ISSN 2077–9879

Eurasian Mathematical Journal

2018, Volume 9, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http : //publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 515
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

KUSSAINOVA LEILI KABIDENOVNA

(to the 70th birthday)

On May 3, 2018 was the 70th birthday of Leili Kabidenovna Kussainova, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department of Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), Professor (2006), Honorary worker of Education of the Republic of Kazakhstan (2005).

L.K. Kussainova was born in the city of Karaganda. In 1972 she graduated from the Novosibirsk State University (Russian Federation) and then completed her postgraduate studies at the Institute of Mathematics (Almaty). L.K. Kussainova's scientific supervisors were distinguished Kazakh mathematicians T.I. Amanov and M. Otelbayev.

Scientific works of L.K. Kussainova are devoted to investigation of the widths of embeddings of the weighted Sobolev spaces, to embeddings and interpolations of weighted Sobolev spaces with weights of general type.

She has solved the problem of three-weighted embedding of isotropic and anisotropic Sobolev spaces in Lebesgue spaces, the problem of exact description of the Lions-Petre interpolation spaces for a pair of weighted Sobolev spaces.

To solve these problems L.K. Kussainova obtained nontrivial modifications of theorems on Besicovitch-Guzman covers. The first relates to covers by multidimensional parallelepipeds, whereas the second relates to double covers by cubes. These modifications have allowed to obtain the description of the interpolation spaces in the weighted case. Furthermore, by using the double covering theorem the exact descriptions of the multipliers were obtained for a pair of Sobolev spaces of general type.

The maximal operators on a basis of cubes with adjustable side length, which were introduced by L.K. Kussainova, have allowed her to solve the problem of two-sided distribution estimate of widths of the embedding of two-weighted Sobolev spaces with weights of general type in weighted Lebesgue spaces.

Under her supervision 6 theses have been defended: 4 candidates of sciences theses and 2 PhD theses.

The Editorial Board of the Eurasian Mathematical Journal congratulates Leili Kabidenovna Kussainova on the occasion of her 70th birthday and wishes her good health and new achievements in mathematics and mathematical education.

The awarding ceremony
of the Certificate of the Emerging Sources Citation of Index database

In 2016 the Eurasian Mathematical Journal has been included in the Emerging Sources Citation of Index (ESCI) of the "Clarivate Analytics" (formerly "Thomson Reuters") Web of Science. In 2018 the second journal of the L.N. Gumilyov Eurasian National University, namely the Eurasian Journal of Mathematical and Computer Applications was also included in ESCI.

The ESCI was launched in late 2015 as a new database within "Clarivate Analytics". Around 3,000 journals were selected for coverage at launch, spanning the full range of subject areas.

The selection process for ESCI is the first step in applying to the Science Citation Index. All journals submitted for evaluation to the core Web of Science databases will now initially be evaluated for the ESCI, and if successful, indexed in the ESCI while undergoing the more in-depth editorial review. Timing for ESCI evaluation will follow "Clarivate Analytics" priorities for expanding database coverage, rather than the date that journals were submitted for evaluation.

Journals indexed in the ESCI will not receive Impact Factors; however, the citations from the ESCI will now be included in the citation counts for the Journal Citation Reports, therefore contributing to the Impact Factors of other journals. If a journal is indexed in the ESCI it will be discoverable via the Web of Science with an identical indexing process to any other indexed journal, with full citation counts, author information and other enrichment. Articles in ESCI indexed journals will be included in an author's H-Index calculation, and also any analysis conducted on Web of Science data or related products such as InCites. Indexing in the ESCI will improve the visibility of a journal, provides a mark of quality and is good for authors.

To commemorate this important achievement of mathematicians of the L.N. Gumilyov Eurasian National University on June 14, 2018, by the initiative of the "Clarivate Analytics", the awarding ceremony of the Certificate of Emerging Sources Citation Index database of "Clarivate Analytics" to the editorial boards of the Eurasian Mathematical Journal and the Eurasian Journal of Mathematical and Computer Applications was held at the L.N. Gumilyov Eurasian National University. The programme of this ceremony is attached.

