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Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

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Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

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- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

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KUSSAINOVA LEILI KABIDENOVNA

(to the 70th birthday)

On May 3, 2018 was the 70th birthday of Leili Kabidenovna Kussainova, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department of Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), Professor (2006), Honorary worker of Education of the Republic of Kazakhstan (2005).

L.K. Kussainova was born in the city of Karaganda. In 1972 she graduated from the Novosibirsk State University (Russian Federation) and then completed her postgraduate studies at the Institute of Mathematics (Almaty). L.K. Kussainova's scientific supervisors were distinguished Kazakh mathematicians T.I. Amanov and M. Otelbayev.

Scientific works of L.K. Kussainova are devoted to investigation of the widths of embeddings of the weighted Sobolev spaces, to embeddings and interpolations of weighted Sobolev spaces with weights of general type.

She has solved the problem of three-weighted embedding of isotropic and anisotropic Sobolev spaces in Lebesgue spaces, the problem of exact description of the Lions-Petre interpolation spaces for a pair of weighted Sobolev spaces.

To solve these problems L.K. Kussainova obtained nontrivial modifications of theorems on Besicovitch-Guzman covers. The first relates to covers by multidimensional parallelepipeds, whereas the second relates to double covers by cubes. These modifications have allowed to obtain the description of the interpolation spaces in the weighted case. Furthermore, by using the double covering theorem the exact descriptions of the multipliers were obtained for a pair of Sobolev spaces of general type.

The maximal operators on a basis of cubes with adjustable side length, which were introduced by L.K. Kussainova, have allowed her to solve the problem of two-sided distribution estimate of widths of the embedding of two-weighted Sobolev spaces with weights of general type in weighted Lebesgue spaces.

Under her supervision 6 theses have been defended: 4 candidates of sciences theses and 2 PhD theses.

The Editorial Board of the Eurasian Mathematical Journal congratulates Leili Kabidenovna Kussainova on the occasion of her 70th birthday and wishes her good health and new achievements in mathematics and mathematical education.

The awarding ceremony
of the Certificate of the Emerging Sources Citation of Index database

In 2016 the Eurasian Mathematical Journal has been included in the Emerging Sources Citation of Index (ESCI) of the "Clarivate Analytics" (formerly "Thomson Reuters") Web of Science. In 2018 the second journal of the L.N. Gumilyov Eurasian National University, namely the Eurasian Journal of Mathematical and Computer Applications was also included in ESCI.

The ESCI was launched in late 2015 as a new database within "Clarivate Analytics". Around 3,000 journals were selected for coverage at launch, spanning the full range of subject areas.

The selection process for ESCI is the first step in applying to the Science Citation Index. All journals submitted for evaluation to the core Web of Science databases will now initially be evaluated for the ESCI, and if successful, indexed in the ESCI while undergoing the more in-depth editorial review. Timing for ESCI evaluation will follow "Clarivate Analytics" priorities for expanding database coverage, rather than the date that journals were submitted for evaluation.

Journals indexed in the ESCI will not receive Impact Factors; however, the citations from the ESCI will now be included in the citation counts for the Journal Citation Reports, therefore contributing to the Impact Factors of other journals. If a journal is indexed in the ESCI it will be discoverable via the Web of Science with an identical indexing process to any other indexed journal, with full citation counts, author information and other enrichment. Articles in ESCI indexed journals will be included in an author's H-Index calculation, and also any analysis conducted on Web of Science data or related products such as InCites. Indexing in the ESCI will improve the visibility of a journal, provides a mark of quality and is good for authors.

To commemorate this important achievement of mathematicians of the L.N. Gumilyov Eurasian National University on June 14, 2018, by the initiative of the "Clarivate Analytics", the awarding ceremony of the Certificate of Emerging Sources Citation Index database of "Clarivate Analytics" to the editorial boards of the Eurasian Mathematical Journal and the Eurasian Journal of Mathematical and Computer Applications was held at the L.N. Gumilyov Eurasian National University. The programme of this ceremony is attached.

Astana June 14, 2018

Venue: L.N. Gumilyov Eurasian National University
Astana, Satpayev street 2, Room 259

14:30- 15:00	Visit to the Museum of the history of Education, Museum of L.N. Gumilyov, Museum of writing
15:00-15:10	*Opening speech of moderator* **A. Moldazhanova** – the First Vice-Rector, Vice-Rector for Academic Works of L.N. Gumilyov Eurasian National University
15:10-15:20	**Oleg Utkin** - Managing Director of Clarivate Analytics in Russia and the CIS
15:20-15:30	*Certification award ceremony of the Eurasian Mathematical Journal, the Eurasian Journal of Mathematical and Computer Applications in international database*
15:30-15:45	**Kordan Ospanov** – Deputy Editor-in-Chief of the Eurasian Mathematical Journal. *History and perspectives of development of the scientific journal Eurasian Mathematical Journal*
15:45-16:00	**Kazizat Iskakov** – Deputy Editor-in-Chief of the Eurasian Journal of Mathematical and Computer Applications. *History and perspectives of development of the scientific journal Eurasian Journal of Mathematical and Computer Applications.*
16:00-16:10	*Closing Ceremony* *Memory photo*
16:10-16:30	*Coffee break for visitors*
16:40-17:20	**Lyaziza Mukasheva** - Official representative of Clarivate Analytics in the Central Asian region *Seminar for editors of scientific journals Scientific library of L.N. Gumilyov Eurasian National University room 104*

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ADDITIVE ESTIMATES FOR DISCRETE HARDY-TYPE OPERATORS

A. Kalybay, S. Shalginbayeva

Communicated by L.K. Kussainova

Key words: additive inequality, Hardy-type inequality, matrix operator, space of sequences.

