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KUSSAINOVA LEILI KABIDENOVNA

(to the 70th birthday)

On May 3, 2018 was the 70th birthday of Leili Kabidenovna Kussainova, member of the Editorial Board of the Eurasian Mathematical Journal, professor of the Department of Fundamental Mathematics of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), Professor (2006), Honorary worker of Education of the Republic of Kazakhstan (2005).

L.K. Kussainova was born in the city of Karaganda. In 1972 she graduated from the Novosibirsk State University (Russian Federation) and then completed her postgraduate studies at the Institute of Mathematics (Almaty). L.K. Kussainova's scientific supervisors were distinguished Kazakh mathematicians T.I. Amanov and M. Otelbayev.

Scientific works of L.K. Kussainova are devoted to investigation of the widths of embeddings of the weighted Sobolev spaces, to embeddings and interpolations of weighted Sobolev spaces with weights of general type.

She has solved the problem of three-weighted embedding of isotropic and anisotropic Sobolev spaces in Lebesgue spaces, the problem of exact description of the Lions-Petre interpolation spaces for a pair of weighted Sobolev spaces.

To solve these problems L.K. Kussainova obtained nontrivial modifications of theorems on Besicovitch-Guzman covers. The first relates to covers by multidimensional parallelepipeds, whereas the second relates to double covers by cubes. These modifications have allowed to obtain the description of the interpolation spaces in the weighted case. Furthermore, by using the double covering theorem the exact descriptions of the multipliers were obtained for a pair of Sobolev spaces of general type.

The maximal operators on a basis of cubes with adjustable side length, which were introduced by L.K. Kussainova, have allowed her to solve the problem of two-sided distribution estimate of widths of the embedding of two-weighted Sobolev spaces with weights of general type in weighted Lebesgue spaces.

Under her supervision 6 theses have been defended: 4 candidates of sciences theses and 2 PhD theses.

The Editorial Board of the Eurasian Mathematical Journal congratulates Leili Kabidenovna Kussainova on the occasion of her 70th birthday and wishes her good health and new achievements in mathematics and mathematical education.

The awarding ceremony
of the Certificate of the Emerging Sources Citation of Index database

In 2016 the Eurasian Mathematical Journal has been included in the Emerging Sources Citation of Index (ESCI) of the "Clarivate Analytics" (formerly "Thomson Reuters") Web of Science. In 2018 the second journal of the L.N. Gumilyov Eurasian National University, namely the Eurasian Journal of Mathematical and Computer Applications was also included in ESCI.

The ESCI was launched in late 2015 as a new database within "Clarivate Analytics". Around 3,000 journals were selected for coverage at launch, spanning the full range of subject areas.

The selection process for ESCI is the first step in applying to the Science Citation Index. All journals submitted for evaluation to the core Web of Science databases will now initially be evaluated for the ESCI, and if successful, indexed in the ESCI while undergoing the more in-depth editorial review. Timing for ESCI evaluation will follow "Clarivate Analytics" priorities for expanding database coverage, rather than the date that journals were submitted for evaluation.

Journals indexed in the ESCI will not receive Impact Factors; however, the citations from the ESCI will now be included in the citation counts for the Journal Citation Reports, therefore contributing to the Impact Factors of other journals. If a journal is indexed in the ESCI it will be discoverable via the Web of Science with an identical indexing process to any other indexed journal, with full citation counts, author information and other enrichment. Articles in ESCI indexed journals will be included in an author's H-Index calculation, and also any analysis conducted on Web of Science data or related products such as InCites. Indexing in the ESCI will improve the visibility of a journal, provides a mark of quality and is good for authors.

To commemorate this important achievement of mathematicians of the L.N. Gumilyov Eurasian National University on June 14, 2018, by the initiative of the "Clarivate Analytics", the awarding ceremony of the Certificate of Emerging Sources Citation Index database of "Clarivate Analytics" to the editorial boards of the Eurasian Mathematical Journal and the Eurasian Journal of Mathematical and Computer Applications was held at the L.N. Gumilyov Eurasian National University. The programme of this ceremony is attached.

Astana June 14, 2018

Venue: L.N. Gumilyov Eurasian National University
Astana, Satpayev street 2, Room 259

14:30- 15:00	Visit to the Museum of the history of Education, Museum of L.N. Gumilyov, Museum of writing
15:00-15:10	*Opening speech of moderator* **A. Moldazhanova** – the First Vice-Rector, Vice-Rector for Academic Works of L.N. Gumilyov Eurasian National University
15:10-15:20	**Oleg Utkin** - Managing Director of Clarivate Analytics in Russia and the CIS
15:20-15:30	*Certification award ceremony of the Eurasian Mathematical Journal, the Eurasian Journal of Mathematical and Computer Applications in international database*
15:30-15:45	**Kordan Ospanov** – Deputy Editor-in-Chief of the Eurasian Mathematical Journal. *History and perspectives of development of the scientific journal Eurasian Mathematical Journal*
15:45-16:00	**Kazizat Iskakov** – Deputy Editor-in-Chief of the Eurasian Journal of Mathematical and Computer Applications. *History and perspectives of development of the scientific journal Eurasian Journal of Mathematical and Computer Applications.*
16:00-16:10	*Closing Ceremony* *Memory photo*
16:10-16:30	*Coffee break for visitors*
16:40-17:20	**Lyaziza Mukasheva** - Official representative of Clarivate Analytics in the Central Asian region *Seminar for editors of scientific journals Scientific library of L.N. Gumilyov Eurasian National University room 104*

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EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF p-HAMILTONIAN SYSTEMS

M.R. Heidari Tavani

Communicated by V.I. Burenkov

Key words: periodic solutions, p-Hamiltonian systems, critical point theory,variational methods.