Astana June 14, 2018

Venue: L.N. Gumilyov Eurasian National University
Astana, Satpayev street 2, Room 259

14:30- 15:00	Visit to the Museum of the history of Education, Museum of L.N. Gumilyov, Museum of writing
15:00-15:10	*Opening speech of moderator*
	A. Moldazhanova – the First Vice-Rector, Vice-Rector for Academic Works of L.N. Gumilyov Eurasian National University
15:10-15:20	**Oleg Utkin** - Managing Director of Clarivate Analytics in Russia and the CIS
15:20-15:30	*Certification award ceremony of the Eurasian Mathematical Journal, the Eurasian Journal of Mathematical and Computer Applications in international database*
15:30-15:45	**Kordan Ospanov** – Deputy Editor-in-Chief of the Eurasian Mathematical Journal. *History and perspectives of development of the scientific journal Eurasian Mathematical Journal*
15:45-16:00	**Kazizat Iskakov** – Deputy Editor-in-Chief of the Eurasian Journal of Mathematical and Computer Applications. *History and perspectives of development of the scientific journal Eurasian Journal of Mathematical and Computer Applications.*
16:00-16:10	*Closing Ceremony*
	Memory photo
16:10-16:30	*Coffee break for visitors*
16:40-17:20	**Lyaziza Mukasheva** - Official representative of Clarivate Analytics in the Central Asian region *Seminar for editors of scientific journals Scientific library of L.N. Gumilyov Eurasian National University room 104*

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 9, Number 2 (2018), 34 – 43

ON THE SYMBOL OF NONLOCAL OPERATORS ASSOCIATED WITH A PARABOLIC DIFFEOMORPHISM

N.R. Izvarina

Communicated by V.I. Burenkov

Key words: Sobolev spaces, pseudodifferential operators, symbol, ellipticity, parabolic diffeomorphism.

AMS Mathematics Subject Classification: 35S05, 47G30, 46E39.

Abstract. We study the ellipticity of the symbols of operators associated with parabolic diffeomorphisms of spheres and we show, that if for some smoothness exponent of the Sobolev space the symbol of an operator is invertible, then the symbol is invertible for all exponents of Sobolev spaces.

DOI: https://doi.org/10.32523/2077-9879-2018-9-2-34-43

1 Introduction

The object of this study is the symbol of elliptic operators associated with actions of a discrete group G on a smooth manifold. Such operators are referred to as G-operators below and presented as linear combinations of compositions of pseudodifferential operators (PDO) and shift operators. Previously they were studied, for example, by A.B. Antonevich [1]. The essential difference between the theory of elliptic G-operators and a similar theory of PDO is that the ellipticity of such operators and their Fredholm property depend on the smoothness exponent s of the Sobolev spaces H^s. Thus, it is natural to describe the values of exponents s, for which the symbol of such operators is invertible in the relevant Sobolev spaces and, as a corollary, the G-operator is elliptic. It is known, that, in the case of isometric diffeomorphisms, the Fredholm property of operators does not depend on s, so it is interesting to consider problems for diffeomorphisms, which do not preserve a Riemannian metric. In [2, 3] the first steps in the study of nonisometric actions were done. In particular, it was shown for dilations of spheres that the set of s, for which a G-operator is elliptic, is a finite or (semi)infinite interval. In the present paper we consider shifts along the trajectories of a nonisometric parabolic diffeomorphism of an $m-$dimensional sphere.

The paper consists of several parts. Statement of the problem and expressions for the symbol are given in Section 2, Section 3 contains the main theorem and its proof. An example in Section 4 illustrates this theorem. In this example we consider an operator, associated with a diffeomorphism of the circle $\mathbb{S}^1$.

2 Statement of the problem

On an m-dimensional sphere $\mathbb{S}^m$, let us consider a parabolic diffeomorphism

$$g : \mathbb{S}^m \to \mathbb{S}^m, \quad x \mapsto x + e, \tag{2.1}$$

which has one fixed point $x_0 = \infty$. Here $\mathbb{S}^m \setminus x_0$ is identified with the space $\mathbb{R}^m$ with the coordinates x; $e \in \mathbb{R}^m$ is a given nonzero vector. Diffeomorphism (2.1) is a shift along the vector e.

Consider operators of the form

$$D = \sum_k D_k T^k : H^s(\mathbb{S}^m) \to H^{s-d}(\mathbb{S}^m), \tag{2.2}$$

where $\sum_k$ is a finite sum, T is the shift operator corresponding to diffeomorphism (2.1), $Tu(x) = u(x+e)$, $s \in \mathbb{R}$, D_k is a PDO of order d on $\mathbb{S}^m$, $k \in \mathbb{Z}$.