AMS Mathematics Subject Classification: 26D10, 26D15, 39B82.

Abstract. We establish necessary and sufficient conditions for the validity of weighted additive estimates of the norms of the discrete Hardy operators.

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1 Introduction

Let $f \geq 0$ be a sequence of real numbers $f = \{f_i\}_{i=1}^{\infty}$ with non-negative terms.

Let $v > 0$, $u \geq 0$ and $w \geq 0$ be weight sequences. Let P^+ and P^- be the discrete Hardy operators:

$$(P^+f)_i = \sum_{j=1}^{i} f_j \quad \text{and} \quad (P^-f)_i = \sum_{j=i}^{\infty} f_j, \quad i \geq 1.$$

Let A^+ and A^- be matrix operators of the form:

$$(A^+f)_i = \sum_{j=1}^{i} a_{i,j} f_j \quad \text{and} \quad (A^-f)_i = \sum_{j=i}^{\infty} a_{j,i} f_j, \quad i \geq 1,$$

where $a_{i,j} \geq 0$ for $i \geq j \geq 1$ and $a_{i,j} = 0$ for $i < j$.

In papers [5] -[7] under certain conditions on the elements $(a_{i,j})$ the authors have found necessary and sufficient conditions for the validity of the inequality:

$$\|uA^{\pm}f\|_q \leq C\left(\|vf\|_p + \|wP^{\pm}f\|_p\right), \quad f \geq 0,$$

where $1 < p, q < \infty$ and $\|\cdot\|_q$ is the standard norm of the space l_q. In particular, the case $A^{\pm} \equiv P^{\pm}$ has been studied in work [7] for $1 < p \leq q < \infty$.

Here we investigate the following weighted additive estimates for the discrete Hardy operators $P^{\pm}$:

$$\|uP^+f\|_q \leq C\left(\|vf\|_p + \|wA^+f\|_r\right), \quad f \geq 0, \tag{1.1}$$

$$\|uP^-f\|_q \leq C\left(\|vf\|_p + \|wA^-f\|_r\right), \quad f \geq 0. \tag{1.2}$$

Let us note that some continuous analogues of inequality (1.1) have been studied in works [2] -[4].

2 Main results

Let $\frac{1}{p}+\frac{1}{p'}=1$. To formulate the main results we need to introduce the following sequences and quantities:

$$\varphi_i^+=\left[\min_{1\le k\le i}\left\{\left(\sum_{j=k}^{i}v_j^{-p'}\right)^{-\frac{1}{p'}}+\left(\sum_{j=k}^{\infty}w_j^ra_{j,k}^r\right)^{\frac{1}{r}}\right\}\right]^{-1},\quad i\ge 1,$$

$$\varphi_i^-=\left[\inf_{i\le k}\left\{\left(\sum_{j=i}^{k}v_j^{-p'}\right)^{-\frac{1}{p'}}+\left(\sum_{j=1}^{k}w_j^ra_{k,j}^r\right)^{\frac{1}{r}}\right\}\right]^{-1},\quad i\ge 1,$$

$$D^+=\sup_{i\ge 1}\left(\sum_{j=i}^{\infty}u_j^q\right)^{\frac{1}{q}}\varphi_i^+,$$

$$D^-=\sup_{i\ge 1}\left(\sum_{j=1}^{i}u_j^q\right)^{\frac{1}{q}}\varphi_i^-.$$

The main results of the paper are the Theorems 2.1 and 2.2.

Theorem 2.1. *Let* $1<\max\{p,r\}\le q<\infty$ *and* $a_{i,k}\le a_{i,j}$ *for* $1\le j\le k\le i$. *Then inequality* (1.1) *holds if and only if* $D^+<\infty$. *Moreover,* $D^+\approx C$, *where* $C>0$ *is the least constant in* (1.1).

Proof. Necessity. Let inequality (1.1) hold with the least constant $C>0$.

