and belongs to $G^{\Re}(\mathbb{E}^n)$. Moreover by Theorem 2.1 $\operatorname{supp}(E * f) \subset \operatorname{supp} E + \operatorname{supp} f \subset \bar{\Omega}_N$. Then by Theorems 2.1 and 3.1 we have $P(D)(E * f) = [P(D)E] * f = \delta^0(f) = f$. □

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