To define the symbol of operator D, we need to compute a special density associated with the diffeomorphism g (see [3]). To calculate this density we can the following formula:

$$\mu_{x,\xi,s}(n) = \left|\det \frac{\partial g^n}{\partial x}\right| \left| \left(\left(\frac{\partial g^n}{\partial x} \right)^T \right)^{-1} \xi \right|^{2s}, \qquad (x,\xi) \in T_0^* \mathbb{S}^m. \tag{2.3}$$

Density (2.3) has different expressions depending on whether $(x = \infty)$ or not. Let us consider both cases.

1. Let us compute the density at $x_0 = \infty$. The diffeomorphism g^n in the pair of coordinate charts x and x', where $x' = \frac{x}{|x|^2}$ are the coordinates in a neighbourhood of $x_0 = \infty$, will have the form

$$g^n(x') = \frac{\frac{x'}{|x'|^2} + ne}{\left|\frac{x'}{|x'|^2} + ne\right|^2} = x' + O(|x'|^2). \tag{2.4}$$

Hence

$$\left.\frac{\partial g^n}{\partial x'}\right|_{x'=0} = \mathrm{Id} \ \text{ is the identity matrix.}$$

As $\mathrm{Id} = \mathrm{Id}^T = \mathrm{Id}^{-1}$, and $\det(\mathrm{Id}) = 1$, so density (2.3) at $x = \infty$ is equal to

$$\mu_{x,\xi,s}(n) = 1. \tag{2.5}$$

2. If $x \neq \infty$, then g^n in the pair of coordinate charts x and x', where $x' = \frac{x}{|x|^2}$ are the coordinates in a neighbourhood of the point $x_0 = \infty$, is presented as

$$x \mapsto x' = g'(x) = \frac{x+e}{|x+e|^2}. \tag{2.6}$$

We set $e = (1, 0, ..., 0)^T$ for simplicity. Then the differential of g^n is equal to

$$\frac{\partial g^n}{\partial x} = \begin{pmatrix} \frac{\partial g_1^n}{\partial x_1} & \frac{\partial g_1^n}{\partial x_2} & \cdots & \frac{\partial g_1^n}{\partial x_m} \\ \frac{\partial g_2^n}{\partial x_1} & \frac{\partial g_2^n}{\partial x_2} & \cdots & \frac{\partial g_2^n}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m^n}{\partial x_1} & \frac{\partial g_m^n}{\partial x_2} & \cdots & \frac{\partial g_m^n}{\partial x_m} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{|x+ne|^2} - \frac{2(x_1+n)^2}{|x+ne|^4} & \frac{-2(x_1+n)x_2}{|x+ne|^4} & \dots & \frac{-2(x_1+n)x_m}{|x+ne|^4} \\ \frac{-2x_2(x_1+n)}{|x+ne|^4} & \frac{1}{|x+ne|^2} - \frac{2x_2^2}{|x+ne|^4} & \dots & \frac{-2x_2x_m}{|x+ne|^4} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-2x_m(x_1+n)}{|x+ne|^4} & \frac{-2x_mx_2}{|x+ne|^4} & \dots & \frac{1}{|x+ne|^2} - \frac{2x_m^2}{|x+ne|^4} \end{pmatrix}.$$

Hence we express the Jacobian as

$$\det\left(\frac{\partial g^n}{\partial x}\right) = -|x+ne|^{-2m},$$

$$\left(\left(\frac{\partial g^n}{\partial x}\right)^T\right)^{-1} = \left(\frac{\partial g^n}{\partial x}\right)^{-1} =$$

$$= \begin{pmatrix} -(x_1+n)^2 + x_2^2 ... + x_m^2 & -2(x_1+n)x_2 & \dots & -2(x_1+n)x_m \\ -2x_2(x_1+n) & (x_1+n)^2 - x_2^2 + ... + x_m^2 & \dots & -2(x_2+n)x_m \\ \vdots & \vdots & \ddots & \vdots \\ -2x_m(x_1+n) & -2x_mx_2 & \dots & (x_1+n)^2 + ... - x_m^2 \end{pmatrix}. \quad (2.7)$$