AMS Mathematics Subject Classification: 35J40, 35J35, 34B15.

Abstract. Based on some variational methods for smooth functionals defined on reflexive Banach spaces, the existence of periodic solutions for a class of p-Hamiltonian systems is established.

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1 Introduction

Consider the following p-Hamiltonian system

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda\nabla F(t,u) + \mu\nabla G(t,u) + h(t) \qquad a.e.\ \ t\in[0,T], \\ u(0)-u(T) = u'(0)-u'(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $p > 1$, $A : [0, T] \to \mathbb{R}^{N\times N}$ is a continuous map from the interval $[0, T]$ to the set of Nth-order symmetric matrices, $\lambda > 0$, $\mu \geq 0$ and $F, G : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ are functions measurable with respect to t, for all $x \in \mathbb{R}^N$, continuously differentiable in x, for almost every $t \in [0, T]$, and satisfying the following standard summability condition:

$$\sup_{|x|\le a} \max\{|F(\cdot,x)|, |G(\cdot,x)|,\ |\nabla F(\cdot,x)|, |\nabla G(\cdot,x)|\} \in L^1([0,T]) \tag{1.2}$$

for any $a > 0$. Also suppose that $h \in L^1(0, T; \mathbb{R}^N)$.

It is clear that if $\nabla F, \nabla G$ are assumed to be continuous in $[0, T] \times \mathbb{R}^N$, then condition (1.2) is satisfied. A special case of dynamical systems are Hamiltonian systems. This type of equations play an important role in fluid mechanics and gas dynamics. For the study of Hamiltonian systems see [20, 23]. In recent years, the existence of at least three periodic solutions for Hamiltonian systems have been studied in many papers (see [6, 12, 13, 15, 16, 17, 27, 28, 36] and the references therein). For example in [17], the authors proved the existence of periodic solutions for the following second-order Hamiltonian system

$$\begin{cases} -\ddot{u}(t) - q(t)\dot{u}(t) + A(t)u(t) = \lambda\nabla F(t,u(t)) + \mu\nabla G(t,u(t)) \qquad a.e.\ \ t\in[0,T], \\ u(0)-u(T) = \dot{u}(0)-\dot{u}(T) = 0. \end{cases} \tag{1.3}$$

by variational methods in the critical point theory. In problem (1.3), $q \in L^1(0, T; \mathbb{R})$. In fact, for $q = 0$ problem (1.3) is a special case of problem (1.1) when $p = 2$. A special case of problem (1.1) when $\mu = 0$, $\lambda = 1$, and $A(t) = 0$ has been investigated in [32]. In [34] the authors, studied the existence of at least three periodic solutions for problem (1.1) in the case of $h(t) = (0, ..., 0)$ using two theorems due respectively to Ricceri (see reference [17] in [34]) and Averna-Bonanno (see reference [13] in [34]).

In this paper, using two kinds of Critical Points Theorems obtained in [1] and [7] which we recall in the next section (Theorem 2.1 and Theorem 2.2), we ensure the existence of at least three weak solutions or one weak solution for problem (1.1) (see Theorem 3.1 and Theorem 3.3). This theorem has been used successfully employed to establish the existence of at least three solutions for perturbed boundary value problems in the papers [3, 4, 9, 14, 19].

2 Preliminaries

Our main tool are two Critical Points Theorems that we recall here in a convenient form. In the first theorem the coercivity of the functional $\Phi - \lambda\Psi$ is required. This theorem has been obtained in [7], and it is a more precise version of Theorem 3.2 of [2]. In the second theorem a special case of the Palais-Smale condition is assumed and it has been obtained in [1].

Theorem 2.1 ([7],Theorem 3.6). *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a functional coercive continuously Gâteaux differentiable and sequentially weakly lower semi-continuous, whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a functional continuously Gâteaux differentiable, whose Gâteaux derivative is compact. Moreover let $\Phi(0) = \Psi(0) = 0$.*

Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$ such that

(a_1) $\dfrac{\sup_{x\in\Phi^{-1}(-\infty,r]}\Psi(x)}{r} < \dfrac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(a_2) *for each* $\lambda \in \Lambda_r := \left]\dfrac{\Phi(\overline{x})}{\Psi(\overline{x})}, \dfrac{r}{\sup_{x\in\Phi^{-1}(-\infty,r]}\Psi(x)}\right[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Definition 1. Fix $r_1, r_2 \in [-\infty, +\infty]$ with $r_1 < r_2$. A Gâtuax differentiable function $I = \Phi - \Psi$ satisfies the Palais-Smale condition cut off lower at r_1 and upper at r_2 (in short ${}^{[r_1]}(PS)^{[r_2]}$ -condition) if any sequence $\{u_n\}$ such that
(a) $\{I(u_n)\}$ is bounded,
(b) $\lim\limits_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
(c) $r_1 < \Phi(u_n) < r_2 \quad \forall n \in N$,
has a convergent subsequence.