Let us take $f_i=v_i^{-p'}$ for $1\le t\le i\le z$ and $f_i=0$ for $1\le i<t$, $i>z$ and $t\le z$. Then

$$\|uP^+f\|_q\ge\left(\sum_{i=z}^{\infty}u_i^q\right)^{\frac{1}{q}}\sum_{i=t}^{z}v_i^{-p'},\tag{2.1}$$

$$\|vf\|_p=\left(\sum_{i=t}^{z}v_i^{-p'}\right)^{\frac{1}{p}},\tag{2.2}$$

$$\|wA^+f\|_r\le\left(\sum_{i=t}^{\infty}w_i^ra_{i,t}^r\right)^{\frac{1}{r}}\sum_{i=t}^{z}v_i^{-p'}.\tag{2.3}$$

From (1.1), (2.1), (2.2) and (2.3) it follows that

$$\left(\sum_{i=z}^{\infty}u_i^q\right)^{\frac{1}{q}}\le C\left(\left(\sum_{i=t}^{z}v_i^{-p'}\right)^{-\frac{1}{p'}}+\left(\sum_{i=t}^{\infty}w_i^ra_{i,t}^r\right)^{\frac{1}{r}}\right).$$

In view of independence of the left-hand side of the obtained inequality on $t:1\le t\le z$, we have

$$\left(\sum_{i=z}^{\infty}u_i^q\right)^{\frac{1}{q}}\le C(\varphi_z^+)^{-1},\quad\forall z\ge 1,$$

from which it follows that

$$D^+\le C.\tag{2.4}$$

Sufficiency. Let $D^+ < \infty$ and let $f = \{f_i\} \geq 0$ be a sequence, for which the right-hand side of (1.1) is finite. We can assume without loss of a generality that $f_1 > 0$.

Let $k_1 = \sup\{k \in \mathbb{Z} : 2^k \leq f_1\}$. Then

$$2^{k_1} \leq f_1 < 2^{k_1+1}. \tag{2.5}$$

Assume that $t_0 = 1$ and $t_1 = \sup\{i \in \mathbb{N} : \sum\limits_{j=1}^{i} f_j < 2^{k_1+1}\}$. Then $t_0 \leq t_1 \leq \infty$ and

$$2^{k_1} \leq \sum_{j=1}^{i} f_j < 2^{k_1+1} \quad \text{for} \quad t_0 \leq i \leq t_1. \tag{2.6}$$

Let $k_n = k_{n-1} + 1$ for $n \geq 2$ and $T_n = \{i \in \mathbb{N} : 2^{k_n} \leq \sum\limits_{j=1}^{i} f_j\}$. Moreover, let $t_n = \inf T_n$ if $T_n \neq \emptyset$ and $t_n = \infty$ if $T_n = \emptyset$.

We see that

$$\sum_{j=1}^{t_n-1} f_j < 2^{k_n} \leq \sum_{j=1}^{t_n} f_j \quad \text{for} \quad t_n < \infty \quad \text{and} \quad \sum_{j=1}^{t_n-1} f_j < 2^{k_n} \quad \text{for} \quad t_n = \infty. \tag{2.7}$$

Moreover, if $t_n < t_{n+1} \leq \infty$, then

$$2^{k_n} \leq \sum_{j=1}^{i} f_j < 2^{k_{n+1}} \quad \text{for} \quad t_n \leq i < t_{n+1}. \tag{2.8}$$

Let $N = \sup\{n \geq 0 : t_n < \infty\}$.

From the definition of D^+ for any $s, k : 1 \leq s \leq k$ we have:

$$\sum_{i=k}^{\infty} u_i^q \leq (D^+)^q \left(\left(\sum_{j=s}^{k} v_j^{-p'} \right)^{-\frac{1}{p'}} + \left(\sum_{j=s}^{\infty} w_j^r a_{j,s}^r \right)^{\frac{1}{r}} \right)^q. \tag{2.9}$$

Let us estimate $\|uP^+ f\|_q$. We separately consider the following cases: $N = 0$, $N = 1$, $N = 2$ and $N \geq 3$.

If $N = 0$ we have $t_1 = \infty$, then from (2.9) and (2.5) it follows that

$$\|uP^+ f\|_q^q = \sum_{j=1}^{t_1} u_j^q (P^+ f)_j^q \leq 2^{q(k_1+1)} \sum_{j=1}^{\infty} u_j^q$$

$$\leq 2^q (D^+)^q \left[2^{k_1} v_1 + \left(\sum_{j=1}^{\infty} w_j^r a_{j,1}^r 2^{rk_1} \right)^{\frac{1}{r}} \right]^q \ll (D^+)^q \left[v_1 f_1 + \left(\sum_{j=1}^{\infty} w_j^r a_{j,1}^r f_1^r \right)^{\frac{1}{r}} \right]^q$$

$$\leq (D^+)^q \left[\left(\sum_{i=1}^{\infty} (v_i f_i)^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty} w_j^r \left(\sum_{i=1}^{j} a_{j,i} f_i \right)^r \right)^{\frac{1}{r}} \right]^q = (D^+)^q \left(\|vf\|_p + \|wA^+ f\|_r \right)^q. \tag{2.10}$$

Let $N = 1$. Then $1 \leq t_1 < \infty$, $t_2 = \infty$ and

$$\|uP^+ f\|_q^q \leq \sum_{j=1}^{t_1} u_j^q (P^+ f)_j^q + \sum_{j=t_1}^{t_2-1} u_j^q (P^+ f)_j^q = F_1 + F_2. \tag{2.11}$$