So, up to equivalence as $n \to \infty$, density (2.3) is equal to

$$\mu_{x,s}(n) =$$

$$= |x+ne|^{-2m} \left| \begin{pmatrix} -(x_1+n)^2 + x_2^2 ... + x_m^2 & -2(x_1+n)x_2 & \dots & -2(x_1+n)x_m \\ -2x_2(x_1+n) & (x_1+n)^2 - x_2^2 + ... + x_m^2 & \dots & -2(x_2+n)x_m \\ \vdots & \vdots & \ddots & \vdots \\ -2x_m(x_1+n) & -2x_mx_2 & \dots & (x_1+n)^2 + ... - x_m^2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{pmatrix} \right|^{2s} \sim$$

$$\sim |n|^{4s-2m} \left| \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{pmatrix} \right|^{2s} \sim |n|^{2(2s-m)}. \quad (2.8)$$

Now, using the definition of the symbol (e.g., see [3]), we obtain the symbol of operator (2.2). It has different expressions depending on whether $x = \infty$ or not. So, at a point $(x, \xi) \in T_0^*\mathbb{S}^m$, $x \neq \infty$ we have the symbol

$$\sigma(D)(x,\xi) = \sum_k \sigma(D_k)(x+ne,\xi)\mathcal{T}^k : l^2(\mathbb{Z}, \mu_{x,\xi,s}) \to l^2(\mathbb{Z}, \mu_{x,\xi,s-d}), \quad (2.9)$$

where $\mathcal{T}u(n) = u(n+1)$ is the shift operator of sequences, and the space $l^2(\mathbb{Z}, \mu_{x,\xi,s})$ consists of sequences $\{u(n)\}, n \in \mathbb{Z}$, square summable with respect to the density $\mu_{x,\xi,s}$, which we previously obtained in (2.8).

At $x = \infty$, we define the symbol as

$$\sigma(D)(\infty,\xi') = \sum_k \sigma(D_k)(\infty,\xi')\mathcal{T}^k : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad (2.10)$$

where $\xi' = (-\xi_1, \xi_2 ..., \xi_m)$. Symbol (2.10) does not depend on the smoothness exponent s of Sobolev space.

Lemma 2.1. *There exists the limit*

$$\lim_{n\to\infty} |n|^{-2d}\sigma(D_k)(x+ne,\xi),$$

which we denote by $\sigma(D_k)(\infty,\xi)$.

Proof. We change the coordinates: $x' = \dfrac{x}{|x|^2}$ and set $x'_n = \dfrac{x+ne}{|x+ne|^2}$. Then using (2.7), we get

$$\xi'_n = \left(\left(\frac{\partial g^n}{\partial x}\right)^T\right)^{-1} \xi \sim |n|^2 \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \xi = |n|^2 \xi'. \tag{2.11}$$

Hence,

$$\begin{aligned} |n|^{-2d}\sigma(D_k)(x+n,\xi) = |n|^{-2d}\sigma'(D_k)(x'_n,\xi'_n) \sim \\ \sim |n|^{-2d}\sigma'(D_k)\left(\frac{x+ne}{|x+ne|^2}, |n|^2\xi'\right) = |n|^{-2d}|n|^{2d}\sigma'(D_k)\left(\frac{x+ne}{|x+ne|^2},\xi'\right), \end{aligned} \tag{2.12}$$

where $\sigma'(D_k)$ is the symbol of operator D_k defined for points with coordinates x' and ξ' in a neighbourhood of infinity. As $n \to \infty$, the last expression has a limit equal to $\sigma'(D_k)(0,\xi')$. $\square$

Note that (2.10) is a difference operator with constant coefficients.

3 Main result

The following theorem is the main result of the present paper.

Theorem 3.1. *Let operator (2.2) satisfy ellipticity conditions for some* $s = s_0$*, that is symbols (2.9) and (2.10) are invertible for* $s = s_0$ *and all* $(x,\xi) \in T_0^*\mathbb{S}^m$*. Then symbols (2.9) and (2.10) are invertible for all* $s \in \mathbb{R}$.

Proof. This statement is obvious for symbol (2.10), as this symbol does not depend on s. The invertibility of symbol (2.9) is equivalent to the following three conditions: symbol (2.9) is Fredholm; the kernel of symbol (2.9) is trivial; the index of symbol (2.9) is equal to zero.