Remark 1. In Definition 1 if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ the Palais-Smale condition is denoted by $(PS)^{[r_2]}$, while if $r_1 \in \mathbb{R}$ and $r_2 = +\infty$ it is denoted by ${}^{[r_1]}(PS)$.

To introduce the next theorem let X be a nonempty set and $\Phi, \Psi : X \to \mathbb{R}$, be two functionals. For all $r \in \mathbb{R}$, we define

$$\varphi(r) := \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(v)}{r - \Phi(v)}, \tag{2.1}$$

Theorem 2.2 ([1],Theorem 5.2). *Let X be a real Banach space and $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals with Φ bounded from below . Fix $r > \inf_X \Phi$ and assume that, for each $\lambda \in \left]0, \dfrac{1}{\varphi(r)}\right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition .Then for each $\lambda \in \left]0, \dfrac{1}{\varphi(r)}\right[$, there is $u_{0,\lambda} \in \Phi^{-1}(]-\infty, r[)$ such that $I_\lambda(u_{0,\lambda}) \leqslant I_\lambda(u)$ for all $u \in \Phi^{-1}(]-\infty, r[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.*

We assume that the matrix A satisfies the following conditions:

(i) $A(t) = (a_{kl}(t))$, $k = 1, \dots, N$, $l = 1, \dots, N$, is a symmetric matrix with $a_{kl} \in L^\infty[0,T]$ for any $t \in [0,T]$,

(ii) there exists a positive constant $\underline{\delta}$ such that $\langle A(t)|x|^{p-2}x, x\rangle \geq \underline{\delta}\,|x|^p$ for all $x \in \mathbb{R}^N$ and $t \in [0,T]$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$ and in the other hand we know that $\langle A(t)|x|^{p-2}x, x\rangle \leq \bar{\delta}\,|x|^p$ for any $x \in \mathbb{R}^N$ and for every $t \in [0,T]$ where $\bar{\delta} \leq \sum_{i,j=1}^{N} \|a_{ij}\|$ ([34]) .

Let us recall some basic concepts. Denote

$$E = \{u : [0,T] \to \mathbb{R}^N,\ u \text{ is absolutely continuous},\ \ u(0) = u(T),\ \ u' \in L^p([0,T], \mathbb{R}^N)\}.$$

Assume that E is equipped with the following norm

$$\|u\|_E = \left(\int_0^T (|u'(t)|^p + |u(t)|^p) dt \right)^{\frac{1}{p}},\ \forall\ u \in E.$$

Also , we consider E with the norm

$$\|u\| = \left(\int_0^T [|u'(t)|^p + \langle A(t)|u(t)|^{p-2}u(t), u(t)\rangle] dt \right)^{\frac{1}{p}}.$$

The Banach space E is a separable and reflexive. Obviously, E is also a uniformly convex Banach space.

Due to the inequality

$$\min\{1, \underline{\delta}\}\|u\|_E^p \leq \|u\|^p \leq \max\{1, \bar{\delta}\}\|u\|_E^p,$$

the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_E$ (see [34]).

Since $(E, \|\cdot\|)$ is compactly embedded in $C([0,T], \mathbb{R}^N)$ (see [20]), there exists a positive constant

$$c \leq c_0 = \sqrt[q]{2}\max\{T^{\frac{1}{q}}, T^{-\frac{1}{p}}\}(\min\{1, \underline{\delta}\})^{-\frac{1}{p}} \tag{2.2}$$

where $q = \frac{p}{p-1}$ (see [34]), such that

$$\|u\|_\infty \leq c \parallel u \parallel, \tag{2.3}$$

where $\|u\|_\infty = \max_{t \in [0,T]} \mid u(t) \mid$.

Now we present the following lemma, which is required in the proof of the main theorem of this paper.

Lemma 2.1. *Let $I : X \to X^*$ be the operator defined by*

$$I(u)(v) = \int_0^T \langle |u'(t)|^{p-2}u'(t), v'(t)\rangle dt + \int_0^T \langle A(t)|u(t)|^{p-2}u(t), v(t)\rangle dt,$$

for every $u, v \in X$. Then I admits a continuous inverse on X^*.