The sum F_1 is estimated as in (2.10). Let us estimate F_2. Using (2.9), (2.6) and (2.8) we obtain

$$F_2 = \sum_{j=t_1}^{t_2-1} u_j^q (P^+f)_j^q \le 2^{qk_2} \sum_{j=t_1}^{\infty} u_j^q$$

$$\le 2^q (D^+)^q \left[2^{k_1} \left(\sum_{i=1}^{t_1} v_i^{-p'} \right)^{-\frac{1}{p'}} + \left(\sum_{j=1}^{\infty} w_j^r a_{j,1}^r 2^{rk_1} \right)^{\frac{1}{r}} \right]^q$$

$$\ll (D^+)^q \left[\left(\sum_{i=1}^{t_1} f_i \right) \left(\sum_{i=1}^{t_1} v_i^{-p'} \right)^{-\frac{1}{p'}} + \left(\sum_{j=1}^{\infty} w_j^r a_{j,1}^r f_1^r \right)^{\frac{1}{r}} \right]^q$$

$$\le (D^+)^q \left[\left(\sum_{i=1}^{\infty} (v_i f_i)^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty} w_j^r \left(\sum_{i=1}^{j} a_{j,i} f_i \right)^r \right)^{\frac{1}{r}} \right]^q$$

$$\le (D^+)^q \left(\|vf\|_p + \|wA^+f\|_r \right)^q . \tag{2.12}$$

Thus, if we combine estimates (2.10) for F_1 and (2.12) for F_2, we have

$$\|uP^+f\|_q \ll D^+ \left(\|vf\|_p + \|wA^+f\|_r \right) . \tag{2.13}$$

If $N = 2$, we have

$$\|uP^+f\|_q^q \le \sum_{j=1}^{t_1} u_j^q (P^+f)_j^q + \sum_{j=t_1}^{t_2-1} u_j^q (P^+f)_j^q + \sum_{j=t_2}^{t_3} u_j^q (P^+f)_j^q = F_1 + F_2 + F_3.$$

Here and in the sequel we assume that $\sum\limits_{j=c}^{d} = 0$ if $c > d$.

The sum F_1 is estimated as in (2.10). If $t_1 > t_2 - 1$, then $F_2 = 0$. If $t_2 > t_1$, then F_2 is estimated as in (2.12). Therefore, regardless of $F_2 = 0$ or $F_2 \neq 0$, it is estimated as in (2.12). Thus, we need to estimate only F_3. From (2.7) for $s \ge 2$ we have

$$2^{k_{s-1}} = 2^{k_s} - 2^{k_{s-1}} \le \sum_{j=1}^{t_s} f_j - \sum_{j=1}^{t_{s-1}-1} f_j = \sum_{j=t_{s-1}}^{t_s} f_j . \tag{2.14}$$

From (2.6), (2.9) and (2.14) we get

$$F_3 = \sum_{j=t_2}^{t_3} u_j^q (P^+f)_j^q \le 2^{qk_3} \sum_{j=t_2}^{\infty} u_j^q$$

$$\le 2^{2q} (D^+)^q \left[2^{k_1} \left(\sum_{i=t_1}^{t_2} v_i^{-p'} \right)^{-\frac{1}{p'}} + \left(\sum_{j=t_1}^{\infty} w_j^r a_{j,t_1}^r 2^{rk_1} \right)^{\frac{1}{r}} \right]^q$$

$$\ll (D^+)^q \left[\left(\sum_{i=t_1}^{t_2} f_i \right) \left(\sum_{i=t_1}^{t_2} v_i^{-p'} \right)^{-\frac{1}{p'}} + \left(\sum_{j=t_1}^{\infty} w_j^r a_{j,t_1}^r \left(\sum_{i=1}^{t_1} f_i \right)^r \right)^{\frac{1}{r}} \right]^q$$

$$\le (D^+)^q \left[\left(\sum_{i=t_1}^{t_2} (v_i f_i)^p \right)^{\frac{1}{p}} + \left(\sum_{j=t_1}^{\infty} w_j^r \left(\sum_{i=1}^{j} a_{j,i} f_i \right)^r \right)^{\frac{1}{r}} \right]^q$$

$$\leq (D^+)^q \left(\|vf\|_p + \|wA^+f\|_r\right)^q . \tag{2.15}$$

The last estimate (2.15) for F_3, together with the estimates for F_1 and F_2, gives (2.13).