1. First, we prove the Fredholm property of symbol (2.9) for all s. For this purpose we replace operator (2.9) with an isomorphic one. Consider the commutative diagram

$$\begin{array}{ccc} l^2(\mathbb{Z},\mu_{x,s}) & \xrightarrow{\sum_k \sigma(D_k)(x+ne,\xi)\mathcal{T}^k} & l^2(\mathbb{Z},\mu_{x,s-d}) \\ {\scriptstyle\sqrt{\mu_{x,s}}}\big\downarrow & & \big\downarrow{\scriptstyle\sqrt{\mu_{x,s-d}}} \\ l^2(\mathbb{Z}) & \xrightarrow{\sqrt{\mu_{x,s-d}}\sum_k \sigma(D_k)(x+ne,\xi)\mathcal{T}^k \frac{1}{\sqrt{\mu_{x,s}}}} & l^2(\mathbb{Z}). \end{array}$$

Then operator (2.9) is isomorphic to the operator

$$\begin{aligned} \sigma'(D)(x,\xi) = (1+|n|)^{2(s-d)-m}\left(\sum_k \sigma(D_k)(x+ne,\xi)\mathcal{T}^k\right)(1+|n|)^{m-2s} = \\ = (1+|n|)^{-2d}\sum_k \left(\frac{1+|n|}{1+|n+k|}\right)^{2s-m} \sigma(D_k)(x+ne,\xi)\mathcal{T}^k : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}). \end{aligned} \tag{3.1}$$

It is obvious that the Fredholm property of operator (2.10) does not depend on the smoothness exponent s. Let us show that the Fredholm property of operator (2.9) does not depend on s too. To this end, it suffices to show that operators (3.1) and (2.10) differ by a compact operator. Indeed, let us consider the difference of these operators

$$(1+|n|)^{-2d}\sum_k\left(\frac{1+|n|}{1+|n+k|}\right)^{2s-m}\sigma(D_k)(x+ne,\xi)\mathcal{T}^k-\sum_k\sigma(D_k)(\infty,\xi')\mathcal{T}^k=$$
$$=\sum_k\left((1+|n|)^{-2d}\left(\frac{1+|n|}{1+|n+k|}\right)^{2s-m}\sigma(D_k)(x+ne,\xi)-\sigma(D_k)(\infty,\xi')\right)\mathcal{T}^k. \tag{3.2}$$

As coefficients of operator (3.2) have zero limits (by Lemma 2.1)

$$|n|^{-2d}\left(\frac{1+|n|}{1+|n+k|}\right)^{2s-m}\sigma(D_k)(x+ne,\xi')-\sigma(D_k)(\infty,\xi)\to 0 \quad \text{as} \quad n\to\infty,$$

the difference of operators (3.1) and (2.10) is a compact operator. Hence, the Fredholm property of operator (2.9) does not depend on s.

2. Second, let us show that the kernel of operator (2.9) is independent of s. Consider solutions of the following equation

$$\left(\sum_k\sigma(D_k)(x+ne,\xi)\mathcal{T}^k\right)u(n)=0, \quad \{u(n)\}\in l^2(\mathbb{Z},\mu_{x,s}). \tag{3.3}$$

The following theorem due to Poincaré enables us to describe the behavior of solutions of equation (3.3) at infinity.

Theorem 3.2 (Poincaré)**.** *Given a linear homogeneous difference equation*

$$u(n+k)+a_{k-1}(n)u(n+k-1)+a_{k-2}(n)u(n+k-2)+...+a_0(n)u(n)=0 \tag{3.4}$$

such that

1) there exist finite limits of its coefficients

$$\lim_{n\to+\infty}a_i(n)=a_i, i=0,...,k-1;$$

2) the roots of the characteristic equation

$$\lambda^k+a_{k-1}\lambda^{k-1}+a_{k-2}\lambda^{k-2}+...+a_1\lambda+a_0=0 \tag{3.5}$$

satisfy condition $|\lambda_i|\neq|\lambda_j|$, *for* $i\neq j$,

then for any nontrivial solution $u(n)$ *of (3.4) there exists the limit*

$$\lim_{n\to+\infty}\frac{u(n+1)}{u(n)}=\lambda_j \quad \textit{for some} \quad 1\le j\le k. \tag{3.6}$$

Let us apply the Poincaré theorem to our equation (3.3).