Proof. Taking into account formula (2.2) from [25] for $p > 1$, there exists a positive constant C_p such that if $p \geq 2$, then

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq C_p|x - y|^p, \tag{2.4}$$

and if $1 < p < 2$, then

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq C_p \frac{|x - y|^2}{(|x| + |y|)^{2-p}}. \tag{2.5}$$

Now according to inequalities (2.4) and (2.5) for every $u, v \in X$ we have,

$$\langle I(u) - I(v), u - v\rangle \geq C_p\|u - v\|^p,$$

which shows that I is strictly monotone. So using ([33], Theorem 26.A(d)), I^{-1} exists and is continuous. □

Definition 2. A function $u \in E$ is a weak solution to problem (1.1), if

$$\int_0^T \langle |u'(t)|^{p-2}u'(t), v'(t)\rangle dt + \int_0^T \langle A(t)|u(t)|^{p-2}u(t), v(t)\rangle dt$$

$$-\lambda \int_0^T \langle \nabla F(t, u(t)), v(t)\rangle dt - \mu \int_0^T \langle \nabla G(t, u(t)), v(t)\rangle dt - \int_0^T \langle h(t), v(t)\rangle dt = 0$$

for every $v \in E$.

3 Main results

In this section, we use the following notation:

$$F^\theta := \int_0^T \sup_{|x|\leq\theta} F(t, x)dt, \quad t \in [0, T], \quad \forall\theta > 0, \tag{3.1}$$

$$G^\theta := \int_0^T \sup_{|x|\leq\theta} G(t, x)dt, \quad t \in [0, T], \quad \forall\theta > 0 \tag{3.2}$$

and

$$G_{x_0} := \int_0^T \inf_{x\in B} G(t, x)dt, \tag{3.3}$$

where $x_0 \in \mathbb{R}^N$ and $B = \{x \in \mathbb{R}^N \ : \ 0 \leq |x| \leq |x_0|\}$.

In order to introduce our first result, fix $\theta > 0$, $0 \neq x_0 \in \mathbb{R}^N$ such that

$$\frac{\bar{\delta}\, T|x_0|^p}{\displaystyle\int_0^T F(t, x_0)dt} < \frac{\theta^p}{c^p F^\theta},$$

where $c > 0$ is from inequality (2.3), and let

$$\Lambda := \left]\frac{\bar{\delta}\, T|x_0|^p}{p\displaystyle\int_0^T F(t, x_0)dt}, \frac{\theta^p}{pc^p F^\theta}\right[, \tag{3.4}$$

$$\delta^*_{\lambda,G} := \min\left\{\frac{\theta^p - pc^p\lambda F^\theta - pc^p\|h\|_{L^1[0,T]}\theta}{p\, c^p G^\theta}, \frac{\bar{\delta}|x_0|^p T - p\lambda\displaystyle\int_0^T F(t, x_0)dt - p\displaystyle\int_0^T \langle h(t), x_0\rangle dt}{p\, G_{x_0}}\right\}, \tag{3.5}$$

and

$$\overline{\delta}_{\lambda,G} := \min\left\{ \delta^*_{\lambda,G}, \frac{1}{\max\left\{0, pc^pT \limsup\limits_{|x|\to+\infty} \frac{\sup_{t\in[0,T]} G(t,x)}{|x|^p}\right\}} \right\}. \tag{3.6}$$

Here we agree to read $\dfrac{1}{0}$ as $+\infty$.

Now, we formulate our main result.

Theorem 3.1. *Let $F : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ satisfy assumption (1.2) and $F(t,0) = 0$ for all $t \in [0,T]$. Assume that the following conditions hold:*
(A_1) there exist positive constant θ and a point $x_0 \in \mathbb{R}^N$ with $\theta < |x_0|c(\underline{\delta}\ T)^{\frac{1}{p}}$, such that

$$\frac{c^pF^\theta}{\theta^p} < \frac{\int_0^T F(t,x_0)dt}{\overline{\delta}\, T|x_0|^p};$$

(A_2) $\limsup\limits_{|x|\to+\infty} \dfrac{\sup_{t\in[0,T]} F(t,x)}{|x|^p} \le 0.$

Then for every $\lambda \in \Lambda$, given by (3.4) and for every function $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying assumption (1.2) with $G(t,0) = 0$, for all $t \in [0,T]$ and such that

$$\limsup_{|x|\to+\infty} \frac{\sup_{t\in[0,T]} G(t,x)}{|x|^p} < +\infty, \tag{3.7}$$

there exists $\mu^* > 0$ such that for each $\mu \in [0,\mu^*)$, problem (1.1) has at least three weak solutions.

Proof. Fix $\lambda \in \Lambda$ and $\mu \in [0,\mu^*)$ where $\mu^* = \overline{\delta}_{\lambda,G}$ defined by (3.6). Let

$$\Phi(u) = \frac{1}{p}\|u\|^p, \quad \Psi(u) = \int_0^T (F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t)) + \frac{1}{\lambda}\langle h(t),u\rangle)dt$$

for each $u \in E$. Since the critical points of the functional $\Phi - \lambda\Psi$ on E are weak solutions to problem (1.1), our aim is to apply Theorem 2.1. It is well known that Ψ is continuously Gâteaux differentiable and the differential at the point $u \in E$ is

$$\Psi'(u)(v) = \int_0^T \langle\nabla F(t,u(t)),v(t)\rangle dt + \frac{\mu}{\lambda}\int_0^T \langle\nabla G(t,u(t)),v(t)\rangle dt + \frac{1}{\lambda}\int_0^T \langle h(t),v(t)\rangle dt,$$