If $N \geq 3$, we have

$$\|uP^+f\|_q^q \leq F_1 + F_2 + \widetilde{F}_3 + F_4, \tag{2.16}$$

where

$$\widetilde{F}_3 = \sum_{j=t_2}^{t_3-1} u_j^q (P^+f)_j^q,$$

$$F_4 = \sum_{s=3}^{N} \sum_{j=t_s}^{t_{s+1}-1} u_j^q (P^+f)_j^q.$$

If $t_3 > t_2$, then by (2.8) the value $\widetilde{F}_3$ is estimated similarly to F_3 in (2.15), otherwise $\widetilde{F}_3 = 0$. Thus, regardless of $\widetilde{F}_3 \neq 0$ or $\widetilde{F}_3 = 0$, we have the estimate

$$\widetilde{F}_3 \leq (D^+)^q \left(\|vf\|_p + \|wA^+f\|_r\right)^q . \tag{2.17}$$

To estimate the value F_4 we need to estimate the sum $\sum\limits_{j=t_s}^{t_{s+1}-1} u_j^q (P^+f)_j^q$ for $s \geq 3$ regardless of whether it is zero or nonzero. Using (2.7), (2.9) and (2.14), we have

$$F_4 \leq \sum_{s=3}^{N} 2^{qk_{s+1}} \sum_{j=t_s}^{\infty} u_j^q \leq 2^{2q} \sum_{s=3}^{N} 2^{qk_{s-1}} \sum_{j=t_s}^{\infty} u_j^q$$

$$\leq 2^{2q}(D^+)^q \sum_{s=3}^{N} \left[2^{k_{s-1}} \left(\sum_{i=t_{s-1}}^{t_s} v_i^{-p'} \right)^{-\frac{1}{p'}} + 2 \left(\sum_{j=t_{s-1}}^{\infty} w_j^r a_{j,t_{s-1}}^r 2^{rk_{s-2}} \right)^{\frac{1}{r}} \right]^q$$

$$\ll (D^+)^q \left[\sum_{s=3}^{N} \left(\sum_{i=t_{s-1}}^{t_s} f_i \right)^q \left(\sum_{i=t_{s-1}}^{t_s} v_i^{-p'} \right)^{-\frac{q}{p'}} \right.$$

$$\left. + \sum_{s=3}^{N} \left(\sum_{j=t_{s-1}}^{\infty} w_j^r \left(\sum_{i=t_{s-2}}^{t_{s-1}} a_{j,i} f_i \right)^r \right)^{\frac{q}{r}} \right] = (D^+)^q \left(F_{41} + F_{42}\right). \tag{2.18}$$

By Hölder's and Jensen's inequalities we have

$$F_{41} \leq \sum_{s=3}^{N} \left(\sum_{j=t_{s-1}}^{t_s} (v_j f_j)^p \right)^{\frac{q}{p}} \leq \left(\sum_{s=3}^{N} \sum_{j=t_{s-1}}^{t_s} (v_j f_j)^p \right)^{\frac{q}{p}}$$

$$\leq 2^{\frac{q}{p}} \left(\sum_{j=t_2}^{t_N} (v_j f_j)^p \right)^{\frac{q}{p}} \ll \|vf\|_p^q. \tag{2.19}$$

Again by Jensen's inequality we have

$$F_{42} \leq \left(\sum_{s=3}^{N} \sum_{j=t_{s-1}}^{\infty} w_j^r \left(\sum_{i=t_{s-2}}^{t_{s-1}} a_{j,i} f_i \right)^r \right)^{\frac{q}{r}}$$

$$\leq \left(\sum_{s=3}^{N+1}\sum_{k=s}^{N+1}\sum_{j=t_{k-1}}^{t_k-1} w_j^r \left(\sum_{i=t_{s-2}}^{t_{s-1}} a_{j,i}f_i\right)^r\right)^{\frac{q}{r}} = \left(\sum_{k=3}^{N+1}\sum_{j=t_{k-1}}^{t_k-1} w_j^r \sum_{s=3}^{k}\left(\sum_{i=t_{s-2}}^{t_{s-1}} a_{j,i}f_i\right)^r\right)^{\frac{q}{r}}$$

$$\leq \left(\sum_{k=3}^{N+1}\sum_{j=t_{k-1}}^{t_k-1} w_j^r \left(\sum_{s=3}^{k}\sum_{i=t_{s-2}}^{t_{s-1}} a_{j,i}f_i\right)^r\right)^{\frac{q}{r}} \leq 2^q \left(\sum_{k=3}^{N+1}\sum_{j=t_{k-1}}^{t_k-1} w_j^r \left(\sum_{i=t_1}^{t_{k-1}} a_{j,i}f_i\right)^r\right)^{\frac{q}{r}}$$

$$\ll \left(\sum_{k=3}^{N+1}\sum_{j=t_{k-1}}^{t_k-1} w_j^r \left(\sum_{i=1}^{j} a_{j,i}f_i\right)^r\right)^{\frac{q}{r}} \leq \|wA^+f\|_r^q. \tag{2.20}$$

If we combine the estimates for all cases $N = 0$, $N = 1$, $N = 2$ and $N \geq 3$, we get that (1.1) holds with the estimate $C \ll D^+$ for the least constant $C > 0$ in (1.1), which together with (2.4) gives $C \approx D^+$. □

Theorem 2.2. *Let $1 < \max\{p, r\} \leq q < \infty$ and $a_{i,k} \leq a_{j,k}$ for $1 \leq k \leq i \leq j$. Then inequality (1.2) holds if and only if $D^- < \infty$. Moreover, $D^- \approx C$, where $C > 0$ is the least constant in (1.2).*