Now we obtain invertibility condition for symbol (2.9). We use the Fourier transform for operator (2.10) and get the function

$$\sum_k\sigma(D_k)(\infty,\xi')e^{i\varphi k}, \quad 0<\varphi\le 2\pi. \tag{3.7}$$

We can also write the characteristic polynomial for a finite-difference equation (2.10)

$$\sigma''(D)(x,\xi') = \sum_k \sigma(D_k)(\infty,\xi')\lambda^k \tag{3.8}$$

and consider the following characteristic equation

$$\sum_k \sigma(D_k)(\infty,\xi')\lambda^k = 0. \tag{3.9}$$

From the invertibility of operator (2.10) it follows that the roots λ_j of equation (3.9) satisfy $\lambda_j \neq e^{i\varphi}$, $\varphi \in [0,2\pi]$. Hence, we get

$$|\lambda_j| \neq 1 \quad \text{for all} \quad j.$$

Let us assume that the trajectory symbol at the fixed point satisfies the conditions of the Poincaré theorem (e.g, see [4]). In particular, all the roots of the characteristic polynomial (3.9) have different absolute values.

The characteristic equation is given by (3.9).

Let $u(n)$ be a nonzero solution of equation (3.3). Then by the Poincaré theorem, we get

$$\frac{u(n+1)}{u(n)} = \lambda_j + \varepsilon_j(n), \qquad \text{where} \quad \varepsilon_j(n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.10}$$

Following [4] we write equalities for all $n = 0, 1, \ldots s-1$

$$\frac{u(1)}{u(0)} = \lambda_j + \varepsilon_j(0),$$

$$\frac{u(2)}{u(1)} = \lambda_j + \varepsilon_j(1),$$

$$\cdots\cdots$$

$$\frac{u(s)}{u(s-1)} = \lambda_j + \varepsilon_j(n-1).$$

Next we multiply them and get

$$u(n) = u(0)\prod_{l=0}^{n-1}[\lambda_j + \varepsilon_j(l)].$$

We replace the product

$$\prod_{l=0}^{n-1}[\lambda_j + \varepsilon_j(l)]$$

by the product of equal binomials $\lambda_j + \eta_j(l)$. We have

$$\lambda_j + \eta_j(n) = \sqrt[n]{\prod_{l=0}^{n-1}[\lambda_j + \varepsilon_j(l)]} = \lambda_j \sqrt[n]{\prod_{l=0}^{n-1}\left[1 + \frac{\varepsilon_j(l)}{\lambda_j}\right]},$$

and (see [4])

$$\lim_{n\to\infty} \lambda_j \sqrt[n]{\prod_{l=0}^{n-1}\left[1 + \frac{\varepsilon_j(l)}{\lambda_j}\right]} = 1.$$

Hence, $\eta_j(l)$ is the mean value of all $\varepsilon_j(l)$, $l = 0, ...n-1$, that is $\eta_j(n) \to 0$ as $n \to \infty$.

Now we denote $u(0)$ by C and get an expression for solutions of equation (3.3) in the following form (see [4])

$$u(n) = C[\lambda_j + \eta_j(n)]^n, \tag{3.11}$$

where λ_j is a root of characteristic equation (3.9).

Lemma 3.1. *If $u \in \ker(\sigma(D))$ belongs to $l^2(\mathbb{Z}, \mu_{s_0})$, then $u \in l^2(\mathbb{Z}, \mu_s)$ for all s.*

Proof. Given $u \in \ker\sigma(D)$, $u(n)$ is presented as in (3.11). Hence $u \in l^2(\mathbb{Z}, \mu_s)$ whenever

$$\sum_j \left| C[\lambda_j + \eta_j(n)]^n \right|^2 |n|^{2(2s-m)} < \infty. \tag{3.12}$$

It remains to show that the convergence of (3.12) does not depend on s. By the d'Alembert's ratio test we have

$$\lim_{n\to\infty} \frac{\left| C[\lambda_j + \eta_j(n+1)]^{n+1} \right|^2 |n+1|^{2(2s-m)}}{\left| C[\lambda_j + \eta_j(n)]^n \right|^2 |n|^{2(2s-m)}} = |\lambda_j|^2. \tag{3.13}$$

Series (3.12) converges if $|\lambda_j| < 1$ and diverges if $|\lambda_j| > 1$. Thus, the convergence of this series does not depend on the smoothness exponent s. This implies the statement of Lemma 2.1. □

3. Finally, we check that

$$\mathrm{ind}\sigma(D)(x, \xi) = \mathrm{ind}\sigma'(D)(x, \xi) = 0. \tag{3.14}$$