for every $v \in E$. Now, we will prove that , Ψ' is compact. Indeed, it is enough to show that Ψ' is strongly continuous on E. For this end, for fixed $u \in E$, let $u_n \to u$ weakly in E as $n \to \infty$, then u_n converges uniformly to u on $[0,T]$ as $n \to \infty$ (because E is compactly embedded in $C([0,T],\mathbb{R}^N)$). Since F and G are continuously differentiable in u, for almost every $t \in [0,T]$ $\nabla F, \nabla G$ are continuous in $\mathbb{R}^{\mathbb{N}}$ for every $t \in [0,T]$, so

$$\nabla F(t,u_n) + \frac{\mu}{\lambda}\nabla G(t,u_n) \to \nabla F(t,u) + \frac{\mu}{\lambda}\nabla G(t,u),$$

as $n \to \infty$. Hence according to the above result and assumption (1.2) we have, $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Thus we proved that Ψ' is strongly continuous on E, which implies that Ψ' is a compact operator by Proposition 26.2 of [33] .

Moreover, Φ is continuously Gâteaux differentiable and the differential at the point $u \in E$ is

$$\Phi'(u)(v) = \int_0^T \langle |u'(t)|^{p-2}u'(t), v'(t)\rangle dt + \int_0^T \langle A(t)|u(t)|^{p-2}u(t), v(t)\rangle dt,$$

for every $v \in E$, Φ is sequentially weakly lower semicontinuous and also,by Lemma 2.1, Φ' admits a continuous inverse on E. Now are aim is to apply Theorem 2.1 to Φ and Ψ. To this end , we will verify conditions (a_1) and (a_2). Put $r = \frac{1}{p}(\frac{\theta}{c})^p$. Bearing in mind (2.3), we see that

$$\begin{aligned}
\Phi^{-1}(]-\infty, r]) &= \{u \in E;\ \Phi(u) \leq r\} \\
&= \left\{u \in E;\ \frac{||u||^p}{p} \leq r\right\} \\
&\subseteq \{u \in E;\ |u(t)| \leq \theta \text{ for each } t \in [0,T]\}.
\end{aligned}$$

Now we have

$$\sup_{\Phi(u)\leq r} \Psi(u) = \sup_{\Phi(u)\leq r} \int_0^T (F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t)) + \frac{1}{\lambda}\langle h(t), u\rangle)dt \leq$$
$$\int_0^T \sup_{|x|\leq\theta} F(t,x)dt + \frac{\mu}{\lambda}\int_0^T \sup_{|x|\leq\theta} G(t,x)dt + \frac{1}{\lambda}\int_0^T \sup_{|x|\leq\theta}\langle h(t), x\rangle dt =$$
$$F^\theta + \frac{\mu}{\lambda}G^\theta + \frac{1}{\lambda}\|h\|_{L^1[0,T]}\theta. \tag{3.8}$$

Since $G(t,0) = 0$ for all $t \in [0,T]$, It is clear that $G^\theta \geq 0$.
Now fix $\bar{x} = x_0$. Clearly $x_0 \in E$. One has

$$\Psi(\bar{x}) = \int_0^T F(t,x_0)dt + \frac{\mu}{\lambda}\int_0^T G(t,x_0)dt + \frac{1}{\lambda}\int_0^T \langle h(t), x_0\rangle dt \geq$$

$$\int_0^T F(t,x_0)dt + \frac{\mu}{\lambda}G_{x_0} + \frac{1}{\lambda}\int_0^T \langle h(t), x_0\rangle dt$$

Again since $G(t,0) = 0$ for all $t \in [0,T]$ we get that $G_{x_0} \leq 0$, and according to assumption (ii) from the previous section and also to the assumption $\theta < |x_0|c(\underline{\delta}\ T)^{\frac{1}{p}}$, we have

$$r < \frac{1}{p}\underline{\delta}\,|x_0|^p T \leq \Phi(\bar{x}) = \frac{1}{p}\|x_0\|^p = \frac{1}{p}\int_0^T \langle A(t)|x_0|^{p-2}x_0, x_0\rangle dt \leq \frac{1}{p}\bar{\delta}|x_0|^p T. \tag{3.9}$$

Therefore, we have

$$\frac{\Psi(\bar{x})}{\Phi(\bar{x})} \geq \frac{\int_0^T F(t,x_0)dt + \frac{\mu}{\lambda}G_{x_0} + \frac{1}{\lambda}\int_0^T \langle h(t), x_0\rangle dt}{\frac{1}{p}\bar{\delta}|x_0|^p T} \tag{3.10}$$

and

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} \leq \frac{F^\theta + \frac{\mu}{\lambda}G^\theta + \frac{1}{\lambda}\|h\|_{L^1[0,T]}\theta}{\frac{1}{p}(\frac{\theta}{c})^p} \tag{3.11}$$