Proof. Necessity can be proved similarly to one in Theorem 2.1. Therefore, we need to prove only sufficiency. Let $D^- < \infty$. Then for any $k, s : 1 \leq k \leq s$ we have

$$\sum_{j=1}^{k} u_j^q \leq (D^-)^q \left[\left(\sum_{i=k}^{s} v_i^{-p'}\right)^{-\frac{1}{p'}} + \left(\sum_{j=1}^{s} w_j^r a_{s,j}^r\right)^{\frac{1}{r}}\right]^q. \tag{2.21}$$

Let $f = \{f_i\} \geq 0$ be a sequence, for which the right-hand side of (1.2) is finite. Then from the condition that $a_{i,k} \leq a_{j,k}$ for $j \geq i \geq k$ we get

$$\infty > \sum_{j=k}^{\infty} a_{j,k} f_j \geq a_{i,k} \sum_{j=i}^{\infty} f_j.$$

Hence, due to the non-triviality of the matrix $(a_{i,j})$ there exist $a_{i,k} > 0$, hence $\sum\limits_{j=1}^{\infty} f_j < \infty$.

Let $k_1 = \inf\{k \in Z : 2^{-k} \leq \sum\limits_{j=1}^{\infty} f_j\}$ and $t_1 = \max\{i \geq 1 : 2^{-k_1} \leq \sum\limits_{j=i}^{\infty} f_j\}$. Then $t_1 \geq 1$,

$$2^{-k_1} \leq \sum_{j=t_1}^{\infty} f_j \quad \text{and} \quad \sum_{j=t_1+1}^{\infty} f_j < 2^{-k_1}. \tag{2.22}$$

Assume that $t_0 = 1$ and $k_n = k_{n-1} + 1$ for $n \geq 2$. Let $t_n = \max\{i \geq 1 : 2^{-k_n} \leq \sum\limits_{j=i}^{\infty} f_j\}$. Then

$$2^{-k_n} \leq \sum_{j=t_n}^{\infty} f_j \quad \text{and} \quad \sum_{j=t_n+1}^{\infty} f_j < 2^{-k_n}. \tag{2.23}$$

Due to (2.22) inequalities (2.23) are valid for $n \geq 1$.

On the basis of (2.23) for $n \geq 1$ we get

$$2^{-k_{n+1}} = 2^{-k_n} - 2^{-k_{n+1}} \leq \sum_{j=t_n}^{\infty} f_j - \sum_{j=t_{n+1}+1}^{\infty} f_j = \sum_{j=t_n}^{t_{n+1}} f_j. \tag{2.24}$$

From the construction of the points t_n it follows that either $t_n \to \infty$ for $n \to \infty$ or there exists $N \geq 2$ such that $t_{N-1} < t_N$ and $t_N = t_n$ for all $n \geq N+1$. In this case $f_i = 0$ for $i \geq N+1$. Therefore, we assume that $t_{N+1} = \infty$ and $t_n = \infty$ for $n \geq N+1$.

Assuming $N \leq \infty$, we have

$$\|uP^- f\|_q^q = \sum_{j=1}^{t_1} u_j^q (P^- f)_j^q + \sum_{n=1}^{N} \sum_{j=t_n+1}^{t_{n+1}} u_j^q (P^- f)_j^q = F_1 + F_2. \tag{2.25}$$

From the definition of k_1 it follows that $(P^- f)_1 < 2^{-k_1+1}$. Therefore, on the basis of (2.21), (2.22) and (2.24) we have

$$F_1 \leq 2^{2q} 2^{-qk_2} \sum_{j=1}^{t_1} u_j^q \ll (D^-)^q \left[2^{-k_2} \left(\sum_{i=t_1}^{t_2} v_i^{-p'} \right)^{-\frac{1}{p'}} + 2^{-k_2} \left(\sum_{j=1}^{t_1} w_j^r a_{t_1,j}^r \right)^{\frac{1}{r}} \right]^q$$

$$\leq (D^-)^q \left[\left(\sum_{i=t_1}^{t_2} f_i \right) \left(\sum_{i=t_1}^{t_2} v_i^{-p'} \right)^{-\frac{1}{p'}} + \left(\sum_{i=t_2}^{\infty} f_i \right) \left(\sum_{j=1}^{t_1} w_j^r a_{t_1,j}^r \right)^{\frac{1}{r}} \right]^q$$

$$\ll (D^-)^q \left[\left(\sum_{i=t_1}^{t_2} (v_i f_i)^p \right)^{\frac{q}{p}} + \left(\sum_{j=1}^{t_1} w_j^r \left(\sum_{i=j}^{\infty} a_{i,j} f_i \right)^r \right)^{\frac{q}{r}} \right] = (D^-)^q (F_{11} + F_{12}). \tag{2.26}$$