By the index formula for operators with stabilizing coefficients, we get

$$\begin{aligned} \mathrm{ind}\sigma'(D)(x, \xi) &= w\Big(\sigma'(D)(+\infty, \xi)\Big) - w\Big(\sigma'(D)(-\infty, \xi)\Big) = \\ &= w\left(\sum_k \sigma(D_k)(+\infty, \xi) e^{i\varphi k}\right) - w\left(\sum_k \sigma(D_k)(-\infty, \xi) e^{i\varphi k}\right), \end{aligned} \tag{3.15}$$

where w is the winding number of a function on $\mathbb{S}^1$. As $\sigma(D_k)(+\infty, \xi) = \sigma(D_k)(-\infty, \xi)$ by Lemma 2.1 so the winding numbers in (3.15) are equal, and we get $\mathrm{ind}\sigma'(D)(x, \xi) = 0$.

The proof of Theorem 3.1 is now complete. □

4 Example

Here we study the Fredholm property of the operator

$$D = 1 + a(x)T : \ H^s(\mathbb{S}^1) \to H^s(\mathbb{S}^1), \tag{4.1}$$

where $Tu(x) = u(x+1)$ is a shift operator, $a(x) \in \mathbb{C}^\infty(\mathbb{S}^1)$. Let us demonstrate independently that Theorem 2.1 holds for this operator.

The symbol of operator (4.1) is presented as

$$\sigma(D)(x) = 1 + a(x+n)\mathcal{T} : \quad l^2(\mathbb{Z}, \mu_s) \to l^2(\mathbb{Z}, \mu_s), \tag{4.2}$$

where $\mathcal{T}u(n) = u(n+1)$ and the density $\mu_{s,n} = |n|^{2(2s-1)}$.

1. First, we study the Fredholm property of symbol (4.2). To this end, we consider the commutative diagram

$$\begin{CD} l^2(\mathbb{Z},\mu_{x,s}) @>{\sum_k \sigma(D_k)(x+ne,\xi)\mathcal{T}^k}>> l^2(\mathbb{Z},\mu_{x,s-d}) \\ @V{\sqrt{\mu_{x,s}}}VV @VV{\sqrt{\mu_{x,s-d}}}V \\ l^2(\mathbb{Z}) @>{\sqrt{\mu_{x,s-d}}\sum_k \sigma(D_k)(x+ne,\xi)\mathcal{T}^k \frac{1}{\sqrt{\mu_{x,s}}}}>> l^2(\mathbb{Z}). \end{CD}$$

and replace operator (4.2) with an isomorphic one

$$\sigma'(D)(x) = (1+|n|)^{2s-1}(1+a(x+n)\mathcal{T})(1+|n|)^{1-2s} = 1+\left(\frac{n}{n-1}\right)^{2s-1} a(x+n)\mathcal{T}.$$

As $n\to\infty$ we obtain the symbol

$$\sigma''(D)(x) = 1+a(\infty)e^{i\varphi}, 0\le\varphi<2\pi, \tag{4.3}$$

which is invertible for $|a(\infty)|\ne 1$. This condition obviously does not depend on s, moreover, it must be noted that operator (4.3) is the symbol of operator (4.1) at $x_0=\infty$. Operators (4.2) and (4.3) differ by a compact operator, thus, this proves that the Fredholm property of operator (4.2) also does not depend on s.

2. Second, we study the kernel of operator (4.2). To this end, we consider a nontrivial solution of the following equation $\sigma(D)(x)u(n)=0$.

Operator (4.2) acts on sequences $\{u(n)\}$ as

$$\Big(1+a(x+n)\mathcal{T}\Big)u(n) = u(n)+a(x+n)u(n-1).$$

Let us find solutions of the equation

$$u(n)+a(x+n)u(n+1)=0. \tag{4.4}$$

We rewrite the equation as

$$u(n)+a(+\infty)u(n+1)+\Big(a(x+n)-a(+\infty)\Big)u(n+1)=0.$$

We make a change $x'=\frac{1}{x+n}$, when x is fixed and $n\to+\infty$ and obtain

$$a(x')-a(0)=a(0)+a'(0)x'+O(x'^2)-a(0)=\frac{C}{n}+O\left(\frac{1}{n^2}\right).$$