Since $\mu < \mu^*$, one has

$$\mu < \frac{\theta^p - pc^p\lambda F^\theta - pc^p\|h\|_{L^1[0,T]}\theta}{p\,c^p G^\theta},$$

this means that

$$\frac{F^\theta + \frac{\mu}{\lambda}G^\theta + \frac{1}{\lambda}\|h\|_{L^1[0,T]}\theta}{\frac{1}{p}(\frac{\theta}{c})^p} < \frac{1}{\lambda}.$$

On the other hand

$$\mu < \frac{\bar{\delta}|x_0|^p\,T - p\lambda\int_0^T F(t,x_0)dt - p\int_0^T \langle h(t), x_0\rangle dt}{p\,G_{x_0}},$$

this means that

$$\frac{\int_0^T F(t,x_0)dt + \frac{\mu}{\lambda}G_{x_0} + \frac{1}{\lambda}\int_0^T \langle h(t), x_0\rangle dt}{\frac{1}{p}\bar{\delta}|x_0|^p\,T} > \frac{1}{\lambda}.$$

Then we have

$$\frac{F^\theta + \frac{\mu}{\lambda}G^\theta + \frac{1}{\lambda}\|h\|_{L^1[0,T]}\theta}{\frac{1}{p}(\frac{\theta}{c})^p} < \frac{1}{\lambda} < \frac{\int_0^T F(t,x_0)dt + \frac{\mu}{\lambda}G_{x_0} + \frac{1}{\lambda}\int_0^T \langle h(t), x_0\rangle dt}{\frac{1}{p}\bar{\delta}|x_0|^p\,T}. \tag{3.12}$$

Hence from (3.10),(3.11) and (3.12) condition (a_1) of Theorem 2.1 is verified. Now we prove that $\Phi - \lambda\Psi$ is coercive for every $\lambda \in \Lambda$.

Since $\mu < \overline{\delta}_{\lambda,G}$, we can fix $l > 0$ such that

$$\limsup_{|x|\to+\infty} \frac{\sup_{t\in[0,T]} G(t,x)}{|x|^p} < l,$$

and $\mu l < \dfrac{1}{pc^pT}$.

Therefore, there exists a function $K \in L^1([0,T])$ such that

$$G(t,x) \leq l|x|^p + K(t), \tag{3.13}$$

for every $t \in [0,T]$ and $x \in \mathbb{R}^N$.

Now, fix $0 < \eta < \dfrac{1}{p\lambda c^pT} - \dfrac{\mu l}{\lambda}$.

By (A_2) there is a function $M_\eta \in L^1([0,T])$ such that

$$F(t,x) \leq \eta|x|^p + M_\eta(t), \tag{3.14}$$

for every $t \in [0,T]$ and $x \in \mathbb{R}^N$. Now, for each $u \in X$, by using (2.3) we have

$$\Phi(u) - \lambda\Psi(u) = \frac{1}{p}\|u\|^p - \lambda\int_0^T (F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t)) + \frac{1}{\lambda}\langle h(t), u(t)\rangle)dt \geq$$

$$\frac{1}{p}\|u\|^p - \lambda\int_0^T (\eta|u|^p + M_\eta(t))dt - \mu\int_0^T (l|u|^p + K(t))dt - \int_0^T \langle h(t), u(t)\rangle dt \geq$$

$$(\frac{1}{p} - \lambda\eta c^pT - \mu l c^pT)\|u\|^p - \lambda\|M_\eta\|_{L^1[0,T]} - \mu\|K\|_{L^1[0,T]} - c\|h\|_{L^1[0,T]}\|u\|,$$

and thus

$$\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty,$$

which means the functional $\Phi-\lambda\Psi$ is coercive, and condition (a_2) of Theorem 2.1 is verified. Since from (3.12) $\lambda\in\left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})},\frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[$, Theorem 2.1 guarantees the existence of three critical points for the functional $\Phi-\lambda\Psi$, and the proof is complete. □

A corollary of Theorem 3.1 is as follows.

Theorem 3.2. *Let $F:\mathbb{R}^N\to\mathbb{R}$ be a function such that $F(0)=0$ and ∇F is continuous in $\mathbb{R}^N$. Moreover, suppose that there exists $x_0\in\mathbb{R}^N$, such that $F(x_0)>0$, and*

$$\liminf_{|\xi|\to0^+}\frac{F(\xi)}{|\xi|^p}=\limsup_{|\xi|\to+\infty}\frac{F(\xi)}{|\xi|^p}=0. \tag{3.15}$$

Then for each $\lambda>\lambda^*:=\frac{\bar{\delta}|x_0|^p}{pF(x_0)}$ and for every $\lambda>\lambda^*$ and for every function $G:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ satisfying assumption (1.2) with $G(t,0)=0$ for all $t\in[0,T]$ and such that

$$\limsup_{|\xi|\to+\infty}\frac{\sup_{t\in[0,T]}G(t,\xi)}{|\xi|^p}<+\infty,$$

there exists $\mu^*>0$ such that, for each $\mu\in[0,\mu^*[$, the problem

$$\begin{cases}-(|u'|^{p-2}u')'+A(t)|u|^{p-2}u=\lambda\nabla F(u)+\mu\nabla G(t,u)+h(t) & a.e.\ t\in[0,T],\\ u(0)-u(T)=u'(0)-u'(T)=0.\end{cases} \tag{3.16}$$

admits at least three weak solutions.