Using (2.21) and (2.23), we get

$$F_2 \leq \sum_{n=1}^{N} 2^{-qk_{n+1}} \sum_{j=t_n+1}^{t_{n+1}} u_j^q$$

$$\leq (D^-)^q \sum_{n=1}^{N} \left[2^{-k_{n+1}} \left(\sum_{i=t_{n+1}}^{t_{n+2}} v_i^{-p'} \right)^{-\frac{1}{p'}} + 2^{-k_{n+1}} \left(\sum_{j=1}^{t_{n+2}} w_j^r a_{t_{n+2},j}^r \right)^{\frac{1}{r}} \right]^q$$

$$\ll (D^-)^q \left[\sum_{n\geq 1} \left(2^{-k_{n+1}} \left(\sum_{i=t_{n+1}}^{t_{n+2}} v_i^{-p'} \right)^{-\frac{1}{p'}} \right)^q + \sum_{n\geq 1} \left(2^{-k_{n+1}} \left(\sum_{j=1}^{t_{n+2}} w_j^r a_{t_{n+2},j}^r \right)^{\frac{1}{r}} \right)^q \right]$$

$$= (D^-)^q (F_{21} + F_{22}). \tag{2.27}$$

By (2.24) we obtain

$$F_{21} \leq 2^q \sum_{n\geq 1} \left(2^{-k_{n+2}} \left(\sum_{i=t_{n+1}}^{t_{n+2}} v_i^{-p'} \right)^{-\frac{1}{p'}} \right)^q$$

$$\ll \sum_{n\geq 1} \left(\left(\sum_{i=t_{n+1}}^{t_{n+2}} f_i \right) \left(\sum_{i=t_{n+1}}^{t_{n+2}} v_i^{-p'} \right)^{-\frac{1}{p'}} \right)^q$$

$$\leq \sum_{n\geq 1} \left(\sum_{i=t_{n+1}}^{t_{n+2}} (v_i f_i)^p \right)^{\frac{q}{p}} \ll \left(\sum_{i=t_2}^{\infty} (v_i f_i)^p \right)^{\frac{q}{p}}.$$

The last estimate, together with (2.26), gives

$$F_{11}+F_{21} \ll \left(\sum_{i=t_1}^{\infty}(v_i f_i)^p\right)^{\frac{q}{p}} \le \|vf\|_p^q. \tag{2.28}$$

Let us estimate F_{22}:

$$\begin{aligned}
F_{22} &\le 2^{2q}\sum_{n\ge1}\left(2^{-k_{n+3}}\left(\sum_{j=1}^{t_{n+2}} w_j^r a_{t_{n+2},j}^r\right)^{\frac{1}{r}}\right)^{q} \ll \sum_{n\ge1}\left(\sum_{j=1}^{t_{n+2}} w_j^r a_{t_{n+2},j}^r\left(\sum_{i=t_{n+2}}^{t_{n+3}} f_i\right)^r\right)^{\frac{q}{r}}\\
&\le \left(\sum_{n\ge1}\sum_{j=1}^{t_{n+2}} w_j^r\left(\sum_{i=t_{n+2}}^{t_{n+3}} a_{i,j} f_i\right)^r\right)^{\frac{q}{r}} = \left(\sum_{n\ge2}\sum_{j=1}^{t_{n+1}} w_j^r\left(\sum_{i=t_{n+1}}^{t_{n+2}} a_{i,j} f_i\right)^r\right)^{\frac{q}{r}}\\
&\ll \left(\sum_{n\ge2}\sum_{s=0}^{n}\sum_{i=t_s}^{t_{s+1}} w_j^r\left(\sum_{i=t_{n+1}}^{t_{n+2}} a_{i,j} f_i\right)^r\right)^{\frac{q}{r}} = \left(\sum_{s\ge0}\sum_{j=t_s}^{t_{s+1}} w_j^r\sum_{n\ge s}\left(\sum_{i=t_{n+1}}^{t_{n+2}} a_{i,j} f_i\right)^r\right)^{\frac{q}{r}}\\
&\le \left(\sum_{s\ge0}\sum_{j=t_s}^{t_{s+1}} w_j^r\left(\sum_{n\ge s}\sum_{i=t_{n+1}}^{t_{n+2}} a_{i,j} f_i\right)^r\right)^{\frac{q}{r}} \ll \left(\sum_{s\ge0}\sum_{j=t_s}^{t_{s+1}} w_j^r\left(\sum_{i=t_{s+1}}^{\infty} a_{i,j} f_i\right)^r\right)^{\frac{q}{r}}\\
&\le \left(\sum_{s\ge0}\sum_{j=t_s}^{t_{s+1}} w_j^r\left(\sum_{i=j}^{\infty} a_{i,j} f_i\right)^r\right)^{\frac{q}{r}} \ll \left(\sum_{j=0}^{\infty} w_j^r\left(\sum_{i=j}^{\infty} a_{i,j} f_i\right)^r\right)^{\frac{q}{r}} = \|wA^- f\|_r^q.
\end{aligned} \tag{2.29}$$

By using the estimate $F_{12} \le \|wA^- f\|_r^q$, from (2.25), (2.26), (2.27), (2.28) and (2.29) we get inequality (1.2) with the estimate $C \ll D^-$ for the least constant C in (1.2). □

3 Application

Let us consider an application of the obtained results.