Here and below C is a constant. We obtain the following equation

$$u(n)+\left(a(+\infty)+\frac{C}{n}+O\left(\frac{1}{n^2}\right)\right)u(n-1)=0.$$

Its solution is of the form

$$u(n)=\prod_{j=1}^{n}\left(-a(+\infty)-\frac{C}{j}-O\left(\frac{1}{j^2}\right)\right)=(a(-\infty))^n\prod_{j=1}^{n}\left(1+\frac{\frac{C}{a(+\infty)}}{j}+O\left(\frac{1}{j^2}\right)\right).$$

Hence,

$$\ln|u(n)| = n\ln|a(+\infty)|+\sum_{j=1}^{n}\ln\left(1+\frac{\frac{C}{a(+\infty)}}{j}+O\left(\frac{1}{j^2}\right)\right),$$

and as $n \to \infty$ this expression is equivalent to

$$\ln|u(n)| \sim n\ln|a(+\infty)| + \sum_{j=1}^{n} \frac{|\frac{C}{a(+\infty)}|}{j} \sim n\ln|a(+\infty)| + \left|\frac{C}{a(+\infty)}\right| \int_1^n \frac{dx}{x} \sim$$

$$\sim n\ln|a(+\infty)| + \left|\frac{C}{a(+\infty)}\right| \ln n.$$

Thus,

$$|u(n)| \sim e^{C - n\ln|a(+\infty)| + \left|\frac{C}{a(+\infty)}\right|} = C|a(+\infty)|^n n^\alpha, \tag{4.5}$$

where $\alpha \in \mathbb{R}$.

Let $u \in \ker\sigma(\mathrm{D})$. Then for the sequences $\{u(n)\}$ we obtain asymptotic behavior (4.5). Hence, $u \in l^2(\mathbb{Z}, \mu_s)$ only when

$$\sum_n C|a(\infty)|^{2n}|n|^{2(\alpha+2s-1)} < \infty. \tag{4.6}$$

We see that the convergence of series (4.6) does not depend on s as $|a(\infty)| \neq 1$.

3. At the last step we investigate the index of operator (4.2). Firstly, we check the equality

$$\mathrm{ind}\sigma'(D)(x) = 0.$$

According to the formula for the index of operators with stabilizing coefficients we get

$$\mathrm{ind}\sigma'(D)(x) = w\Big(\sigma'(D)(+\infty)\Big) - w\Big(\sigma'(D)(-\infty)\Big) = w\Big(1 + a(+\infty)e^{i\varphi}\Big) - w\Big(1 + a(-\infty)e^{i\varphi}\Big).$$

As $a(+\infty) = a(-\infty)$ by the conditions of our problem, so

$$w\Big(\sigma'(D)(+\infty)\Big) = w\Big(\sigma'(D)(-\infty)\Big),$$

and $\mathrm{ind}\sigma'(D)(x) = 0$.

As

$$\mathrm{ind}\sigma'(D)(x) = \dim(\ker\sigma'(D)(x)) - \dim(\mathrm{coker}\sigma'(D)(x)),$$

so $\dim(\mathrm{coker}\sigma'(D)(x)) = 0$, so $\dim(\mathrm{coker}\sigma(D)(x)) = 0$.

Thus, we see that Theorem 2.1 holds for this example and the invertibility of symbol (4.2) does not depend on s.

Acknowledgments

Great gratitude is expressed to the scientific advisor B.Yu. Sternin for his attention paid to this work, as well as for valuable guidance and comments throughout the work process.

The present result was supported by RFBR grant 16-01-00373 (topic no.022004-2-693).

References

[1] A.B. Antonevich, *Strongly nonlocal boundary value problems for elliptic equations.* Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 1, 3-24.

[2] A.Yu. Savin, B.Yu. Sternin, *Nonlocal elliptic operators for the group of dilations.* Sb. Math. 202 (2011), no. 10, 1505–1536.

[3] A.Yu. Savin, *On the symbol of nonlocal operators in Sobolev spaces.* Differential Equations 47 (2011), no. 6, 897–900.

[4] A.O. Gel'fond, *Calculus of finite differences.* Hindustan Publ. Corp., 1971

Natalia Romanovna Izvarina
S.M. Nikol'skii Mathematical Institute
RUDN University
6 Miklukho-Maklay St
117198, Moscow, Russian Federation
E-mail: izvarina-n@mail.ru

Received: 12.02.2017