Proof. Fix $\lambda>\lambda^*:=\frac{\bar{\delta}|x_0|^p}{pF(x_0)}$ and $\mu\in[0,\mu^*[$ where $\mu^*=\bar{\delta}_{\lambda,G}$ defined by (3.6). According to condition (3.15) there is a sequence $\{\theta_n\}\subset]0,+\infty[$ such that $\lim_{n\to\infty}\theta_n=0$ and

$$\lim_{n\to\infty}\frac{\sup_{|\xi|\le\theta_n}F(\xi)}{\theta_n^p}=0.$$

So, there exists $\overline{\theta}>0$ such that

$$\frac{\sup_{|\xi|\le\overline{\theta}}F(\xi)}{\overline{\theta}^p}<\min\left\{\frac{F(x_0)}{\bar{\delta}|x_0|^pc^p},\ \frac{1}{pc^p\lambda}\right\}$$

and $\overline{\theta}<|x_0|c(\underline{\delta}\ T)^{\frac{1}{p}}$. Now the desired result can be obtained from Theorem 3.1. □

Now, we will present an example for Theorem 3.2.

Example 1. Let $T=1, p=3$ and $A(t)=I$, where I is identity matrix of order 3×3. Also let $F(x)=|x|^{14}e^{(1-0.1|x|^2)}$, $G(t,x)=e^{-t}|x|^3$ and $\mathrm{h(t)}=\begin{bmatrix}\mathrm{sint}\\ \mathrm{cost}-1\\ \mathrm{t}\end{bmatrix}$ for all $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ with $|x|=(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}$ and $t\in[0,1]$.

Then for every $\lambda>\lambda^*=\frac{1}{3e^{0.9}}$ all the hypotheses of Theorem 3.2 are satisfied with $\bar{\delta}=1$ and $x_0=(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$.

At this point, with various changes and new assumptions in Theorem 3.1, the existence of one weak solution to problem (1.1) will be proved.

Theorem 3.3. *Let $F : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ satisfy assumption* (1.2) *and $F(t,0) = 0$ for all $t \in [0,T]$. Suppose that $0 \neq x_0 \in \mathbb{R}^N$ is such that $|x_0| < (\frac{p}{T\overline{\delta}})^{\frac{1}{p}}$, $\int_0^T F(t,x_0)dt \geqslant 0$ and $\int_0^T \langle h(t),x_0\rangle dt \geqslant 0$.*

Then for every $\lambda \in \Lambda' = \left]0, \frac{1}{F^{cp^{\frac{1}{p}}}}\right[$, where $F^{cp^{\frac{1}{p}}}$ is defined by (3.1), *and for every non-negative function $G : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ satisfying assumption* (1.2) *with $G(t,0) = 0$, for all $t \in [0,T]$, there exists $\mu^* > 0$ such that for each $\mu \in [0,\mu^*)$ problem* (1.1) *admits at least one weak solution.*

Proof. Our aim is to apply Theorem 2.2 to problem (1.1). For this purpose fix $\lambda \in \Lambda'$ and consider $G^{cp^{\frac{1}{p}}}$ defined by (3.2). Also fix $\mu \in [0,\mu^*[$ where

$$\mu^* = \frac{1 - \lambda F^{cp^{\frac{1}{p}}} - cp^{\frac{1}{p}}\|h\|_{L^1[0,T]}}{G^{cp^{\frac{1}{p}}}}.$$

Let Φ and Ψ be as given in the proof of Theorem 3.1. First we prove that $I_\lambda = \Phi - \lambda\Psi$, satisfies $(PS)^{[r]}$- condition for all $r > 0$. Equivalently, we will prove that any sequence $\{u_n\} \subset E$ satisfying

$$L := \sup_n I_\lambda(u_n) < +\infty \quad , \quad \|I'_\lambda(u_n)\|_{E*} \to 0 \quad \textit{and} \quad \Phi(u_n) < r \quad \forall n \in N, \tag{3.17}$$

contains a convergent subsequence. Since Φ is coercive, from $\Phi(u_n) = \frac{1}{p}\|u_n\|^p < r$, $\forall n \in N$ one has that$\{u_n\}$ is bounded in E and hence by the Eberlian-Smulyan Theorem, passing to a subsequence if necessary we can assume that $u_n \rightharpoonup u_0$. Now since Ψ' is compact then $\Psi'(u_n) \to \Psi'(u_0)$. But from (3.17) we have $I'_\lambda(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) \to 0$. This implies that $u_n \to \Phi'^{-1}(\lambda\Psi'(u_0))$ (because Φ' is a homeomorphism) and finally according to the uniqueness of the weak limit, $u_n \to u_0$ in E and so I_λ satisfies $(PS)^{[r]}$-condition.