Let $k \ge j \ge 1$, $n > 1$ and

$$(k-j+1)_{(1)}^{n-1} = (k-j+1)(k-j+2)\dots(k-j+n-1).$$

Let $g = \{g_i\}_{i\in\mathbb{Z}}$ be a sequence such that $g_i = 0$ for $i \le 0$. Then for $k \ge 1$ we have

$$g_k = \frac{1}{(n-1)!}\sum_{j=1}^{k}(k-j+1)_{(1)}^{n-1}\Delta^n g_j,$$

where $\Delta g_i = g_i - g_{i-1}$, $\Delta^n g_i = \Delta(\Delta^{n-1} g_i)$, $\Delta^0 g_i \equiv g_i$ and $n \ge 1$.

According to [1] the sequence $\{g_i\}$ is n-convex for $n \ge 1$ if $\Delta^n g_i \ge 0$ for $i \ge 1$. Thus, 1-convexity and 2-convexity mean the non-decrease and usual convexity of the sequence $\{g_i\}$ for $i \ge 1$, respectively.

Let B_n be a class of all sequences $g = \{g_i\}_{i\in\mathbb{Z}}$ such that $g_i = 0$ for $i \le 0$ and n-convex for $i \ge 1$.

In (1.1) we assume that $f_i = \Delta^n g_i$ and $a_{k,j} = (k-j+1)_{(1)}^{n-1}$. It is obvious that $a_{k,j} \le a_{k,i}$ for $1 \le i \le j \le k$. Then we have

$$(P^+ f)_i = \sum_{i=1}^{j} f_i = \sum_{i=1}^{j}\Delta^n g_i = \Delta^{n-1} g_j, \quad j \ge 1.$$

Therefore, inequality (1.1) can be rewritten in the form:

$$\|u\Delta^{n-1}g\|_q \leq C\left(\|v\Delta^n g\|_p + \|wg\|_r\right), \quad g \in B_n. \tag{3.1}$$

Then on the basis of Theorem 2.1 we have the following statement.

Theorem 3.1. *Let $1 < \max\{r,p\} \leq q < \infty$. Then inequality (3.1) holds if and only if $D_n^+ < \infty$. Moreover, $D_n^+ \approx C$, where $C > 0$ is the least constant in (3.1) and*

$$D_n^+ = \sup_{i\geq 1}\left(\sum_{j=i}^{\infty} u_j^q\right)^{\frac{1}{q}} \varphi_{n,i}^+,$$

$$\varphi_{n,i}^+ = \left[\min_{1\leq k\leq i}\left\{\left(\sum_{j=k}^{i} v_j^{-p'}\right)^{-\frac{1}{p'}} + \left(\sum_{j=k}^{\infty} w_j^r\left((j-k+1)_{(1)}^{n-1}\right)^r\right)^{\frac{1}{r}}\right\}\right]^{-1}.$$

Let us consider sequences $g = \{g_i\}_{i\geq 1}$ such that $\sum\limits_{i=1}^{\infty} g_i < \infty$. Then $g_i \to 0$ as $i \to \infty$. Let $\overline{\Delta} g_i = g_i - g_{i+1}$, $\overline{\Delta}^n g_i = \overline{\Delta}(\overline{\Delta}^{n-1} g_i)$ and $n \geq 1$. Then

$$g_j = \frac{1}{(n-1)!}\sum_{k=j}^{\infty}(k-j+1)_{(1)}^{n-1}\overline{\Delta}^n g_j.$$

Let $\overline{B}_n$ be a class of the sequences $g = \{g_i\}_{i\geq 1}$ such that $\sum\limits_{i=1}^{\infty} g_i < \infty$ and $\overline{\Delta}^n g_i \geq 0$ for $i \geq 1$.

For $\{g_k\} \in \overline{B}_n$ we assume that $f_i = \overline{\Delta}^n g_i$ and $a_{k,j} = (k-j+1)_{(1)}^{n-1}$. Then inequality (1.2) has the form (3.1) for $g \in \overline{B}_n$.

Hence, on the basis of Theorem 2.2 we have the following statement.

Theorem 3.2. *Let $1 < \max\{r,p\} \leq q < \infty$. Then inequality (3.1) holds for $g \in \overline{B}_n$ if and only if $D_n^- < \infty$. Moreover, $D_n^- \approx C$, where $C > 0$ is the least constant in (3.1) and*

$$D_n^- = \sup_{i\geq 1}\left(\sum_{j=1}^{i} u_j^q\right)^{\frac{1}{q}} \varphi_{n,i}^-,$$

$$\varphi_{n,i}^- = \left[\inf_{i\leq k}\left\{\left(\sum_{j=i}^{k} v_j^{-p'}\right)^{-\frac{1}{p'}} + \left(\sum_{j=1}^{k} w_j^r\left((k-j+1)_{(1)}^{n-1}\right)^r\right)^{\frac{1}{r}}\right\}\right]^{-1}.$$

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