Put $r = 1, v = x_0$. Then by (3.9) and the condition $|x_0| < (\frac{p}{T\overline{\delta}})^{\frac{1}{p}}$, one has $\Phi(v) = \Phi(x_0) < 1 = r$. Also if $u \in \Phi^{-1}(]-\infty,1[)$ then $\Phi(u) < 1$ and so $\|u\| < p^{\frac{1}{p}}$. Hence according to (2.1) and (2.3) we get

$$\varphi(r) = \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u) - \Psi(v)}{r - \Phi(v)} \leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,1[)}\Psi(u) - \Psi(x_0)}{1 - \Phi(x_0)} \leq$$

$$\sup_{u\in\Phi^{-1}(]-\infty,1[)} \Psi(u) \leq F^{cp^{\frac{1}{p}}} + \frac{\mu}{\lambda}G^{cp^{\frac{1}{p}}} + \frac{1}{\lambda}cp^{\frac{1}{p}}\|h\|_{L^1[0,T]}. \tag{3.18}$$

Since

$$\mu < \mu^* = \frac{1 - \lambda F^{cp^{\frac{1}{p}}} - cp^{\frac{1}{p}}\|h\|_{L^1[0,T]}}{G^{cp^{\frac{1}{p}}}}$$

one has

$$F^{cp^{\frac{1}{p}}} + \frac{\mu}{\lambda}G^{cp^{\frac{1}{p}}} + \frac{1}{\lambda}cp^{\frac{1}{p}}\|h\|_{L^1[0,T]} < \frac{1}{\lambda}. \tag{3.19}$$

Hence from (3.18) and (3.19) we get

$$\lambda \in \Lambda' \subseteq \left]0, \frac{1}{\varphi(r)}\right[.$$

So all assumptions of Theorem 2.2 are verified. Therefore for every $\lambda \in \Lambda'$ the functional I_λ admits at least one critical point that is a weak solution to problem (1.1). $\square$

Now we point out the following applications of Theorem 3.3 when F does not depend on t.

Theorem 3.4. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a non-negative function with $F(0) = 0$ and ∇F be continuous in $\mathbb{R}^N$ and such that*

$$|\nabla F(x)| \leq a_0 + a_1|x|^{p-1} \tag{3.20}$$

for all $x \in \mathbb{R}^N$ where a_0 and a_1 are constants.

Then for every $\lambda \in \,]0, \lambda^*[$ where $\lambda^* = \left(T(a_0 c p^{\frac{1}{p}} + a_1 c^p)\right)^{-1}$ and for every non-negative function $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying assumption (1.2) with $G(t,0) = 0$, for all $t \in [0,T]$, there exist $\mu^* > 0$ such that for each $\mu \in [0, \mu^*[$ the problem

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda\nabla F(u) + \mu\nabla G(t,u) & a.e. \;\; t \in [0,T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3.21}$$

admits at least one weak solution.

Proof. First, we see that

$$F(x) - F(0) = \int_0^1 \langle \nabla F(sx).x \rangle ds.$$

Then from condition $F(0) = 0$ and (3.20) one has

$$F(x) = \int_0^1 |\nabla F(sx)||x| ds \leqslant \int_0^1 (a_0 + a_1 s^{p-1}|x|^{p-1})|x| ds = a_0|x| + \frac{a_1}{p}|x|^p.$$

Now,we have

$$F^{cp^{\frac{1}{p}}} = \int_0^T \sup_{|x| \leq cp^{\frac{1}{p}}} F(x) dt \leq \int_0^T \sup_{|x| \leq cp^{\frac{1}{p}}} (a_0|x| + \frac{a_1}{p}|x|^p) dt = T(a_0 c p^{\frac{1}{p}} + a_1 c^p) = \frac{1}{\lambda^*}. \tag{3.22}$$

Therefore from (3.22) one has, $\lambda \in]0, \lambda^*[\subseteq \left]0, \dfrac{1}{F^{cp^{\frac{1}{p}}}}\right[$ and the conclusion follows from Theorem 3.3. $\square$

Remark 2. If in Theorem 3.4, $\lambda\nabla F(0) + \mu\nabla G(t,0) \neq (0,0,0,...,0)$ then problem (3.21) has at least one non-trivial weak solution whose norm in E is less than $p^{\frac{1}{p}}$.

Finally, we present the following example to illustrate Theorem 3.4.

Example 2. Let $F(x) = |x|^3(1 - \cos(\ln(1 + |x|^2)))$. By a simple computation it can be shown that $|\nabla F(x)| \leq 8|x|^2$ for all $x \in \mathbb{R}^N$. So (3.20) is satisfied with $a_0 = 0, a_1 = 8$ and $p = 3$. Then for every $\lambda \in \,]0, \frac{1}{8c^3}[$ and for every non-negative function $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying assumption (1.2) with $G(t,0) = 0$, for all $t \in [0,T]$ and $\nabla G(t,0) \neq 0$, there exist $\mu^* > 0$ such that for each $\mu \in]0, \mu^*[$ the problem

$$\begin{cases} -(|u'|u')' + A(t)|u|u = \lambda\nabla F(u) + \mu\nabla G(t,u) & a.e. \;\; t \in [0,1], \\ u(0) - u(1) = u'(0) - u'(1) = 0. \end{cases} \tag{3.23}$$

admits at least one non-trivial weak solution.

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